Further development of the new bound-state formalism

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The new bound-state formalism developed previously is here extended in two important respects. First is the inclusion of self-energy effects and the demonstration of the implementability of the usual renormalization procedure by means of a calculation of the lowest-order self-energy effects comprising the Lamb shift in a hydrogenlike atom. Second is the derivation of the two-body equation for the particle-antiparticle case where annihilation effects enter. This is exemplified by the derivation of the positronium equation and a calculation of a (known) annihilation contribution to its singlet-triplet splitting. The entire development is facilitated by a modified functional formulation which chooses the fully interacting physical system as the unperturbed one.

I. INTRODUCTION

In a previous paper[1] (hereafter referred to as paper I), a new formalism for the description of bound states in quantum field theory was derived within the context of quantum electrodynamics. The basic motivation for this development was to provide an alternative to the Bethe-Salpeter formalism which is free from the undesirable features suffered by the latter. The importance of such an alternative derives not only from the role it plays in dealing with high-precision atomic physics, but also from the very likely applications it will find in the constituent theories of particle structure and interactions once these achieve the required degree of definitiveness and precision.

In this paper, we extend the formalism of paper I in two important respects. First, we include the consideration of self-energy effects and the associated renormalization. The exclusion of these in paper I was deemed an important shortcoming, particularly since the "Hamiltonian" appearance of our two-body equation in the center-of-mass frame might make its renormalizability doubtful. However, as surmised in paper I, the underlying covariance and gauge invariance of our equations allow a convenient contact with the usual renormalization procedure. We shall demonstrate this by stating the principle of renormalization in the present context and reducing the lowest-order self-energy effects in hydrogenlike atoms to the known contributions to the Lamb shift. To stress the structural aspects of our formulation, particularly gauge invariance, we shall calculate the electron self-energy part (the more significant one from the standpoint of testing renormalizability) in the Coulomb gauge and that of the photon in the Feynman gauge.

The second extension mentioned above is the consideration of a particle-antiparticle system such as positronium, and the derivation of the corresponding two-body equation. This equation differs from Eq. (5) of paper I in a term which accounts for annihilation effects. Furthermore, this modification does not harm the desirable features of the equation discussed in paper I. Application of this equation is illustrated by the calculation of the single-photon annihilation contribution to the hyperfine structure of positronium. In addition, an interesting condition, satisfied by the positronium amplitude as a consequence of current conservation, is noted and discussed.

The above developments are here facilitated by a functional formulation which chooses the fully interacting physical system as the unperturbed system, in contradistinction to the conventional formulation which was employed in paper I.[2] This new derivation then allows a direct consideration of two-body amplitudes and avoids the intermediary of two-body Green's functions and the Gell-Mann–Low limiting process.[3]

This paper is organized as follows: Section II contains the derivation of the two-body formalism for the case of a particle-antiparticle system within the context of quantum electrodynamics. In Sec. III we consider self-energy effects and renormalization, and derive the Lamb-shift contributions. In Sec. IV we illustrate the application of the particle-antiparticle equations to positronium. Section V contains our concluding remarks.

II. DERIVATION OF THE PARTICLE-ANTIPARTICLE EQUATIONS

Our first task here is the modification of the usual perturbation scheme for the calculation of functional derivatives. This modification consists in considering the interacting electron-photon system as the unperturbed system, to be perturbed by the external (c-number) current J. The latter function is required to vanish at distant times, so that the system in fact reduces to the physical one

at such times and allows a *rigorous* definition of an *interaction picture*. It is then a matter of standard procedure to obtain the following relation for time-ordered products:

$$^J\langle\beta,\text{out}|T[\psi^J(x_1)\phi^J(x_2)\cdots]|\alpha,\text{in}\rangle^J/{}^J\langle 0,\text{out}|0,\text{in}\rangle^J = \langle\beta,\text{out}|T[S^J\psi(x_1)\phi(x_2)\cdots]|\alpha,\text{in}\rangle/\langle 0|S^J|0\rangle, \tag{1}$$

where

$$S^J = T\left\{\exp\left[-i\int d^4x\, J(x)\cdot\mathfrak{A}(x)\right]\right\}.$$

Here the superscript J denotes the presence of the external source, $\mathfrak{A}$ denotes the electromagnetic field operator, and all states and operators refer to the Heisenberg picture.

