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Classical Yang-Mills potentials

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Classical Coulomb-type configurations of Yang-Mills fields coupled to external sources (charges) are described and discussed. They are shown to be stable against small classical fluctuations in the fields if the Yang-Mills coupling is sufficiently small. In developing the stability analysis, massless charged scalar fields in the presence of a weak Coulomb potential are also shown to be stable to small field fluctuations.

I. INTRODUCTION

The possible role of colored gluons in confining quarks is receiving increasing attention. The famous "infrared slavery" conjecture,[1] that the infrared singularities of Yang-Mills field theory would result in long-range confining forces, has not led to direct calculations of such effects, but models have been proposed which show a connection between color gauge symmetry and confinement.[2-6]

Two examples of such models of quark confinement that work in four-dimensional space-time are the lattice gauge theory and the MIT bag. In the lattice gauge theory confinement is a result of integrations over a full gauge symmetry group.[2] In the MIT bag model confinement is a result of the fact that any color-nonsinglet object must be surrounded by a gauge field, but all such fields are excluded from the exterior of a bag.[3] In both models, other crucial ingredients beyond gauge symmetry are needed for confinement. However, without gauge symmetry, these mechanisms would not exclude states carrying quark quantum numbers.

Because of the central role being played by gauge symmetry in current ideas about quark confinement, it is important to discover whatever one can about the forces mediated by gauge bosons. As a step toward understanding these forces, we study here classical Yang-Mills fields coupled to finite-strength, spatially fixed sources. This is the Yang-Mills analog of electromagnetic fields in the presence of fixed charges. While this is a trivial electromagnetic problem, the non-Abelian case is more interesting. Although we fix the sources in space, giving a preferred Lorentz frame, we couple the sources gauge invariantly, so that the full system of fields and sources retains complete gauge symmetry. The sources may be thought of as the quanta of an infinite-mass field coupled to the gauge field. (Note that they have nothing to do with the infinitesimal gauge-breaking "sources" used to define and calculate Green's functions.)

There are several advantages to using finite-charge, physical, spatially fixed sources to analyze Yang-Mills fields:

(1) The interactions between sources represent physical ("on-shell") effects.

(2) Because the sources have finite charge, even classically they can bring the nonlinear structure of the Yang-Mills field into play.

(3) The sources are only weakly dynamical; although geometrically fixed, their orientation in color space can change.

(4) Fixing the sources in space separates the problem of finding the forces or effective potentials resulting from a Yang-Mills interaction (as a function of source separation) from the complications due to the motion of charged constituents of a composite system.

(5) The presence of several sources introduces spatial structure which regulates the infrared singularities of Yang-Mills field theory.

In Sec. II we discuss classical solutions to the equations for Yang-Mills fields in the presence of external sources. Classical sources are described by a charge vector in the group space. The charge vector is dynamical; as a consequence of its coupling to the non-Abelian gauge field it can change with time. However, we hold the source's location fixed. There are especially simple solutions to the field equations if all of the sources are oriented in commuting directions in the Yang-Mills space [for example, along the I_3 and hypercharge directions of SU(3), corresponding to sources carrying a definite value of charge and hypercharge]. When the vectors are so oriented, it is consistent to make the ansatz that the only nonvanishing components of the Yang-Mills field are along these same directions. This ansatz immediately linearizes the field equations, and reduces them to a set of decoupled copies of Maxwell's equations in the presence of external fixed charges. There is an independent Maxwellian field associated with each of the commuting directions in the Yang-Mills charge space. If the gauge group is SU(n), each source has a definite constant value of $n-1$ independent charges, which give rise to $n-1$ independent static, Coulomb-type fields. The force between sources depends on their separation as $1/r^2$, and is the sum of independent Coulomb forces associated with each of the $n-1$ charges.

These solutions are intrinsically much simpler than the sourceless configurations of Yang-Mills fields, which relate orientations in internal-symmetry space to orientations in geometrical space.[7-9] However, the Coulomb-type solutions require the presence of external sources.

The remainder of this paper is a discussion of the stability of the Coulomb-type field configurations. Our discussion is restricted to the case of small, though finite, charge. The question of stability is important because it addresses the issue of whether or not the classical solutions are relevant to the quantum-mechanical situation. If the Coulomb-type solutions are classically stable, then it is possible for the quantum-mechanical configuration to closely resemble the classical one. On the other hand, if these solutions are classically unstable, the fluctuations, which are inherent in all quantum-mechanical systems, will excite an instability and so result in a completely new and unrelated configuration of fields.

Classical stability is, of course, not a guarantee of quantum stability. A classically stable configuration which corresponds to a local minimum of the energy which is not a global minimum will be unstable to tunneling. In addition, the divergent renormalization needed in quantum field theory can completely change the stability character of a given classical configuration. Classical stability is thus necessary if the classical solution is to resemble the quantum situation, but it by no means ensures such a resemblance. On the other hand, classical instability is a sure sign that the quantum situation will be quite unlike the classical one.

Section III is a preliminary discussion of the stability of Coulomb-type configurations. The masslessness of the charged Yang-Mills fields is a possible source of an instability. Because there is no rest energy associated with field configurations, it might seem that configurations of charged fields which screen the charges carried by the sources would have lower total energy than the sources plus Coulomb fields alone. We first examine this physical situation in a related but somewhat simpler model, the electrodynamics of a massless charged scalar field in the presence of a single external charge. The analogous question is the following: Is the Coulomb field of the external charge stable to fluctuations of the charged scalar field? One might think that the answer is no, because, since the scalar field is massless, each hydrogenic bound-state level of the charged scalar in the external Coulomb field has a lower energy than the Coulomb field alone, and so represents an instability.

