PHYSICAL REVIEW D VOLUME 14, NUMBER 12 15 DECEMBER 1976

Vector coupling and bound states of fermions in three space dimensions*†

Johann Rafelski and Berndt Müller
Argonne National Laboratory, Argonne, Illinois 60439
and Institut für Theoretische Physik der Universität, Frankfurt, Federal Republic of Germany
(Received 17 May 1976)

The behavior of fermions interacting via vector gluons in the strong-coupling limit is investigated. A suitable coupling betweeen the Dirac and the vector-gluon field gives rise to bound-state solutions. The coherent field approximation is employed to find the bound-state masses, which are further discussed by analytical and numerical methods and are found to be positive-definite in the example considered numerically.

In the recently proposed theories of strong quark coupling via colored gluons ("chromodynamics") the interaction of a fermion field with a non-Abelian vector field of the Yang-Mills type plays an important role.[1] Motivated by the problem of quark confinement we make a step in the proposed direction and consider an Abelian, massive vector field in interaction with a spinor field. This is the simplest case, in which we can study the properties of bound-state solutions in three space dimensions involving vector fields. In contrast to the one-space-dimensional models where an inherent nonlinearity of the field is sufficient to generate stable extended (soliton) solutions, the analog in the three-dimensional space requires the presence of at least two interacting fields.[2]

We wish to discuss here the problem caused by the arbitrarily strong binding possible with a strong vector interaction. Our main result will be to show that in a schematic model with an attractive vector interaction, the total energy of the localized fermionic bound state remains positive as a function of the coupling strength. This happens because the contribution to the energy by the meson field compensates the negative part of the bound fermion energy.

It is well known that a three-dimensional fermion field of mass m subject to a sufficiently strong *external* vector potential can exhibit particle states bound by more than $2m$.[3] This leads to the spontaneous production of pairs, while the energy of the process is provided by the external field.[4] The manifestation of this phenomenon is the existence of a particle resonance which is embedded in the continuum of antiparticle scattering states.[3] We will call such states *supercritical* and the associated coupling *strong*.

To illustrate the behavior of a fermion field coupled strongly to a vector-meson field we consider the model Hamiltonian (with $A\cdot A \equiv A^2 = A_0{}^2 - \vec{A}\cdot\vec{A}$), where $W(A^2)$ is an arbitrary potential energy of the vector field A_μ,

$$H=\int d^3x\,\psi^\dagger[\vec{\alpha}\cdot\vec{p}+\beta m-g_V(\vec{\alpha}\cdot\vec{A}-A^0)]\psi - \tfrac{1}{2}\int d^3x\,[\pi_\mu{}^2+(\nabla A_\mu)^2+W(A\cdot A)-W(0)]\,, \quad (1)$$

which leads to the equations of motion

$$[\vec{\alpha}\cdot\vec{p}+\beta m-g_V(\vec{\alpha}\cdot\vec{A}-A^0)]\psi=\frac{i\partial\psi}{\partial t}\,, \quad (2a)$$

$$\pi_\mu-\Delta A_\mu+A_\mu\frac{\partial W}{\partial A^2}=+g_V\bar{\psi}\gamma_\mu\psi\,. \quad (2b)$$

We also have the auxiliary condition $\langle \mathrm{ps}|\partial_\mu A^\mu|\mathrm{ps}\rangle = 0$ for any physical state $|\mathrm{ps}\rangle$.

For strong interactions it is most convenient to choose as the basis for the expansion of the Fermi field ψ the complete set of solutions of the Dirac equation (ϵ stands for both discrete and continuous indices):

$$[\vec{\alpha}\cdot(\vec{p}-g_V\vec{A}_c)+\beta m+g_V A_c^0]\psi_\epsilon=E_\epsilon\psi_\epsilon\,, \quad (3)$$

where the c-number field A_c^μ is the mean field $A_c^\mu=\langle A^\mu\rangle$. The field operator is expanded in the quasiparticle Fock space as

$$\psi(\vec{x},0)=\int_m^\infty b_E\psi_E(\vec{x})dE+\sum_{m>E_k>-m}b_k\psi_k(\vec{x}) + \int_{-\infty}^{-m}d_E^\dagger\psi_E(\vec{x})dE \quad (4)$$

whenever no strongly coupled vector field is present.[5] For our purpose a different expansion is needed when a resonance of Eq. (3) is embedded in the negative energy continuum,[4] as illustrated in Fig. 1. Then we have

$$\psi(x,t)=\int_m^\infty b_E\psi_E dE+\sum_{m>E_k>-m}b_k\psi_k + b_0\int_{-\infty}^{-m}a(E)\psi_E dE + \int_{-\infty}^{-m}d_{E'}^\dagger\left[\int_{-\infty}^{-m}h_{E'}(E)\psi_E\,dE\right]dE'\,, \quad (4')$$

