PHYSICAL REVIEW D VOLUME 14, NUMBER 12 15 DECEMBER 1976

Extended hadron model based on the modified sine-Gordon equation

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(Received 30 March 1976)

A baglike potential is proposed which is based on a slight modification of the sine-Gordon equation. The potential has one absolute minimum and one slightly high local minimum for a period. On the basis of this potential, an extended hadron model is obtained in one space dimension and the views of MIT and SLAC bag models are unified. The exact solution has two kinks and is obtained for the scalar field equation which obeys our potential. The quark field is confined between these two kinks. The scalar and quark field solutions are analytic with respect to the one-dimensional space variable. Stability is proved classically for the two-kink solution.

I. INTRODUCTION

There have been several attempts to give the extended domain on elementary particles. In the MIT[1] bag model, the hadron is regarded as a finite region of space and is called the bag. The potential is taken as a positive constant B inside the bag and is zero outside. This mechanism explains quark confinement. But in this model the Lagrangian has a clear cut at the boundary of the bag. Hence the model is not a field theory. We want to construct the field-theoretical realization of the MIT bag model. Creutz[2] takes one quartic potential with two unequal local minimums and applies the absolute minimum of the potential for the vacuum state and the slightly higher local minimum for the hadron (bag) state. He takes the scalar field $\phi(x)=\beta$ for $x<R$ and $\phi(x)=0$ for $x>R+\Delta R$. Here R is the radius of the bag and ΔR is the width of the boundary. He takes the limit $\Delta R\to 0$ in order to neglect the effect of the boundary. Then he finds that the local maximum of the potential grows to infinity and that this separates the bag and the vacuum states. But he assumes the existence of the smooth scalar and quark fields which realize the stable bag state. This assumption is unclear. If we construct explicitly the exact analytic solutions of the scalar and quark field equations in the whole space of one dimension, we can see the existence of the smooth fields. In this paper it will be proved that the exact analytic solutions can be constructed and that these solutions give the stable bag state. Also, Vinciarelli[3] gives the numerical solutions of the nonlinear scalar field equation for the quartic potential with two unequal local minimums in the spherically symmetric case and of the corresponding quark field equation. His estimation is numerical, and so it is unclear whether his solutions realize the stable bag state or not.

On the other hand, Bardeen *et al.*[4] consider the quartic potential with two equal minimums. In this case, the exact analytic solution of the nonlinear scalar field equation is known in one space dimension. The exact solution of the quark field equation is also found, which has one peak localized at one boundary of the hadron and corresponds to the solution of the scalar field equation. The solution $\phi(x)$ of the nonlinear field equation is monotonic and has one kink and two limits, $\phi(\infty)=f$, $\phi(-\infty)=-f$, while the extended hadron has two boundaries in one space dimension. But it does not appear that the quartic potential with two equal minimums admits such a solution.

In this situation it seems useful to consider the scalar field $\phi(x)$ which is monotonic and has two kinks and two limits, $\phi(\infty)=f$, $\phi(-\infty)=-f$. We take the sine-Gordon equation as the candidate for the nonlinear scalar field equation, because this equation has the period with respect to ϕ and so appears to admit the two-kink solution with the monotonic property. In fact, it will be shown that the slight modification of the sine-Gordon equation gives the expected solution. Also, on the corresponding potential, the field $\phi(x)$ is for the vacuum state at the absolute minimum in the spatially far regions, i.e., outside of the bag, and is for the bag state at the slightly higher local minimum in the neighborhood of the spatial origin, i.e., inside of the bag. This is exactly the situation in the MIT bag model. Hence we will unify the views of the MIT and SLAC bag models.

We can explain the quark-confinement mechanism on the basis of two exact solutions: One is for the modified sine-Gordon equation of the neutral scalar field and the other is for the quark field equation, in which the quark field ψ interacts with the scalar field ϕ through the coupling $\bar{\psi}\psi\phi$. Two positions of kinks are interpreted as two boundaries of the extended hadron in one space dimension. The scalar field is approximately zero inside the boundaries and is approximately a nonzero constant outside. The quark field is nearly constant inside the boundaries and is negligibly small outside. Therefore the quark-confinement mechanism can be explained.