These arguments notwithstanding, we show that for sufficiently small charge, $\alpha|Z|<\frac{1}{2}$, the Coulomb field is classically stable. The result is obtained in two ways, both of which relay on linearizing the field equations about the classical solution, and finding the eigenfrequencies of small fluctuations. The resulting stability equation is the Klein-Gordon equation with a $1/r$ potential. In this method of testing stability, if all the eigenfrequencies are real, the system is classically stable, while a complex eigenfrequency signals an instability. Such an instability is a mode which grows exponentially with time, and which is associated with a negative-energy quantum-mechanical bound state. The first argument for stability uses the fact that any given solution of the linearized field equation is an analytic function of the charge parameter. We use this analyticity as a means of extrapolating away from zero charge, where all the eigenfrequencies are real, and show that at least a finite extrapolation is needed for a complex eigenfrequency to develop. The second proof of stability is more direct. We simply recognize that the stability equation can be transformed into a standard, well-studied equation (Whittaker's), and show from the known properties of solutions to this equation that there are no complex eigenfrequencies for $\alpha|Z|<\frac{1}{2}$. This proof is obviously much simpler, and we would not bother with the first were it not for the fact that we have only been able to generalize the less direct argument to the Yang-Mills case.

The mystery of the missing hydrogenic wave functions is easily resolved. When the mass of the Klein-Gordon field is taken to be zero, the binding energy of each hydrogenic bound state also goes to zero, in fact proportional to the scalar field mass. Furthermore, the wave functions themselves vanish in the zero-mass limit, and the bound states completely disappear from the spectrum.

It is known that beyond a certain critical charge, even a massive Klein-Gordon field becomes unstable.[10,11] Beyond this charge, the massless field will likewise be unstable.

In Sec. IV we show that for a finite, though undetermined, range of charge, the Coulomb-type configuration of Yang-Mills fields due to a single source is classically stable. As in the Klein-Gordon example, the argument proceeds by linearizing the Yang-Mills field equations about the static classical solution, and examining the eigenfrequencies of small fluctuations. Again, a complex eigenfrequency would signal a classical instability. Here we use the fact that a given solution to the linearized field equations is an analytic function of the Yang-Mills coupling constant to extrapolate solutions away from zero coupling and show that at least a finite extrapolation is necessary before a complex eigenfrequency could develop.

The analysis of the Yang-Mills case is a great deal more complicated than the Klein-Gordon example. The obvious source of complication is the fact that the Yang-Mills field has spin 1, which means that each mode of fluctuation will be described by two radial functions (rather than one, as in the scalar case) which satisfy a set of two coupled second-order equations. The second, and more serious, complication arises from the fact that the coupling between the two radial wave equations is singular. This results in the system of differential equations having a singular point which moves to the origin in the limit of zero- coupling constant. The effect of this singularity is to make the analysis of even the zero-coupling-constant system nontrivial (where by zero coupling constant one understands the $g \to 0$ limit of the finite-coupling-constant system). In order to show, even in the zero-coupling limit, that there are no complex eigenfrequencies, it is necessary to analytically solve a reduced but still nontrivial fourth-order differential equation, which we do. Having explicitly found the zero-coupling fluctuations, we are able to extrapolate away from zero and show that there is a finite range of coupling with which there are no complex eigenfrequencies. Because we lack closed-form solutions to the finite-coupling fluctuation equations, we have no quantitative estimates of the extent of this range.

The stability considerations presented in this paper do not apply to the case of large charge and coupling constant. Additional analysis is needed to find the characteristics of the instabilites expected there.

We conclude this Introduction with the remark that the question of whether or not these classical solutions are quantum mechanically stable is likely to be more subtle than we have indicated. Quantum mechanically, the effective coupling constant depends on the distance scale. Since the known instability of massless Klein-Gordon particles in very strong Coulomb fields results from the long-distance part of the potential (the effect is present even with a cutoff Coulomb potential), presumably it is the infrared limit of the effective Yang-Mills coupling which will determine the relevance of the consideration presented here. If this limit is finite, the kind of stability analysis presented here might determine the quantum stability of Yang-Mills charges. If the infrared limit is divergent, a more refined analysis is needed, though the analogy with scalar electrodynamics suggests that an instability should be expected.

II. CLASSICAL FIELDS AND SOURCES

In this section we will describe Coulomb-type configurations of classical Yang-Mills fields due to the presence of static classical sources. A classical Yang-Mills field is specified by the value of each component of the field multiplet. A classical stationary source is specified by its fixed location $\vec{x}_i$ in space and by a vector e_i^a that has a given magnitude (the total Yang-Mills charge carried by the source) and which points in some direction in the internal Yang-Mills charge space. For SU(n), e_i^a has n^2-1 components. Classically each component of e_i is well defined, and the group structure is carried by Poisson bracket relations among these components.

The total (4-vector) current density due to an assembly of classical sources is given by

$$j_\mu^a(\vec{x}) = \delta_{\mu 0} \sum_i e_i^a \delta^3(\vec{x} - \vec{x}_i) . \tag{2.1}$$

It is often convenient to arrange the n^2-1 components of an SU(n) Yang-Mills multiplet in an $n \times n$ matrix. For example, we define the matrix e_i by

$$e_i = \tfrac{1}{2} \sum_a \lambda^a e_i^a , \tag{2.2}$$

where the matrices $\frac{1}{2}\lambda^a$ are the generators of the fundamental representation of SU(n). In the following, A_μ, $F_{\mu\nu}$, j_μ, and e_i stand for such traceless $n \times n$ matrices formed from the n^2-1 Yang-Mills potentials, fields, currents, and charges.