The object of interest here is the amplitude corresponding to the electron-positron system:

$$\chi^J_{\alpha\beta}(x,y) = Z^{J\,J}\langle 0,\text{out}|T[\psi^{(e)J}_\alpha(x)\psi^{(p)J}_\beta(y)]Q^J|P,\text{in}\rangle^J,$$

where

$$(Z^J)^{-1} = {}^J\langle 0,\text{out}|0,\text{in}\rangle^J,$$

$$Q^J = 1 - Z^J|0,\text{in}\rangle^{J\,J}\langle 0,\text{out}|.$$

Here $\psi^{(e)J}(\psi^{(p)J})$ denotes the electron (positron) field operator, and $|P,\text{in}\rangle^J$ stands for the positronium state of total momentum P^μ, all in the presence of J. The corresponding objects in the absence of J are, of course, the physical ones. Note that

$$Q^J|0,\text{in}\rangle^J = 0, \quad {}^J\langle 0,\text{out}|Q^J = 0, \quad Q|0\rangle = 0,$$
$$\langle 0|Q = 0, \quad Q|P,\text{in}\rangle = |P,\text{in}\rangle,$$

implying that in the limit $J\to 0$, (2) reduces to

$$\chi_{\alpha\beta}(x,y) = \langle 0|T[\psi^{(e)}_\alpha(x)\psi^{(p)}_\beta(y)]P,\text{in}\rangle, \tag{2}$$

which is the customary definition of the positronium amplitude.

The reason for the insertion of Q^J in the definition of χ^J is to allow for a suitable description of (virtual) annihilation processes [see Eq. (3) below], and it may be traced to the antisymmetry property of the two-electron Green's function.[4]

We may now proceed to evaluate the necessary functional derivatives using the alternative expression for χ^J provided by Eq. (1):

$$\chi^J_{\alpha\beta}(x,y) = \langle 0|T[S^J\psi^{(e)}_\alpha(x)\psi^{(p)}_\beta(y)]|P,\text{in}\rangle/\langle 0|S^J|0\rangle - \langle 0|T[S^J\psi^{(e)}_\alpha(x)\psi^{(p)}_\beta(y)]|0\rangle\langle 0|S^J|P,\text{in}\rangle \times[\langle 0|S^J|0\rangle]^{-2}. \tag{3}$$

Note that the above expression for χ^J would be meaningless within the customary functional formulation where the unperturbed system is noninteracting and therefore without bound states.

Using the standard procedures of the functional formulation,[2] we obtain Eqs. (1) and (2) of paper I, and proceed to evaluate the quantity (see paper I for notation)

$$R = n^\mu(\gamma^{(e)}_\mu\mathfrak{L}^{(e)}_x + \gamma^{(p)}_\mu\mathfrak{L}^{(p)}_y)\chi^J(x,y).$$

Recall that R originates in virtual annihilation effects and was absent in the cases considered in paper I. It is given by

$$R = e[n^\mu\gamma^{(e)}_\mu\gamma^{(e)}_\nu F^\nu(x) - n^\mu\gamma^{(p)}_\mu\gamma^{(p)}_\nu F^\nu(y)]\,S^J(x,y)C, \tag{4}$$

where

$$F^\mu(x) = \frac{\delta}{\delta J_\mu(x)}[Z^{J\,J}\langle 0,\text{out}|P,\text{in}\rangle^J],$$

and C is the charge conjugation matrix $i\gamma^2\gamma^0$. The differentiation involved in the expression for F may be performed to give

$$F^\mu(x) = -iZ^{J\,J}\langle 0,\text{out}|\mathfrak{A}^\mu(x) - A^\mu(x)|P,\text{in}\rangle^J = -iZ^{J\,J}\langle 0,\text{out}|\mathfrak{A}^\mu(x)Q^J|P,\text{in}\rangle^J.$$