In Sec. II the preliminary framework is given. In Sec. III the modification of the sine-Gordon equation is proposed and the properties of the exact solution are examined. In Sec. IV the stability is proved classically for our exact solution. In Sec. V the quark field equation is investigated and is solved exactly. In Sec. VI the quark-confinement mechanism is explained.

II. PRELIMINARY FRAMEWORK

We start with the model Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2-V(\phi)-\bar{\psi}i\gamma^\mu\partial_\mu\psi-M\bar{\psi}\psi-g\bar{\psi}\psi\phi\ . \tag{2.1}$$

Here ϕ is the neutral scalar field, $V(\phi)$ is the potential with two unequal local minimums, ψ is the set of quark fields,

$$\psi=\sum_{i=1}^{n}\psi_i\ , \tag{2.2}$$

where n denotes the number of quarks, and M is the total bare mass of the quarks. Then the Euler-Lagrange equations are

$$(\partial_t^{\,2}-\vec{\nabla}^2)\phi+V'(\phi)=-g\bar{\psi}\psi \tag{2.3}$$

and

$$(i\gamma^0\partial_t-i\vec{\gamma}\cdot\vec{\nabla}+g\phi+M)\psi=0\ . \tag{2.4}$$

We consider time-independent exact solutions of Eqs. (2.3) and (2.4) in one space dimension under the subsidiary condition

$$\bar{\psi}\psi=0\ . \tag{2.5}$$

Then Eqs. (2.3) and (2.4) reduce to

$$\frac{d^2}{dx^2}\,\phi(x)=V'(\phi) \tag{2.6}$$

and

$$i\gamma^1\frac{d}{dx}\,\psi(x)=[g\phi(x)+M]\psi(x)\ . \tag{2.7}$$

The Dirac matrices γ^μ can be taken as

$$\gamma^0=\sigma^2=\begin{pmatrix}0&-i\\ i&0\end{pmatrix}, \tag{2.8}$$

$$\gamma^1=i\sigma_1=\begin{pmatrix}0&i\\ i&0\end{pmatrix}; \tag{2.9}$$

and

$$\bar{\psi}\psi=\psi^\dagger\gamma^0\psi\ . \tag{2.10}$$

The condition (2.5) is accomplished for

$$\psi\equiv\begin{pmatrix}u\\ v\end{pmatrix}=u(x)\begin{pmatrix}1\\ \rho\end{pmatrix} \tag{2.11}$$

with real ρ.

III. THE MODIFIED SINE-GORDON EQUATION

We consider the sine-Gordon Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi_0)^2-\frac{m^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}}{m}\,\phi_0\right)\right] \tag{3.1}$$

in one space dimension. It is well known that the sine-Gordon equation

$$\frac{d^2}{dx^2}\,\phi_0(x)=\frac{m^3}{\sqrt{\lambda}}\sin\left[\frac{\sqrt{\lambda}}{m}\,\phi_0(x)\right] \tag{3.2}$$

has the solution

$$\phi_0(x)=\frac{4m}{\sqrt{\lambda}}\tan^{-1}e^{\pm m(x-x_0)}\ . \tag{3.3}$$

This solution has only one kink at $x=x_0$. For simplicity we set

$$\sqrt{\lambda}=m=1\ . \tag{3.4}$$

Now let us construct the two-kink solution. Making use of the one-kink solution (3.3), we take

$$\phi(x)=\phi_1(x)-\phi_2(x)\ , \tag{3.5}$$

where $\phi_1(x)$ and $\phi_2(x)$ are given by

$$\tan[\phi_1(x)/4]=e^{x+R} \tag{3.6}$$

and

$$\tan[\phi_2(x)/4]=e^{-x+R}\ . \tag{3.7}$$

Here $2R$ is the distance between two kinks and R is supposed to be the radius of the bag. Then we obtain

$$\tan\left[\frac{\phi(x)}{4}\right]=\frac{\tan(\phi_1/4)-\tan(\phi_2/4)}{1+\tan(\phi_1/4)\tan(\phi_2/4)}=\frac{\sinh x}{\cosh R}\ . \tag{3.8}$$

The value of $\phi(x)$ is -2π at $x=-\infty$, and is nearly zero around $x=0$. Hence we see the period 4π of ϕ, and we consider $\phi(x)$ between -2π and 2π.