The equations satisfied by the Yang-Mills fields are, in this notation,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] , \tag{2.3}$$

$$\partial_\nu F_{\mu\nu} - ig[A_\nu, F_{\mu\nu}] + gj_\mu = 0 . \tag{2.4}$$

The commutators, of course, refer only to the matrix structure. The "external" current j_μ must satisfy the relation

$$\partial_\mu j_\mu - ig[A_\mu, j_\mu] = 0 . \tag{2.5}$$

This relation is a consequence of the field equations, and may be thought of as a consistency condition on any source which is coupled to a gauge field. Just as Maxwell's equations alone require that the charge coupled to the electromagnetic field be conserved, so the Yang-Mills field equation alone requires that the source current satisfy the extended conservation law, Eq. (2.5). An important implication of this condition is that it may be inconsistent with gauge invariance to demand that the orientations of the charge vectors do not change. Even classically these orientations must be regarded and treated as dynamical quantities.

These equations are invariant under the classical gauge transformation

$$\begin{aligned} A_\mu &\to \omega^{-1} A_\mu \omega + \frac{i}{g}\omega^{-1}\partial_\mu \omega , \\ F_{\mu\nu} &\to \omega^{-1} F_{\mu\nu}\omega , \\ j_\mu &\to \omega^{-1} j_\mu \omega , \end{aligned} \tag{2.6}$$

where ω is an arbitrary, unitary, space-time dependent, $n \times n$ matrix.

These coupled equations provide a complete classical description of a Yang-Mills field interacting with sources. Our discussion of both the classical solutions and their stability will use nothing more. However, this description of the sources is not canonical, and so is not directly suitable for quantization. In order to quantize, we would introduce canonical variables for the sources. The choice of variables amounts to a choice of the irreducible representation of the gauge group according to which a quantum source transforms. Classically there is no need to resolve this issue, since the charges e_i are not quantized.

Coulomb-type potentials exist when the charge vectors of each of the sources e_i are parallel or point in commuting directions in the Yang-Mills space, that is, if for each pair of sources

$$[e_i, e_j] = 0 . \tag{2.7}$$

If we then look for solutions to the field equations with potentials A_μ that have components in Yang-Mills space only in the directions specified by the charges e_i, we observe that since these directions commute, the field equations [(2.3) and (2.4)] reduce to Maxwell's equations with fixed charges. The resulting static potential is

$$\begin{aligned} A_0(\vec{x}) &= g \sum_i \frac{e_i}{4\pi |\vec{x} - \vec{x}_i|} , \\ \vec{A}(\vec{x}) &= 0 . \end{aligned} \tag{2.8}$$

Note that this elementary solution is compatible with the extended charge conservation law [Eq. (2.5)], because, by assumption, the charge vectors e_i all commute. However, had we started with noncommuting charge vectors, no static solution with only a scalar potential would have been possible. If the scalar potential at the location of a Yang-Mills charge has any finite component in a direction which does not commute with its charge vector, the vector will rotate in charge space according to Eq. (2.5).

The static, commuting charge, solution [Eq. (2.8)] satisfies the Lorentz and radiation gauge conditions,

$$\partial_\mu A_\mu = \vec{\nabla} \cdot \vec{A} = 0 . \tag{2.9}$$

It also has the property that the (gauge-dependent) charge and current densities of the gauge field are zero:

$$\mathcal{J}_\mu = -i[A_\nu, F_{\mu\nu}] = 0 . \tag{2.10}$$

Thus the total charge of the system is just the sum of the charges carried by the sources:

$$Q = \int d^3x(\mathcal{J}_0 + j_0) = \sum_i e_i . \tag{2.11}$$

The total charges Q are *invariant* under local gauge transformations which vanish at spatial infinity (i.e., $\omega \to 1$), but the division between the charge carried by the sources and the charge carried by the Yang-Mills field is gauge dependent. If one has a solution to the Yang-Mills equations with noncommuting charges and no space charge carried by the fields, one can always make a local gauge transformation which rotates the charges into commuting ones. However, such a gauge transformation will introduce a space charge density $\mathcal{J}_0$, rather than produce a solution in the form of Eq. (2.8).

As an explicit example consider a single source with charge e at the origin, and a resulting potential as in Eq. (2.8). Under a gauge transformation localized in a finite region of radius a about the origin, and constant therein,

$$\omega(x) = \exp[i\gamma\theta(a - |\vec{x}|)] , \tag{2.12}$$

where γ is a constant Hermitian matrix, the charge vector is transformed by

$$e \to \exp(-i\gamma)e \exp(i\gamma) , \tag{2.13}$$

and the transformed gauge potential produces a shell of charge.

$$\mathcal{J}_0 = \frac{i}{a^2} \int_0^1 d\xi \exp(+i\gamma\xi)[i\gamma, e]\exp(-i\gamma\xi) \times \delta(|\vec{x}| - a) . \tag{2.14}$$

Although Coulomb-type potentials result only from the simplest configurations of charges, these configurations are exactly the classical analogs of the basic quark configurations inside mesons and baryons. The hypothesis of the colored quark-gluon model is that mesons are made from a quark and an antiquark in a color-singlet state, while baryons are made from three quarks in a color-singlet state, and that color interactions are mediated through a color octet of Yang-Mills fields. The classical equivalent of a color-singlet state is a configuration in which the total Yang-

Mills charge is zero. Thus corresponding to the quark and antiquark in a meson are two sources with charge vectors e_1 and $e_2 = -e_1$, which are not only commuting, but are parallel vectors as well. Corresponding to the quarks in a baryon are three sources with vectors e_1, e_2, and e_3, satisfying

$$e_1 + e_2 + e_3 = 0 . \tag{2.15}$$

If the quarks are the standard eigenstates of color isospin and hypercharge, the vectors are

$$\begin{aligned} e_1 &= \tfrac{1}{4}\lambda_3 + \frac{1}{4\sqrt{6}}\lambda_8 , \\ e_2 &= -\tfrac{1}{4}\lambda_3 + \frac{1}{4\sqrt{6}}\lambda_8 , \\ e_3 &= -\frac{1}{2\sqrt{6}}\lambda_8 , \end{aligned} \tag{2.16}$$

which also commute.