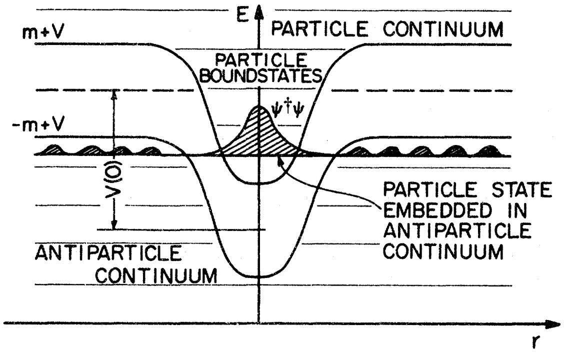

FIG. 1. Schematic description of the Dirac equation spectrum. Notice the localized particle solutions embedded in the antiparticle continuum well.

where $a(E)$ and $h_{E'}(E)$ are suitable functions that describe the resonance in the negative energy continuum (see the Appendix of Ref. 4). The extraction of the resonance from the negative-energy-continuum wave functions has been discussed in detail in related work.[5] If more than one resonance is embedded in the continuum, a generalization of Eq. (4′) is in order.

We would like to stress the difference between our self-consistent field problem and the case of an external field.[4,6] We will show that the neutral vacuum, characterized by $b_n|0\rangle = d_n|0\rangle = 0$, is in our case a *stable*, zero-energy state also for the strong coupling, since the total energy of the excited state $b_0^\dagger|0\rangle$ is positive in contradistinction to the situation with external vector potentials.

Our choice of the mean field A_c^μ was directed by our desire to eliminate from the ground state most contributions from the virtual quasiparticle excitations. Then A_c^μ is determined from

$$-\Delta A_c^\mu + A_c^\mu \left(\frac{\partial W}{\partial A^2}\right)_{A_c^2} = g_V \langle p|\tfrac{1}{2}[\bar\psi, \gamma^\mu \psi]|p\rangle . \qquad (5)$$

We postpone the discussion of the validity of the mean-field approximation. The condition for its validity is known for the case of strong coupling of meson fields to a static source.[7] We consider the state $b_0^\dagger|0\rangle = |p\rangle$ as a trial state. Then Eq. (5) becomes (neglecting virtual vacuum polarization[4])

$$-\Delta A_c^\mu + A_c^\mu \left(\frac{\partial W}{\partial A^2}\right)_{A_c^2} = g_V \bar\psi_0 \gamma^\mu \psi_0 . \qquad (6)$$

ψ_0 is the coefficient of b_0 in Eq. (4′) or the lowest-energy particle wave function in Eq. (4).

At this point, following the spirit of our introductory remarks, we modify our approach to allow bound fermion states in the presence of an Abelian vector field only. Therefore we arbitrarily change the sign on the right-hand side of Eq. (6) and call the vector field attractive;

$$-\Delta A_c^\mu + A_c^\mu \left(\frac{\partial W}{\partial A^2}\right)_{A_c^2} = -g_V \bar\psi_0 \gamma^\mu \psi_0 . \qquad (6')$$

We believe that this change can be justified in particular channels when the internal SU(3) symmetry of the fields involved is included in our discussion, in analogy to the different possible signs of the z component of the isospin. This change also means that the energy of the interacting fields is

$$\begin{aligned} E_V &= E_{\mathrm{Dirac}} + E_{\mathrm{meson}} \\ &= \int d^3x\, \psi_0^\dagger [\vec\alpha \cdot \vec p + \beta m - g_V(\vec\alpha \cdot \vec A - A^0)]\psi_0 \\ &\quad + \tfrac{1}{2}\int d^3x [(\nabla A_c^\mu)^2 + W(A_c \cdot A_c) - W(0)] \\ &= \int d^3x\, \mathcal{H} . \end{aligned} \qquad (7)$$

Considering that we are interested in the strong coupling, we believe that this is a satisfactory mock-up of the picture of a colored octet of SU(3) vector gluons that presumably mediates the interaction between the quarks in hadrons.