Our field $\phi(x)$ leads to

$$\frac{d\phi}{dx}=2\sin\frac{\phi}{2}\coth x\ . \tag{3.9}$$

Therefore we obtain the field equation for the neutral scalar field $\phi(x)$:

$$\frac{d^2\phi}{dx^2}=\sin\phi-(\cosh R)^{-2}\left(\sin\phi+2\sin\frac{\phi}{2}\right)\ . \tag{3.10}$$

This equation reduces to the usual sine-Gordon equation when two kinks are far away: $R\to\infty$. So we call this equation (3.10) the modified sine-

Gordon equation. This equation is given by the potential

$$V(\phi)=1-\cos\phi + (\cosh R)^{-2}\left(3+\cos\phi+4\cos\frac{\phi}{2}\right). \tag{3.11}$$

This potential is invariant for the discrete symmetry $\phi\to-\phi$ and has the period 4π. The absolute minimum is zero at $\phi=-2\pi$ and 2π and the slightly high local minimum is $8(\cosh R)^{-2}$ at $\phi=0$. Our potential has the values 0.0012687 at $\phi=0$ and 2.0031718 at $\phi=\pi$ for the concrete parameter values $m=1$ GeV and $R=1$ fm $=5.06769$ GeV^{-1}. Hence it is difficult to distinguish our potential and the usual sine-Gordon potential $1-\cos\phi$ in terms of a picture. So the figure of our potential $V(\phi)$ is not given.

Next we estimate the total energy of our solution:

$$\begin{aligned} E&=\int_{-\infty}^{\infty}dx\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+V(\phi)\right]\\ &=\int_{-2\pi}^{2\pi}d\phi[2V(\phi)]^{1/2}\\ &=\frac{16m^3}{\lambda}+\frac{32m^4R}{\lambda\sinh(2mR)}\ . \end{aligned} \tag{3.12}$$

Here we write explicitly λ and m for comparison with other works. This value reduces to $16m^3/\lambda$ as $R\to\infty$. This is twice that of the case of the usual sine-Gordon equation. The reason is that the period is twice that for the usual case. Hence we obtained the exact analytic solution (3.8) of the neutral scalar field equation (3.10) for the baglike potential (3.11) in the space of one dimension. This solution has two kinks and the total energy (3.12).

IV. STABILITY OF THE NEUTRAL SCALAR FIELD SOLUTION

In this section we prove the stability of the solution (3.8) of the neutral scalar field equation (3.10). We consider the following quantity for a small number α and an arbitrary differentiable function $\eta(x)$ with the boundary conditions $\eta(\infty)=\eta(-\infty)=0$:

$$\begin{aligned} E(\alpha)&\equiv\int_{-\infty}^{\infty}dx\left(\frac{1}{2}\left\{\frac{d}{dx}[\phi(x)+\alpha\eta(x)]\right\}^2+V(\phi+\alpha\eta)\right)\\ &=\int_{-\infty}^{\infty}dx[\tfrac{1}{2}\phi'^2+V(\phi)]+\alpha\int_{-\infty}^{\infty}dx[\phi'\eta'+V'(\phi)\eta]\\ &\quad+\frac{\alpha^2}{2}\int_{-\infty}^{\infty}dx[\eta'^2+V''(\phi)\eta^2]+O(\alpha^3)\\ &\equiv E+\alpha J_1+\frac{\alpha^2}{2}J_2+O(\alpha^3)\ . \end{aligned} \tag{4.1}$$