The classical force produced between two sources with $e_1 = -e_2$ is an attractive $1/r^2$ force, just the Coulomb force. The forces on the three sources mimicking the quarks in a baryon are the sum of two Coulomb forces, one in which the role of charge is played by the third component of color isospin and another in which it is played by the color hypercharge.

III. CLASSICAL STABILITY

In this section we will discuss the (classical) stability of the Coulomb-type configurations of Yang-Mills fields with sources with respect to small alterations in the fields such as will be induced by quantum fluctuations. To maintain gauge invariance, it is necessary to simultaneously consider small fluctuations of the source vectors e_i in internal-symmetry space, since these are linked to field fluctuation through the extended conservation law

$$\partial_\mu j_\mu - ig[A_\mu, j_\mu] = 0 . \tag{3.1}$$

For at least some fluctuations in A_μ, since the source locations are fixed, their orientations will change with time.

To discuss the classical stability of this system, it is not necessary to express the source vectors e_i in terms of canonical variables. Classically, the coupled equations [Eqs. (2.1)–(2.4)] for the Yang-Mills fields and currents j_μ are complete and deterministic. Since for the Coulomb-type configuration the potentials and sources are static, the stability question for the fields is a purely static stability problem. Furthermore, the particular choice of canonical variables to represent the sources has no effect on the stability of the fields or their modes of fluctuation.

The price of avoiding introducing canonical source variables is that the method of testing stability by computing the second variation of the Hamiltonian with respect to field and source fluctuation is not available to us. Instead, we must examine the time dependence of fluctuations directly from the field equations. If all modes of fluctuation have oscillatory time dependence, the solution is stable; but if some modes grow exponentially with time (complex frequency) the system is unstable.

Classically, the canonical variables that produce a static charge oscillate with time. However, their equations of motion are linear (in the source variables), so that the modes of fluctuation of the canonical source variables are found by solving a small-vibration problem with constant coefficients but with an oscillatory forcing term, rather than with oscillatory coefficients.

The result of the analysis will be that for sufficiently small values of the Yang-Mills coupling constant and the source charge, the field due to a single charge is (classically) stable. The restriction to a single charge is merely technical: The spherical symmetry of the classical fields reduces the stability problem to ordinary differential equations. Presumably other charge configurations are also stable, if the coupling constant is small enough.

The phase "sufficiently small" coupling and charge has the following precise meaning. We will show that there is a finite (though undetermined) range about zero of the parameter $g^2|e|$ ($|e|$ is the total charge of the source) within which the field and charge configurations are stable with respect to infinitesimal classical fluctuations.

The restriction to small charge certainly limits the physical applicability of the result. However, the result is quite sufficient to proceed to a semiclassical approximation, which, as is well known, is valid only for small coupling constant.[12]

The stability argument for the Yang-Mills field will be based on the analyticity properties of the field fluctuations as a function of the source strength, as a means of extrapolating from the case of free-field fluctuations. Naturally, the spin of the fluctuating field complicates the argument in several ways, and it will be instructive to first examine a physically similar spinless example. The example we will analyze is a charged Klein-Gordon field in an external Coulomb potential. Its equation of motion is

$$[(\partial_0 - iq/r)^2 - \nabla^2]\varphi = 0 . \tag{3.2}$$

The φ fluctuations are analogs of the charged Yang-Mills field fluctuations; the q/r potential is an analog of the static Yang-Mills field.

This Klein-Gordon equation is itself the equation satisfied by small fluctuations of the field φ

about the Coulomb classical solution of a system consisting of the electromagnetic field, a massless charged scalar field, and an external charge of magnitude $4\pi q/e$ at the origin. In the Lorentz gauge the coupled equations are

$$\Box^2 A_\mu = 4\pi q\, \delta_{\mu 0}\delta^3(\vec{r}) - ie\varphi^* \overleftrightarrow{\partial}_\mu \varphi - 2e^2 A_\mu \varphi^*\varphi, \qquad (3.3)$$
$$(\partial_\mu - ieA_\mu)^2\varphi = 0,$$

and the static solution about which small fluctuations are being examined is

$$A_0 = q/r, \qquad (3.4)$$
$$\vec{A} = \varphi = 0.$$

Since we are interested in whether the fluctuations of φ remain small with time or increase in magnitude, we represent the time dependence of each mode of vibration by an exponential,

$$\varphi = e^{i\omega t}\Phi(\vec{r}). \qquad (3.5)$$

If all the frequencies of vibration ω are real, the fluctuations remain bounded and the classical solution [Eq. (3.4)] is stable. If any of the frequencies are complex, then at least one mode will grow exponentially in time and that classical solution is unstable.

To solve the fluctuation equation (3.2) we separate the angular dependence of Φ by writing

$$\Phi(\vec{r}) = \frac{1}{r} W(r) Y_m^l(\hat{r}). \qquad (3.6)$$

The resulting equation for W is

$$\left[\frac{d^2}{dr^2} + \omega^2 - \frac{2\omega q}{r} + \frac{q^2 - l(l+1)}{r^2}\right] W = 0. \qquad (3.7)$$

Let us examine the reality of the eigenfrequencies in two ways: first by using the analyticity of the solutions as a function of q, and second by using the exact solutions of the equation. We will then generalize the first procedure to the Yang-Mills case.