On the other hand, using the definition of the current vector, we obtain

$$-e\,\text{tr}[\gamma^\mu\chi^J(x,x)C] = Z^{J\,J}\langle 0,\text{out}|j^\mu(x)Q^J|P,\text{in}\rangle^J. \tag{5}$$

As is well known, the above quantity is a potentially singular one, and a more careful definition involving a line-integral factor and a special limiting procedure must be applied in cases of singular behavior.[5]

Using the field equation

$$\mathfrak{A}_\mu(x) = \int d^4x' D^0_{\mu\nu}(x-x')[j^\nu(x') + J^\nu(x')] + \mathfrak{A}^{\text{free}}_\mu(x),$$

and the properties of Q^J, we can convert (5) into

$$F_\mu(x) = ie\int d^4x' D^0_{\mu\nu}(x-x')\,\text{tr}[\gamma^\nu\chi^J(x',x')C]. \tag{6}$$

Equation (6) completes the evaluation of R. We now change the functional variable to A, and collect the set of relations complementing the positronium equation:

$$n^\mu(\gamma_\mu^{(e)}\mathfrak{L}_x^{(e)}+\gamma_\mu^{(p)}\mathfrak{L}_y^{(p)})\chi^A(x,y)=en^\mu[\gamma_\mu^{(e)}\gamma_\nu^{(e)}F^\nu(x)-\gamma_\mu^{(p)}\gamma_\nu^{(p)}F^\nu(y)]S^A(x,y)C,$$

$$\mathfrak{L}_x^{(e),(p)}=\gamma_\mu^{(e),(p)}\left[p_x^\mu\mp eA^\mu(x)\mp ie\int d^4z\,D_\nu^{A\mu}(z,x)\frac{\delta}{\delta A_\nu(z)}\right]-m,$$

$$F_\mu(x)=ie\int d^4z\,D^0_{\mu\nu}(x-z)\ \mathrm{tr}[\gamma^\nu\chi^A(z,z)C], \tag{7}$$

$$D^A_{\mu\nu}(x,y)=D^0_{\mu\nu}(x-y)-ie\int d^4z\,d^4z'D^0_{\mu\lambda}(x-z)\ \mathrm{tr}\left[\gamma^\lambda\frac{\delta S^A(z,z)}{\delta A_\sigma(z')}\right]D^A_{\sigma\nu}(z',y),$$

$$\mathfrak{L}_x^{(e)}S^A(x,y)=\delta^4(x-y).$$

In the above equations, D^0 is the free photon propagator in any gauge. Equation (7) is the two-body equation for positronium. As mentioned before, it maintains the property of reducing to a single-time, Schrödinger-type equation in the center-of-mass frame.

Before applying (7), we wish to note an interesting constraint obeyed by χ. Equation (5) in the physical limit reads

$$-e\ \mathrm{tr}[\gamma^\mu\chi(x,x)C]=\langle 0|j^\mu(x)|P,\mathrm{in}\rangle,$$

which upon the imposition of current conservation, gives

$$\frac{\partial}{\partial x_\mu}\mathrm{tr}[\gamma_\mu\chi(x,x)C]=0.$$

On the other hand, in this limit χ has the structure

$$\chi(x,x)=\exp(-ix_\mu P^\mu)\chi(0,0).$$

The last two relations combine to give the sought-after constraint

$$\mathrm{tr}[n_\mu\gamma^\mu\chi(0,0)C]=0,$$

which, in the center-of-mass frame, reduces to

$$\gamma_{\alpha\beta}{}^2\chi_{\beta\alpha}(0,0)=0\ \text{(center-of-mass frame)}. \tag{8}$$

We prove in the Appendix that χ, as given by our Eq. (7), satisfies this constraint.

III. RENORMALIZATION

As mentioned in the Introduction, our task here is to show that the usual renormalization procedure is implementable within the present formulation. To this end, we shall consider the second-order electron and photon self-energy contributions in a two-body atom, paying particular attention to the structural properties of covariance and gauge invariance.