We say that the system is stable if the conditions

$$J_1=0 \tag{4.2}$$

and

$$J_2>0 \tag{4.3}$$

are satisfied for any small number α. The field equation (2.6) gives us Eq. (4.2) and

$$\phi'''-V''(\phi)\phi'=0\ . \tag{4.4}$$

The form of J_2 is rewritten as

$$J_2=\int_{-\infty}^{\infty}dx\left(\eta'-\frac{\phi''}{\phi'}\eta\right)^2 \tag{4.5}$$

from Eq. (4.4), and

$$\begin{aligned} 0&=\int d\left(\frac{\phi''}{\phi'}\eta^2\right)\\ &=\int_{-\infty}^{\infty}dx\left[2\frac{\phi''}{\phi'}\eta'\eta+\frac{\phi'''}{\phi'}\eta^2-\left(\frac{\phi''}{\phi'}\right)^2\eta^2\right]. \end{aligned} \tag{4.6}$$

Here we used the fact that our solution (3.8) is monotonically increasing and its derivative ϕ' is not zero between the boundaries $x=\infty$ and $x=-\infty$. Then we obtain the estimate (4.3), if η is not proportional to ϕ'. The case in which η is proportional to ϕ' shows the translational invariance of our system. Hence we have proved the stability of our scalar field solution.

V. THE QUARK FIELD EQUATION

In this section we derive the time-independent exact solution of the field equation for the quarks interacting with the neutral scalar field (3.8). The field equation (2.4)

$$(i\gamma^0\partial_t-i\vec{\gamma}\cdot\vec{\nabla}+g\phi+M)\psi=0$$

is separated into the two equations

$$\frac{d}{dx}v(x)+[g\phi(x)+M]u(x)=0 \tag{5.1}$$

and

$$\frac{d}{dx}u(x)+[g\phi(x)+M]v(x)=0 \tag{5.2}$$

in one space dimension under the representation (2.9) of the Dirac matrix. Then we get

$$v(x)\frac{d}{dx}v(x)-u(x)\frac{d}{dx}u(x)=0\ , \tag{5.3}$$

and so

$$[v(x)]^2-[u(x)]^2=\text{constant}\ . \tag{5.4}$$

Then Eqs. (2.11) and (5.4) lead to

$$v(x)=\pm u(x)\ . \tag{5.5}$$

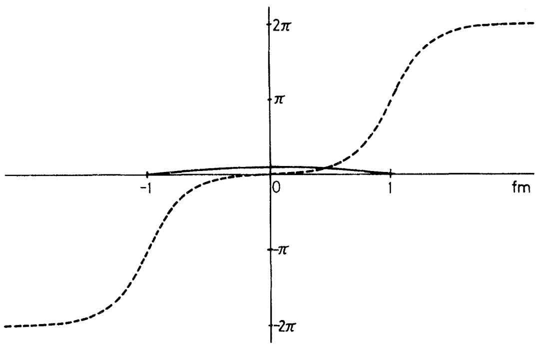

FIG. 1. The exact solution $\phi(x)$ of the neutral scalar field equation for the modified sine-Gordon potential $V(\phi)$ and the exact solution $u(x)$ of the quark field equation. The dashed line refers to $\phi(x)$ and the solid line refers to $u(x)$. The parameters are taken as $g=1$, $M=0$, $\sqrt{\lambda}=m=1$ GeV, and $R=1$ fm$=5.06769$ GeV^{-1}.

So two equations (5.1) and (5.2) are unified as

$$\frac{d}{dx}u(x)\pm[g\phi(x)+M]u(x)=0\ . \tag{5.6}$$

Then from our scalar field solution (3.8), we obtain the exact analytic solution of the quark field equation (5.6)

$$u(x)=A\exp\left[-g\int_0^x dx\,4\tan^{-1}\left(\frac{\sinh mx}{\cosh mR}\right)\right]e^{-Mx}\ , \tag{5.7}$$

where A is an arbitrary constant, and we take the minus sign of $\pm g$ in order to make $u(x)$ finite at $x=\pm\infty$. Here it is noted that the integral in the solution (5.7) cannot be expressed by the elementary functions and that the quark bare mass M is inessential in our considerations.