For small r, the two solutions behave like irrational powers of r:

$$W_\pm \sim r^{\nu_\pm}, \qquad (3.8)$$
$$\nu_\pm = \tfrac{1}{2} \pm [(l+\tfrac{1}{2})^2 - q^2]^{1/2}.$$

At least for $|q| < l + \frac{1}{2}$ we must take the solution with ν_+, the less singular (or nonsingular) solution.

For large r the leading asymptotic behaviors of the solutions are

$$W \sim e^{\pm i\omega r}. \qquad (3.9)$$

For real ω both forms are allowed, but if ω is complex, only the falling exponential may be present. Thus the existence of complex frequencies is reduced to the following question: Is there a complex frequency ω which, when the solution of Eq. (3.7), which is well behaved at the origin (W_+), is extrapolated to $r \to \infty$, has only an exponentially falling part? Such an eigenfrequency would represent an instability of the classical solution.

We can answer this question by observing that if we regard the q-dependent terms in Eq. (3.7) as a perturbation on the $q = 0$ equation, the perturbation is nonsingular (regular). That is, as $q \to 0$, the perturbation vanishes *uniformly* with respect to the remaining terms in the equation. Because the perturbation is regular, the solution W_+ is an analytic function of q, and if we examine W_+ for large r,

$$W_+(r) \sim V_1(q)e^{i\omega r} + V_2(q)e^{-i\omega r}, \qquad (3.10)$$

the functions V_1 and V_2 are analytic in q and are independent of ω. The question is: Are there values of ω and q which, for Imω less than or greater than zero, have V_1 or V_2 (respectively) vanishing? We argue that for a finite range of q, at least, this does not happen.

When $q = 0$, Eq. (3.7) is related to the spherical Bessel equation, and its nonsingular solution is

$$W_+ = r j_l(\omega r). \qquad (3.11)$$

For large argument the spherical Bessel functions have equal amounts of each exponential solution, so V_1 and V_2 have a known relation at $q = 0$:

$$|V_1(q=0)| = |V_2(q=0)|. \qquad (3.12)$$

This relation holds for all ω. Since both functions are analytic in q, there must be some *finite* range of q within which neither function vanishes relative to the other. Within this range no complex eigenfrequencies are possible.

Thus we have argued that there is a finite (though unknown) range of charge about $q = 0$ for which there are no complex eigenfrequencies and so within which the field [Eq. (3.4)] is stable. In no event can this argument be extended past $|q| = \frac{1}{2}$, since beyond this value both solutions of the s-wave equation become equally singular and there is no longer any necessity to pick solution W_+.

Before proceeding to the Yang-Mills case, let us confirm the previous argument by using the exact solution to Eq. (3.7). The solution is a Whittaker function[13] of complex argument:

$$W_+(r) = M_{-iq,\mu}(-2i\omega r), \qquad (3.13)$$

where

$$\mu = +[(l+\tfrac{1}{2})^2 - q^2]^{1/2}. \qquad (3.14)$$

For large r its asymptotic behavior is (Re$\omega > 0$)

$$M_{-iq,\mu} \sim \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu-iq)} e^{-i\pi(1/2+\mu+iq)}(-2i\omega r)^{-iq}e^{i\omega r} + \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu+iq)}(-2i\omega r)^{+iq}e^{-i\omega r}. \quad (3.15)$$

Since $\mu^2+q^2>0$, neither term can vanish. Thus we have not merely confirmed the analyticity argument, but have shown that it holds out to $q=\frac{1}{2}$.

The reader may justly wonder what has become of the hydrogenic bound states, each of which should give rise to an instability by the following argument. An instability is present whenever there is a configuration with lower total energy than that of the state whose stability is at issue. The total energy needed to fill a bound state is the energy to create a pair of quanta, that is, twice the rest mass of the quanta, less the binding energy of the level,

$$\Delta E = 2M - E_B .$$

However, the Klein-Gordon quanta are massless in our case, so ΔE appears to be negative. Hence an instability is associated with each bound-state energy level.

The reason we found no instabilities is that the massless Klein-Gordon equation with a Coulomb potential does not have hydrogenic bound states, a fact which is connected to the scale invariance of the equation. The scale invariance of the massless Klein-Gordon equation (3.2) implies that it has a purely continuous spectrum. As one easily sees from the equation, if $\varphi(\vec{x})$ is an eigenfunction with frequency ω, then $\varphi(\lambda\vec{x})$ is an eigenfunction with frequency $\lambda\omega$. To explicitly see how the discrete levels have disappeared, imagine introducing a mass term into the equation by replacing ∇^2 by $\nabla^2 - M^2$. This new equation does have discrete normalizable states, but because M is the only dimensional parameter in the equation, each of their binding energies is directly proportional to M. When M goes to zero, so the binding energies.

However, these bound levels do not simply move up to zero binding; they actually disappear from the spectrum. The reason is that M not only sets the scale of energy, but $1/M$ sets the scale of distances over which their wave functions ψ are appreciable. Since the normalization remains fixed, as M goes to zero, the wave function must be proportional to $M^{3/2}$,

$$\psi \propto M^{3/2} .$$

So the wave functions themselves vanish in the zero-mass limit.

IV. YANG-MILLS STABILITY

We will now show that the classical Coulomb-type solution of the Yang-Mills equations with a single weak source [Eq. (2.8) with only one term in the sum] is stable with respect to infinitesimal fluctuations. The method will be the most straightforward; we will linearize the full equations about the given static solution and solve for the eigenfrequencies of small vibrations. If they are all real, the solution is stable.