Before we discuss our numerical solutions, we derive a virial theorem for classical interacting fields to gather insight into the qualitative features of our model. Taking the expectation value of the commutator $[\vec x \cdot \vec p, \alpha \cdot \vec p + \beta m + \mathcal{V}]$ between any bound Dirac wave function one finds that

$$\langle \vec\alpha \cdot \vec p\rangle = \langle \vec x \cdot \vec\nabla \mathcal{V}\rangle , \qquad (8)$$

where $\mathcal{V}$ may be any momentum-independent matrix potential. In our model $\mathcal{V}$ is just the vector potential, $g_V\gamma_0\gamma \cdot A_c$. From Eqs. (8) and (6′) it follows that

$$\langle \vec\alpha \cdot (\vec p - g_V \vec A_c) + g_V A_c^0\rangle = +\int d^n x (\vec x \cdot \nabla A_c^\mu - A_c^\mu) \times \left(-\Delta A_c^\mu + A_c^\mu \frac{\partial W}{\partial A^2}\right), \qquad (9)$$

where we have generalized to n space dimensions in order to facilitate comparison with other work.[2]

Equation (9) can be rewritten with some effort to give an expression for the energy, Eq. (7):

$$E_V = +\int d^n x \left\{\frac{n-3}{2}(\nabla A_c^\mu)^2 + \frac{n+1}{2}[W(A_c^2) - W(0)] - A_c^2 \frac{\partial W}{\partial A_c^2}\right\} + m\int d^n x\, \bar\psi_0 \psi_0 . \qquad (10)$$

Even if we had succeeded in eliminating ψ completely from the energy (as is possible for a scalar interaction), Eq. (10) could *not* be used as a basis for a variational principle since the constraint on ψ_0 to be the lowest-energy particle solution of the Dirac equation is not implemented. For strong coupling, Eq. (10) is only an approximation, since ψ_0 is then a superposition of continuum waves and the virial theorem, Eq. (8), does not apply. The error made is small for the range of coupling strength discussed in our numerical investigation below.

The most remarkable result is the disappearance of the kinetic-energy term in the physical space (i.e., for $n=3$). If $W(A^2)$ is a polynomial in A^2, $E_V(A^2)$ does not contain any A^4 term. Except for the case $W=\mu^2A^2$ the minimum of E_V does not coincide with the minimum of W. The effective shift is due to the interaction with the fermion field.

Let us devote the rest of our discussion to the special case $W=\mu^2A^2$. Then Eq. (10) reduces, for $n=3$, to

$$E_V=\int d^3x(\mu^2 A_c{}^2+m\bar{\psi}_0\psi_0)\,. \tag{11}$$

We note a partial cancellation between E_{Dirac} and E_{meson} that has been mentioned previously. The energy E_V is positive-definite if the scalar integral $\int d^3x\,\bar{\psi}_0\psi_0$ is positive. This is a sufficient but not necessary condition. We have not established a lower bound for the scalar integral, but we have not been able to find an example for which it is negative.

The positivity of the bound-state energy will be demonstrated rigorously only by the numerical calculations presented in the following part of the paper for the model defined in the sentence above Eq. (11). The family of variational states considered is described in the vicinity of Eq. (5) and consists of a direct product between a coherent state in the Hilbert space of the meson field with a quasifermion state. It is conceivable that in a different model than that considered here numerically, a negative energy will be found.