VI. THE QUARK-CONFINEMENT MECHANISM

In this section we give an explanation of the quark-confinement mechanism. The hadron is regarded as an extended region of one-dimensional space containing quarks. The quark fields $\psi(x)$ interact with the neutral scalar field $\phi(x)$. The scalar field $\phi(x)$ can be considered as a vessel of the quark fields. We have considered the modified sine-Gordon-type potential of the scalar field $\phi(x)$. This potential has two local minimums in the unit period. The vacuum state is settled at the absolute minimum and the hadron state is around a slightly high local minimum of the potential $V(\phi)$. The hadron state is surrounded by two high potential walls. So any quark must get over one of these high walls in order to appear singly outside of the hadron.

Two typical scalar and quark field solutions (3.8) and (5.7) are shown in Fig. 1. As is shown in Sec. V and Fig. 1, the exact solution tells us that the quark field $u(x)$ is localized inside of the boundaries $x=\pm R$ and is negligible outside. The quark mass is

$$-\frac{\partial^2\mathcal{L}}{\partial\bar{\psi}\,\partial\psi}=g\,\phi(x)+M\ . \tag{6.1}$$

The quark must get over the large-mass wall in order to appear singly. The quark field damps negligibly outside of the hadron, if the quark overcomes the wall. It is noted that the value of M is inessential in our considerations.

VII. DISCUSSION

In this paper we constructed a field-theoretical version of the MIT bag model[1] in one space dimension. The vacuum state is at the absolute minimum of our potential and the hadron (bag) state is at the slightly high local minimum. Our solu- of the neutral scalar field equation is exact, analytic, monotonically increasing, and has the values -2π at the negative infinite region, 2π at the positive infinite region, and approximately zero inside the bag. The quark field solution is exact, analytic, and confined within the bag. It should be noticed that our scalar and quark field solutions are exact and analytic in the whole space of one dimension, while the usual solutions of the MIT bag-type model have a clear cut at the surface of the bag.

Also our model realizes the point of the SLAC bag model.[4] Our model is one field theory with the quark-confinement mechanism. The quark field is confined inside the bag volume according to the classical picture as pointed out by Bardeen *et al.*, although the quark field is not confined at the bag surface as they proposed.

Next we consider the quark mass (6.1). The value is

$$M_I=g\,\phi_I+M=M \tag{7.1}$$

in the interior of the bag, strictly in the center, and is

$$M_E=g\,\phi_E+M=2\pi g\,\frac{m}{\sqrt{\lambda}}+M \tag{7.2}$$

in the exterior of the bag, strictly in the vacuum. Hence the value of the quark mass is infinite outside of the bag at the weak-coupling limit $\lambda\to 0$. This justifies the following quark-confinement mechanism at the weak-coupling limit: The quarks are massless inside of the bag and have a mass of infinite value outside.

We compare the energy density of our baglike model with the phenomenological parameter value.

The energy density B is $V(\phi=0)$ in our model and so $B^{1/4}=0.189$ GeV for our parameter values $\lambda=1$, $m=1$ GeV, and $R=1$ fm. This value can be compared with the phenomenological value[5] $B^{1/4}=0.146$ GeV. This value is attained by the value $R=1.1$ fm. This fact supports the position that our parameter values are reasonable, although our values were taken very simply.

We now note the following. We took the combination $e^{x+R}-e^{-x+R}$ in Sec. III for the two-kink solution. If we take the form $e^{x+R}+e^{-x+R}$, we can obtain the exact analytic solution of the corresponding neutral scalar field equation. But our prescription in Sec. IV cannot prove the stability of this solution, because the derivative of this solution is zero.

To sum up, we constructed the exact analytic solution of the baglike potential in the whole space of one dimension and we explained the quark-confinement mechanism.

Note added in proof. In this paper we discussed the modified sine-Gordon equation. We can also construct[6] the neutral scalar field equation on the basis of the eighth-order polynomial potential

$$V(\phi)=\lambda^2(\phi^2+\alpha^2)^2(\phi^2-\beta^2)^2 ,$$

which is the baglike potential and gives the exact analytic solution with two kinks for the field equation. Although the potential has no period, this solution and the corresponding one of the quark field equation are very similar to our solutions in this paper. Therefore we can also have the quark-confinement mechanism on the basis of the exact analytic solutions of the scalar and quark field equations for the baglike polynomial potential.

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