We shall carry out the calculations for an atom composed of two arbitrary particles (a) and (b), and use the notation of paper I when appropriate. The relevant equations are then (1′), (5), and (6) of paper I. Within the context of these equations, the usual renormalization program may be stated as the requirement that the two-body equation shall, in the limit of large interparticle separation, reduce (exactly) to the (simplified) Breit equation, Eq. (14) of paper I, with masses and charges equal to the corresponding observed quantities. Note that the above requirement of reduction to the Breit equation is nothing but a simultaneous statement of the principles of mass renormalization (which specifies the interaction-free part of the Breit Hamiltonian for two spin-$\frac{1}{2}$ particles) and charge renormalization (which specifies the Coulomb potential for two widely separated charges).

As mentioned before, we shall employ the (obviously unsuitable) Coulomb gauge for the electron self-energy part to dispel any fear that its lack of manifest covariance might disturb the renormalization procedure. Thus starting with the above-mentioned set of equations, we specialize to the center-of-mass frame, Eq. (7) of paper I, replace D^A by D_{CR} (the regularized, free photon propagator in the Coulomb gauge),

$$D_{CR}(x)=-(2\pi)^{-4}\int d^4k\,e^{-ik\cdot x}\left[g^{\mu\nu}+\frac{k^\mu k^\nu-n\cdot k(k^\mu n^\nu+k^\nu n^\mu)}{(k\cdot n)^2-k^2}\right][(k^2+i\epsilon)^{-1}-(k^2-\Lambda^2+i\epsilon)^{-1}],$$

and effect the transformation

$$\chi^A(x,y)=\exp\left\{-i\int_{-\infty}^{t}d\tau[e^{(a)}A^0(\tau,\vec{x})+e^{(b)}A^0(\tau,\vec{y})]\right\}\hat{\chi}^A(x,y).$$

The resulting equation for $\hat{\chi}$ is

$$i\frac{\partial}{\partial t}\hat{\chi}^A(t,\vec{x},\vec{y})=\left(h+V_C+\frac{e^2}{4\pi}\Lambda+\hat{U}^{(a)}+\hat{U}^{(b)}\right)\hat{\chi}^A(t,\vec{x},\vec{y}), \tag{9}$$

where

$$h = h^{(a)} + h^{(b)}, \quad h^{(a)} = \vec{\alpha}^{(a)} \cdot \vec{p}_x + \beta^{(a)} m^{(a)}, \quad V_C = -\frac{e^2}{4\pi} \frac{1}{|\vec{x} - \vec{y}|},$$

$$\hat{U}^{(a)}(t, \vec{x}) = e^{(a)} \left\{ i \int d^4 z \, D^{00}_{CR}(z - x) \frac{\delta}{\delta A^0(z)} - \alpha_i^{(a)} \left[A^i(x) + \int_{-\infty}^{t} d\tau \, \nabla^i A^0(\tau, \vec{x}) + i \int d^4 z \, D^{ij}_{CR}(z - x) \frac{\delta}{\delta A^j(z)} \right] \right\},$$

where $h^{(b)}$ and $\hat{U}^{(b)}$ are correspondingly defined. Note the appearance of the cutoff-dependent term in (9).

The effect we are seeking is obtained by calculating the lowest-order self-energy contribution of $\hat{U}^{(a)} + \hat{U}^{(b)}$ according to the perturbation scheme of paper I, and adding the cutoff term appearing in (9) thereto:

$$\Delta_n^S = ie^2 \int d(\vec{x} - \vec{y}) d\vec{x}' d\vec{y}' dt' \chi^{B\dagger}(\vec{x} - \vec{y}) [S^{(a)}(x, y \,|\, x', y') + S^{(b)}(x, y \,|\, x', y')] \chi^B(\vec{x}' - \vec{y}') + \frac{e^2}{4\pi} \Lambda,$$

where

$$S^{(a)}(x, y \,|\, x', y') = -\int d^4 z \, D^{00}_{CR}(z - x) \frac{\delta}{\delta A^0(z)} G(x, y \,|\, x', y') \vec{\alpha}^{(a)} \cdot \int_{-\infty}^{t'} d\tau \, \vec{\nabla} A^0(\tau, \vec{x}')$$

$$+ \int d^4 z \, \alpha_i^{(a)} D^{ij}_{CR}(z - x) \frac{\delta}{\delta A^j(z)} G(x, y \,|\, x', y') \vec{\alpha}^{(b)} \cdot \vec{A}(x'),$$

and similarly for $S^{(b)}$. Recall that χ^B is a solution of the Breit equation, Eq. (14) of paper I.