We will denote the static solution [Eq. (2.8)] for the Yang-Mills potential by $A_\mu^{(0)}$, the corresponding static charge-current density $j_\mu^{(0)}$, and the fluctuation of A_μ and j_μ by $\hat{A}_\mu$ and $\hat{j}_\mu$,

$$A_\mu = A_\mu^{(0)} + \hat{A}_\mu , \quad j_\mu = j_\mu^{(0)} + \hat{j}_\mu . \quad (4.1)$$

The space components of both $A_\mu^{(0)}$ and $j_\mu^{(0)}$ vanish. Furthermore, since the sources cannot move in space (although their Yang-Mills orientation may fluctuate), the space components of $\hat{j}_\mu$ also vanish. Upon linearizing in the small fluctuation $\hat{A}_\mu$, the field equations become

$$\partial_\mu\partial_\nu\hat{A}_\nu - \Box^2\hat{A}_\mu - ig\partial_\nu[A_\mu^{(0)},\hat{A}_\nu] + ig[A_\mu^{(0)},\partial_\mu\hat{A}_\nu] - ig[A_\nu^{(0)},\partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - ig[A_\mu^{(0)},\hat{A}_\nu] + ig[A_\nu^{(0)},\hat{A}_\mu]] + ig[\partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)},\hat{A}_\nu] + g\hat{j}_\mu = 0 . \quad (4.2)$$

This very complicated equation can be simplified by choosing a convenient gauge and by identifying the internal-symmetry structure of the eigenmodes of fluctuation. The gauge we will use is the background-field gauge,[14]

$$\partial_\mu\hat{A}_\mu - ig[A_\mu^{(0)},\hat{A}_\mu] = 0 . \quad (4.3)$$

This gauge condition, which neither alters nor restricts the physical content of the solutions, reduces the fluctuation equation to

$$-\Box^2 A_\mu + 2ig[A_\nu^{(0)},\partial_\nu\hat{A}_\mu] + g^2[A_\nu^{(0)},[A_\nu^{(0)},\hat{A}_\mu]] + 2ig[\partial_\mu A_\nu^{(0)},\hat{A}_\nu] - 2ig[\partial_\nu A_\mu^{(0)},\hat{A}_\nu] + g\hat{j}_\mu = 0 . \quad (4.4)$$

Only three of these four equations are independent. In fact, the background-field covariant divergence is identically zero,

$$\partial_\mu[\text{Eq. (4.4)}] - ig[A_\mu^{(0)}, [\text{Eq. (4.4)}]] \equiv 0\,, \qquad (4.5)$$

as a consequence of the gauge condition [Eq. (4.3)] and the extended conservation law for the current j_μ [Eq. (2.5)].

The simplification of the internal-symmetry structure results from the fact that since there is only one source, the source and the potential field $A_0^{(0)}$ everywhere in space are proportional to a single matrix e. This implies that the eigenmodes of fluctuation of $\hat{j}_0$ and $\hat{A}_\mu$ are proportional to fixed matrices $\bar{e}$, which are eigenvectors of e under commutation:

$$[e, \bar{e}] = \lambda \bar{e}\,. \qquad (4.6)$$

For any given source matrix e, there are only a finite number of eigenvalues, and any traceless matrix is expressible as a linear combination of $n^2 - 1$ eigenmatrices [for SU(n)] of e.

This characterization of the eigenmodes of fluctuation has a simple physical meaning. To see it, choose a basis for the Yang-Mills internal space in which the source matrix e is diagonal. In this basis the source has definite values of the $n-1$ commuting diagonal SU(n) charges, and zero values of the off-diagonal, charge-changing, generators. The potential setup by the source also has zero values of the charge-changing generators, that is, it is neutral. Each of the fluctuation eigenmodes has a definite value of the charge-changing generators, that is, it carries a definite value of each of the $n-1$ group charges. These values are numerically fixed, since $\hat{A}_\mu$ transforms according to the adjoint representation of the global gauge group. The eigenvalue λ is the sum [over the $n-1$ diagonal charges of SU(n)] of the products of the source charges times the field eigenmode charges.

With a single source at the origin, the background potential is

$$A_0^{(0)} = \frac{g}{4\pi}\frac{e}{|r|}\,, \qquad (4.7)$$

and this potential only appears in commutators multiplied by g. For any eigenmode of fluctuation, corresponding to eigenvalue λ, the charge-strength parameters appear only in the combination $\lambda g^2/4\pi$, which we denote by q:

$$q = \lambda \frac{g^2}{4\pi}\,. \qquad (4.8)$$

If $\hat{A}_{(q)\mu}$ is the coefficient of $\bar{e}$ for an eigenmode with eigenvalue λ,

$$\hat{A}_\mu = \hat{A}_{(q)\mu}\bar{e}\,, \qquad (4.9)$$

the linearized field equation for $\hat{A}_{(q)\mu}$ in the background field gauge becomes [here using the fact that the spatial components of $A_\mu^{(0)}$ vanish]

$$\left[-\left(\partial_0 - \frac{q}{r}\right)^2 + \nabla^2\right]\hat{A}_{(q)\mu} + 2i\left(\partial_\mu \frac{q}{r}\right)\hat{A}_{(q)0} - 2i\delta_{\mu 0}\left(\partial_\nu \frac{q}{r}\right)\hat{A}_{(q)\nu} + g\hat{j}_{(q)\mu} = 0\,. \qquad (4.10)$$

The gauge condition becomes

$$\partial_\mu \hat{A}_{(q)\mu} - i\frac{q}{r}\hat{A}_{(q)0} = 0\,. \qquad (4.11)$$

We are concerned with the stability of the background field with respect to the infinitesimal fluctuations $\hat{A}_\mu$ and $\hat{j}_\mu$, which is equivalent to the question of whether or not the eigenfrequencies of fluctuation are real or complex. Thus we will take as the time dependence

$$\hat{A}_{(q)\mu}, \hat{j}_{(q)\mu} \propto e^{i\omega t}\,, \qquad (4.12)$$

and then analyze the reality of ω. Factoring this time dependence from the equation has the effect of replacing each time derivative ∂_0 by $i\omega$.