We have solved numerically Eqs. (3) and (6′) for the lowest-energy particle state such that the Dirac wave functions are normalized to 1. The general behavior of the solutions is quite different from that reported[8] for scalar interactions.[9] In Fig. 2 we plot the energy of the bound fermion, E_{Dirac} [Eq. (7)], the total energy E_V, and the term $E_{\mathrm{mass}}=m\int d^3x\,\bar{\psi}_0\psi_0$, as a function of the coupling constant g_V for vector-meson masses $\mu=0.2m$ and $0.4m$. The energy is measured in units of the bare fermion mass m. We find that, as predicted

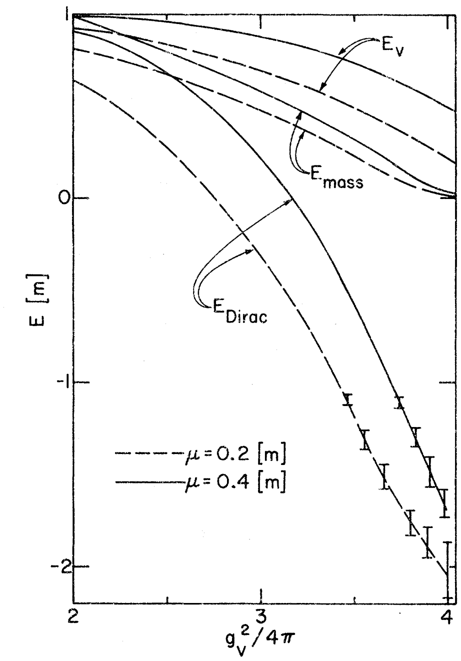

FIG. 2. Numerical results for soliton solutions as a function of the coupling constant: Dashed lines apply to gluon mass $\mu=0.2m$ while full lines are for $\mu=0.4m$. E_{Dirac} denotes the Dirac eigenenergy, E_{mass} corresponds to the $m\int d^3x\,\bar{\psi}_0\psi_0$ term, while E_V is the total energy (mass) of the bound state. The error bars at E_{Dirac} are qualitative signs to stress that E_{Dirac} is no longer a member of the discrete spectrum (beyond $E_{\mathrm{Dirac}}=-m$).

by the virial theorem, the total energy is always positive and very small for coupling constants $g_V{}^2>4$, while the Dirac eigenvalues E_{Dirac} take large negative values. E_{mass} is a negligible contribution to E_V, Eq. (11), for strong coupling. We also found an exact agreement of our results with those expected from the virial theorem, Eq. (11), as long as $E_{\mathrm{Dirac}}>-m$. As mentioned already, for strong coupling the virial theorem is only an approximation. For the range of coupling constants shown in Fig. 2 with $E_{\mathrm{Dirac}}<-m$, the error made using the virial theorem, Eq. (11), to calculate E_V with given A_c^0 is less than 1%.

Our main result is the recognition that the total energy of the strongly bound state lies above that of the neutral vacuum, which therefore is a stable ground state of the theory. Spontaneous production of bound-state pairs by the neutral vacuum is forbidden in the strong-coupling limit, although the energy of each bound fermion, E_{Dirac}, is smaller than $-m$. The localized bound state is the lowest-energy state of the charged sector of the Hilbert space. We believe that this behavior is symptomatic for similar and more complicated interactions—in particular, the non-Abelian vector gluons. Further, we have presented a new form of the virial theorem for interacting classical fields.

One of us (J.R.) would like to thank F. Coester for stimulating and fruitful discussions and J. W. Clark for critical reading of the manuscript.

---

*Work performed under the auspices of the U.S. Energy Research and Development Administration.

†Work supported by Bundesministerium für Forschung und Technologie, by the Deutsche Forschungsgemeinschaft, and by the Gesellschaft für Schwerionen (GSI).

[1]M. Gell-Mann, Institute for Advanced Study, Princeton, report, 1974 (unpublished).

[2]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975), Sec. II.

[3]B. Müller, H. Peitz, J. Rafelski, and W. Greiner, Phys. Rev. Lett. 28, 1235 (1972), and references therein.

[4]J. Rafelski, B. Müller, and W. Greiner, Nucl. Phys. B68, 585 (1974), Sec. III.

[5]See Ref. 4, Sec. II.

[6]L. P. Fulcher and A. Klein, Ann. Phys. (N.Y.) 84, 335 (1974).

[7]G. Wentzel, Helv. Phys. Acta 38, 65 (1965), and references therein. For a simple introduction to this field see also F. Coester, Helv. Phys. Acta 17, 35 (1944).

[8]J. Rafelski, Phys. Rev. D 14, 2358 (1976).

[9]Although the solutions exhibit qualitatively acceptable form factors, their size and other matrix elements of the radial wave function do not allow their immediate use as a model of quark confinement and hadronic structure, as was the case with scalar interaction (Ref. 8).