First we consider those terms that involve A^0, which we denote by $S_L^{(a)} + S_L^{(b)}$. These are

$$S_L^{(a)} + S_L^{(b)} = -\int dz^0 \theta(t' - z^0) G(x, y \,|\, x', y') \left[\vec{\alpha}^{(a)} \cdot \vec{\nabla}_{x'} D^{00}_{CR}(z^0 - t, \vec{x}' - \vec{x}) + \vec{\alpha}^{(b)} \cdot \vec{\nabla}_{y'} D^{00}_{CR}(z^0 - t, \vec{y}' - \vec{y}) \right].$$

The quantity in the brackets can be recognized as a commutator involving $h + V_C$. Using this fact, one can transform the last equation to

$$S_L^{(a)} + S_L^{(b)} = G(x, y \,|\, x', y') [D^{00}_{CR}(t' - t, \vec{x}' - \vec{x}) + D^{00}_{CR}(t' - t, \vec{y}' - \vec{y})]$$

$$+ 2i \int dz^0 \theta(t' - z^0) \delta(\vec{x} - \vec{x}') \delta(\vec{y} - \vec{y}') D^{00}_{CR}(z^0 - t', 0).$$

The last integral is just the negative of the cutoff term that appeared before. Thus this spurious term, which originates in "longitudinal photons" attending the use of the Coulomb gauge, is canceled by a corresponding contribution, leaving

$$\Delta_n^S = \int dt' d(\vec{x} - \vec{y}) d\vec{x}' d\vec{y}' \chi^{B\dagger}(\vec{x} - \vec{y}) \times [\hat{S}^{(a)}(x, y \,|\, x', y') + \hat{S}^{(b)}(x, y \,|\, x', y')] \chi^B(\vec{x}' - \vec{y}'), \tag{10}$$

where

$$\hat{S}^{(a)} = D^{\mu\nu}_{CR}(t' - t, \vec{x}' - \vec{x}) \gamma_0^{(a)} \gamma_\mu^{(a)} G(x, y \,|\, x', y') \gamma_0^{(a)} \gamma_\nu^{(a)},$$

and similarly for $\hat{S}^{(b)}$.

At this juncture we note that we essentially have the usual result in (10), except that our photon propagator is in the Coulomb gauge, and our wave functions are the two-body amplitudes of the Breit equation. Actually, it is an easy matter to show that the contribution of the gauge terms in (10) vanishes identically, effectively replacing D_{CR} by the corresponding Feynman propagator D_{FR}. The expression (10) may now be renormalized according to the program stated before, namely, reduction to the Breit equation at large distances. We shall not pursue this program further, but refer to standard calculations that reduce (10) to the Lamb-shift contributions within suitable approximations.[6]

We shall now proceed with the outline of the computation of the vacuum polarization contribution. This calculation is almost trivial, since the relevant renormalization concerns only the photon propagator and not the two-body equation. Nevertheless, it will serve to demonstrate the use of a manifestly covariant gauge in our calculations.

Referring to the two equations preceding Eq. (9) above, we now use D_F' (the Feynman propagator including the lowest-order radiative correction) in place of D_{CR} and effect the appropriate transformation to arrive at the Breit equation with the potential

$$V_C'(\vec{x} - \vec{y}) = -e^2 \int d^4 z \, \gamma_0^{(a)} \gamma_\mu^{(a)} D_F'^{\mu 0}(z - x) \frac{\delta}{\delta B^0(z)} \int_{-\infty}^{t} d\tau \, B^0(\tau, \vec{y}) - e^2 \int d^4 z \, \gamma_0^{(b)} \gamma_\mu^{(b)} D_F'^{\mu 0}(z - y) \frac{\delta}{\delta A^0(z)} \int_{-\infty}^{t} d\tau \, A^0(\tau, \vec{x})$$