There is no practical advantage in maintaining formal Lorentz covariance, since the background field defines a preferred Lorentz frame. If we denote the time and space components of the fluctuations by

$$\hat{A}_{(q)\mu} = (iB, \vec{A})e^{i\omega t}\,, \qquad \hat{j}_{(q)\mu} = (i\tilde{e}\delta^3(\vec{x}), \vec{0})e^{i\omega t}\,, \qquad (4.13)$$

the time-independent eigenvalue equations for the fluctuation become

$$\left[\left(\omega - \frac{q}{r}\right)^2 + \nabla^2\right]\vec{A} + 2\left(\vec{\nabla}\frac{q}{r}\right)B = 0\,, \qquad (4.14a)$$

$$\left[\left(\omega - \frac{q}{r}\right)^2 + \nabla^2\right]B - 2\left(\vec{\nabla}\frac{q}{r}\right)\cdot\vec{A} + q\tilde{e}\delta^3(\vec{x}) = 0 \qquad (4.14b)$$

and the gauge condition becomes

$$\left(\omega - \frac{q}{r}\right)B - \vec{\nabla}\cdot\vec{A} = 0\,. \qquad (4.15)$$

Finally, we must maintain the extended conservation law for the sources, which to first order in the fluctuations gives the extremely singular constraint.

$$\left[\left(\omega - \frac{q}{r}\right)\tilde{e} + \frac{4\pi}{g}qB\right]\delta^3(\vec{x}) = 0\,. \qquad (4.16)$$

As we remarked earlier, the four equations (4.14)–(4.16) are not independent. In fact, Eq. (4.14b) is a linear combination of the divergence of Eq. (4.14a), the gauge condition (4.15), and the current conservation condition (4.16). Therefore, we need not consider it separately, and we will henceforth ignore it. Also, the conservation condition simply fixes $\tilde{e}$ in terms of B, but has no effect on the eigenvalues, so we will ignore it as well. This leaves us with the complete system Eqs. (4.14a) and Eq. (4.15), whose spectrum we will now analyze.

In order to exploit the spherical symmetry of the background potential, which ensures that the fluctuation eigenmodes have definite angular momentum, we express $\vec{A}$ and B in terms of spherical harmonics:

$$\vec{A} = \sum_{\pm} P_{\pm}(r)\vec{Y}_{j,j\pm1,m}(\hat{r}) , \qquad B = Q(r)Y_{j,m}(\vec{r}) . \tag{4.17}$$

The $\vec{Y}_{jlm}$ are vector spherical harmonics, defined by

$$\vec{Y}_{j,l,m} = \sum_{\mu} \langle jm | l\, m-\mu\ 1\mu\rangle Y_{l,m-\mu}\vec{\xi}_{\mu} . \tag{4.18}$$

The vectors $\vec{\xi}_{\mu}$ are fixed numerical vectors normalized to 1.

Vector spherical harmonics are eigenfunctions of both orbital and total angular momentum. Some useful relations connecting them to the ordinary spherical harmonics are

$$\begin{aligned}
\hat{r}Y_{j,m} &= \sum_{\pm} a_{\pm}\vec{Y}_{j,j\pm1,m} , \\
\hat{r}\cdot\vec{Y}_{j,j\pm1,m} &= a_{\pm}Y_{j,m} , \\
\vec{\nabla}\cdot\vec{Y}_{j,j+1,m} &= \frac{j+2}{r}a_{+}Y_{j,m} , \\
\vec{\nabla}\cdot\vec{Y}_{j,j-1,m} &= -\frac{j-1}{r}a_{-}Y_{j,m} ,
\end{aligned} \tag{4.19}$$

with

$$a_{+} = -\left(\frac{j+1}{2j+1}\right)^{1/2} , \qquad a_{-} = \left(\frac{j}{2j+1}\right)^{1/2} . \tag{4.20}$$

These relations reduce the fluctuation equation and the gauge condition to

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \left(\omega - \frac{q}{r}\right)^2 - \frac{(j\pm1)(j+1\pm1)}{r^2}\right]P_{\pm} - \frac{2q}{r^2}a_{\pm}Q = 0 , \tag{4.21a}$$

$$\left(\omega - \frac{q}{r}\right)Q = a_{+}\left(\frac{d}{dr} + \frac{j+2}{r}\right)P_{+} + a_{-}\left(\frac{d}{dr} - \frac{j-1}{r}\right)P_{-} . \tag{4.21b}$$

These are a set of two coupled second-order equations for P_+ and P_-. The function Q is just an auxiliary variable.

We shall now argue that for a finite range of coupling q about $q=0$ all the eigenvalues ω of these equations are real, by showing that if ω is chosen complex, any solution which is well behaved at the origin diverges exponentially as $r\to\infty$.