$$= -e^2 \gamma_0^{(a)} \gamma_\mu^{(a)} \int_{-\infty}^{0} d\xi \, D_F'^{\mu 0}(\xi, \vec{y} - \vec{x}) - e^2 \gamma_0^{(b)} \gamma_\mu^{(b)} \int_{-\infty}^{0} d\xi \, D_F'^{\mu 0}(\xi, \vec{x} - \vec{y}), \tag{11}$$

where D_F' is given by

$$D_F'^{\mu\nu}(x-y)=g^{\mu\nu}D_F(x-y)-ie\lim_{A,B\to 0}\int d^4z\,d^4z'D_F(x-z)\ \mathrm{tr}\left[\gamma^\mu\frac{\delta S^A(z,z)}{\delta A_\nu(z')}+\gamma^\mu\frac{\delta S^B(z,z)}{\delta B_\nu(z')}\right]D_F(z'-y).$$

Obviously D_F' is nothing but the usual Feynman propagator augmented by the single-loop corrections contributed by particles (a) and (b), respectively. The renormalization required here is therefore quite the same as in the usual theory, and we take the standard result

$$D_F'^{\mu\nu}(x)=-(2\pi)^{-4}\int d^4k\frac{e^{-ik\cdot x}}{k^2+i\epsilon}\left[g^{\mu\nu}+\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)I(k^2)\right],$$

where

$$I(k^2)=\frac{e^2}{2\pi^2}\int_0^1 d\xi(1-\xi)\xi\ln\left\{\left[1-\frac{\xi(1-\xi)k^2}{(m^{(a)})^2-i\epsilon}\right]\left[1-\frac{\xi(1-\xi)k^2}{(m^{(b)})^2-i\epsilon}\right]\right\}.$$

Inserting this result in (11), and performing some simple steps, we obtain

$$V_C'(\vec{x}-\vec{y})=e^2(2\pi)^{-3}\int d^4k\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon}[1+I(k^2)]\delta(k^0)+(\vec{\alpha}^{(a)}-\vec{\alpha}^{(b)})\cdot\vec{\nabla}_x f(|\vec{x}-\vec{y}|),$$

where we have left f unspecified, since, in the resulting equation

$$i\frac{\partial}{\partial t}\chi^B(t,\vec{x},\vec{y})=[h_x^{(a)}+h_y^{(b)}+V_C'(\vec{x}-\vec{y})]\chi^B(t,\vec{x},\vec{y}),$$

the second piece of V_C' can be gauged away via the transformation

$$\chi^B=\exp(-if)\hat{\chi}^B.$$

The modified Coulomb potential is thus given by the first term of V_C', which is easily recognized as containing the usual vacuum polarization correction.

IV. APPLICATION OF THE POSITRONIUM EQUATION

In this section we shall derive the single-photon annihilation contribution to the singlet-triplet splitting of positronium using the equations derived in Sec. II.

The application procedure, as before, consists in extracting the Coulomb potential in the center-of-mass frame, discarding all functional derivatives remaining on the left-hand side of (7), and setting the functional variables equal to zero. The result is

$$\left(i\frac{\partial}{\partial t}-h-V_C\right)\hat{\chi}=e[\gamma_0^{(e)}\gamma_\nu^{(e)}F^\nu(x)-\gamma_0^{(p)}\gamma_\nu^{(p)}F^\nu(y)]S(x-y)C\quad(x^0=y^0=t).\tag{12}$$

The desired energy shift, Δ_n^A, is obtained by computing the contribution of the right-hand side of (12) to the lowest order. Thus

$$\Delta_n^A=\int d(\vec{x}-\vec{y})\chi_n^{B\dagger}(\vec{x}-\vec{y})\exp(i\epsilon_n x^0)e\,[\gamma_0^{(e)}\gamma_\nu^{(e)}F^\nu(x)-\gamma_0^{(p)}\gamma_\nu^{(p)}F^\nu(y)]S(x-y)C\quad(x^0=y^0=t),\tag{13}$$

where now

$$F^\mu(x)=ie\int d^4z\,D_C^{\mu\nu}(x-z)\ \mathrm{tr}[\gamma_\nu\exp(-i\epsilon_n z^0)\chi_n^B(0)C].$$

Here ϵ_n is the energy eigenvalue corresponding to χ_n^B. We further approximate Δ_n^A by inserting S^0 (the free electron propagator) for S and $u_{\alpha\beta}\phi_n$ (the Pauli-Schrödinger hydrogenic wave function with reduced mass $m/2$) for $\chi_{\alpha\beta}^B$. Here $u_{\alpha\beta}$ is a nonrelativistic bispinor whose nonzero elements (the upper-left block) form the corresponding Pauli bispinor. Effecting these substitutions, we obtain

$$F^\mu\cong[-ie\phi_n(0)/4m^2]\gamma_{\alpha\beta}^\mu u_\beta^{(e)}u_\lambda^{(p)}C_{\lambda\alpha},$$

where we have also approximated ϵ_n by $2m$. Note that as a consequence of current conservation, Eq. (8), F^0 is identically zero.

Next we calculate $S^0(x-y)$ for $x^0=y^0$ and find that

$$S(0,\vec{x}-\vec{y})=(16i\pi^3)^{-1}(m+i\vec{\gamma}\cdot\vec{\nabla}_x)\times\int d\vec{k}\,e^{i\vec{k}\cdot(\vec{x}-\vec{y})}(m^2+\vec{k}^2)^{-1/2}.$$

It can be easily verified that the term involving $\vec{\gamma}$ in the above expression does not contribute. Therefore, the spatial integral in (13) will involve the factor

$$(16i\pi^3)^{-1}\int d(\vec{x}-\vec{y})\phi_n^*(\vec{x}-\vec{y})d\vec{k}\,e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\times(1+k^2/m^2)^{-1/2},$$

which is easily approximated to be $(2i)^{-1}\phi_n^*(0)$. Equation (13) may then be written

$$\Delta_n^A \cong -(2i)^{-1}\phi_n^*(0)\vec{\Gamma}\cdot\mathrm{tr}(u^\dagger\vec{\alpha}C - u^\dagger C\vec{\alpha}^t)$$
$$= (e^2/4m^2)\,|\phi_n(0)|^2\,\mathrm{tr}(u^\dagger\vec{\alpha}C)\cdot\mathrm{tr}(\vec{\gamma}uC).$$

Since u is effectively a two-dimensional matrix, the above trace factors reduce to

$$\mathrm{tr}(u^\dagger\vec{\sigma}\sigma^2)\cdot\mathrm{tr}(\sigma^2\vec{\sigma}u),$$

where u is now to be considered a Pauli bispinor. This bispinor is found to be proportional to σ^2 $(\sigma^{0,1,3})$ for singlet (triplet) states. Thus the above factor is equal to $S(S+1)$, with S the total spin, whence follows the usual result.

V. CONCLUSION

In this paper we have completed the development of the basic elements of a new treatment of the bound-state problem which was begun in paper I. We have demonstrated that the standard renormalization procedure is implementable within this treatment. Our two-body equations, Eq. (5) of paper I for a general situation and Eq. (7) here for a particle-antiparticle system, have been applied to a wide spectrum of standard bound-state calculations, and a number of *known* results have thereby been obtained. The aim of these calculations has been the demonstration that the present formalism is well capable of dealing with the customary bound-state problems.

Throughout we have stressed the fact that a single formalism, without further assumptions, modifications, or "special treatments," is being applied in all situations. Thus the unity and completeness of the formalism, which is characteristically lacking in other treatments, has been underlined. Other features of the formalism, particularly in comparison with the Bethe-Salpeter treatment, have been remarked upon throughout, especially in paper I. Paramount among these have been the properties of covariance and gauge invariance.

As remarked above, we have thus far been introducing a new formulation by means of applications to old problems; the future development should obviously concern new applications. Of course, it will be these that will determine how useful and effective the new formalism is.

APPENDIX

Here we shall verify the constraint derived in the text, Eq. (8). Consider Eq. (7) in the center-of-mass frame, where it assumes the single-time form

$$\left[i\frac{\partial}{\partial t} - e\phi^0(x) + e\phi^0(y)\right]\chi^A(x,y) = \{\vec{\alpha}^{(e)}\cdot[\vec{p}_x - e\phi(x)] + \beta^{(e)}m + \vec{\alpha}^{(p)}\cdot[\vec{p}_y + e\phi(y)] + \beta^{(p)}m\}\chi^A(x,y)$$
$$+ e\{[F^0(x) - F^0(y)] - [\vec{\alpha}^{(e)}\cdot\vec{F}(x) - \vec{\alpha}^{(p)}\cdot F(y)]\}S^A(x,y)C \qquad (x^0 = y^0 = t),$$

where

$$\phi^\mu(x) = A^\mu(x) + i\int d^4z\, D_\nu^{A\mu}(z,x)\frac{\delta}{\delta A_\nu(z)}.$$

Recall that the Dirac matrices labeled (e) $[(p)]$ act on the first [second] index of χ. Using this fact and the identities

$$\gamma^2\vec{\alpha}\gamma^2 = -\vec{\alpha}^t, \quad \gamma^2\beta\gamma^2 = \beta^t,$$

we can write the above equation in the usual matrix notation as follows:

$$\left[i\frac{\partial}{\partial t} - e\phi^0(x) + e\phi^0(y)\right]\chi^A\gamma^2 - e[F^0(x) - F^0(y)]S^AC\gamma^2$$
$$= m[\beta, \chi^A\gamma^2] + [\vec{p}_x - e\phi(x)]\cdot\vec{\alpha}\,\chi^A\gamma^2 - [\vec{p}_y + e\phi(y)]\cdot\chi^A\gamma^2\vec{\alpha} - e\vec{F}(x)\cdot\vec{\alpha}SC\gamma^2 + e\vec{F}(y)\cdot SC\gamma^2\vec{\alpha}.$$

At this juncture we take the trace of this equation and get

$$\left[i\frac{\partial}{\partial t} - e\phi^0(x) + e\phi^0(y)\right]\mathrm{tr}[\chi^A\gamma^2] = [\vec{p}_x + \vec{p}_y - e\vec{\phi}(x) + e\vec{\phi}(y)]\cdot\mathrm{tr}[\vec{\alpha}\chi^A\gamma^2] + ie[F_\mu(x) - F_\mu(y)]\cdot\mathrm{tr}[\gamma^\mu S^A].$$

The occurrence of the combination $\vec{p}_x + \vec{p}_y$ allows the elimination of the relative space coordinate, leaving the center-of-mass coordinate $\vec{R} = (\vec{x} + \vec{y})/2$. Equivalently, we may simply let $y = x$ (since already $y^0 = x^0$),

and replace $\vec{p}_x + \vec{p}_y$ by $\vec{p}_x$. Then

$$i\frac{\partial}{\partial t}\,\mathrm{tr}[\chi^A(x,x)\gamma^2] = \vec{p}_x \cdot \mathrm{tr}[\vec{\alpha}\chi^A(x,x)\gamma^2].$$

Finally, we pass to the physical limit $A = 0$. In this limit

$$\chi(x,x) = \exp(-ix_\mu P^\mu)\chi(0,0),$$

and in the center-of-mass frame, where P^μ has only a time component, the result given by Eq. (8) is obtained.

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[1]M. H. Partovi, Phys. Rev. D 12, 3887 (1975).
[2]Here reference is being made to the use of perturbation methods for the derivation of functional equations and not Schwinger's own method, which is based on his quantum dynamical principle.
[3]M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).
[4]R. Karplus and A. Klein, Phys. Rev. 87, 848 (1952), Eq. (2.11).
[5]See J. Schwinger, Phys. Rev. Lett. 3, 296 (1959). This remark applies equally well to Eq. (2) of paper 1.
[6]J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Cambridge, Massachusetts, 1955), p. 346.