Near $r=0$, for finite q, all four solutions of the system [Eq. (4.21)] have a power dependence on r, and the powers are independent of ω:

$$P_{\pm}(r) \sim p_{\pm}r^{\nu} . \tag{4.22}$$

The power ν can take the four values

$$\nu = -1 \pm \sqrt{\kappa} \pm \tfrac{1}{2} , \tag{4.23}$$

where

$$\kappa = (j+\tfrac{1}{2})^2 - q^2 . \tag{4.24}$$

For each value of ν, the ratio of the constants p_+ and p_- is fixed by

$$[\nu(\nu+1) + 2(\nu+j+2)a_{+}^{2} - (j+1)(j+2) + q^2]p_{+} = -2(\nu-j+1)a_{+}a_{-}p_{-} . \tag{4.25}$$

The $q\to0$ limits of these behaviors are

$$\begin{pmatrix}P_+\\P_-\end{pmatrix} \sim \begin{pmatrix}a_-\\-a_+\end{pmatrix}r^{j} , \quad \begin{pmatrix}0\\1\end{pmatrix}r^{j-1} , \quad \begin{pmatrix}a_-\\-a_+\end{pmatrix}r^{-j-1} , \quad \begin{pmatrix}1\\0\end{pmatrix}r^{-j-2} . \tag{4.26}$$

However, the limit $q\to0$ is not uniform, because the expression for Q [Eq. (4.21b)] has a singularity at $r=q/\omega$. If $q=0$, the equations (4.21a) decouple, and the solutions are spherical Bessel functions. The small-r behaviors of the four solutions are

$$\begin{pmatrix}P_+\\P_-\end{pmatrix} \sim \begin{pmatrix}1\\0\end{pmatrix}r^{j+1} , \quad \begin{pmatrix}0\\1\end{pmatrix}r^{j-1} , \quad \begin{pmatrix}0\\1\end{pmatrix}r^{-j} , \quad \begin{pmatrix}1\\0\end{pmatrix}r^{-j-2} . \tag{4.27}$$

The set of $r\to0$ behaviors in Eq. (4.26) is the $q\to0$ limit of the $r\to0$ limit of the solutions of the system, Eq. (4.21). The set of $r\to0$ behaviors in Eq. (4.27) is the $r\to0$ limit of the $q\to0$ limit of the same system. The discrepancy between these orders of limits,

$$\lim_{q\to0}\lim_{r\to0} \text{ vs } \lim_{r\to0}\lim_{q\to0} ,$$

reflects the nonuniformity in r of the $q\to0$ limit. This nonuniformity is in turn a direct result of the singularity at $r=q/\omega$ in Eq. (4.21b), which moves to $r=0$ when q vanishes.

As in the Klein-Gordon discussion, we want to know whether, for a small finite range of q, there are any complex eigenfrequencies ω for which a solution to Eq. (4.21) which is well behaved at $r=0$ does not diverge exponentially as $r\to\infty$. Such an eigenfrequency would represent a classical instability. That is to say, we are interested in the large-r behavior of the two well-behaved solutions in Eq. (4.26),

$$\begin{pmatrix}P_+\\P_-\end{pmatrix}\sim\begin{pmatrix}a_-\\-a_+\end{pmatrix}r^j,\ \begin{pmatrix}0\\1\end{pmatrix}r^{j-1},$$

because the physically relevant limit prescription will first take the $r\to 0$ limit and afterward examine small q.

Sadly, it is the other order of limits which is easily analyzable, since when q vanishes the system Eq. (4.21) just reduces to two copies of the spherical Bessel equation. It is thus trivial to extrapolate each of the four solutions described by Eq. (4.27) to infinity. It is less trivial to extrapolate the $q,r\to 0$, $r\ll q$ behavior [Eq. (4.26)] to $q,r\to 0$, $r\gg q$ [Eq. (4.27)], but it is exactly this connection which we must make in order to complete the connection between the small- and large-r behaviors, and so we find the large-r behavior of the two well-behaved solutions of Eq. (4.26).

In the limit $q\to 0, r\to 0$, with their ratio arbitrary, the system [Eq. (4.21)] simplifies slightly but does not decouple. The only length parameter which survives in this limit is q/ω. If we scale r by this parameter, defining

$$x=\omega r/q\,, \tag{4.28}$$

and pass to the limit, the system has no arbitrary parameters:

$$\left[\begin{pmatrix}\mathfrak{D}-\frac{(j+1)(j+2)}{x^2} & 0\\ 0 & \mathfrak{D}-\frac{j(j-1)}{x^2}\end{pmatrix}-\frac{2}{x(x-1)}\begin{pmatrix}a_+^2\left(\frac{d}{dx}+\frac{j+2}{x}\right) & a_+a_-\left(\frac{d}{dx}-\frac{j-1}{x}\right)\\ a_+a_-\left(\frac{d}{dx}+\frac{j+2}{x}\right) & a_-^2\left(\frac{d}{dx}-\frac{j-1}{x}\right)\end{pmatrix}\right]\begin{pmatrix}P_+\\P_-\end{pmatrix}=0 \tag{4.29}$$

with

$$\mathfrak{D}=\frac{1}{x^2}\frac{d}{dx}x^2\frac{d}{dx}\,. \tag{4.30}$$

While this looks nearly as formidable as the original system, it is in fact exactly soluble in terms of the most elementary functions.

To solve it, first note that $\binom{0}{1}x^{j-1}$ and $\binom{1}{0}x^{-j-2}$ are the behaviors of solutions both as $x\to 0$ [Eq. (4.26)] and as $x\to\infty$ [Eq. (4.27)]. The $x\to\infty$ limit is governed by the first matrix in Eq. (4.29), while both matrices contribute equally as $x\to 0$. Thus we see that the above expressions are exact solutions. This in turn prompts us to define new variables by

$$\begin{pmatrix}P_+\\P_-\end{pmatrix}=\begin{pmatrix}x^{-j-2} & 0\\ 0 & x^{j-1}\end{pmatrix}\begin{pmatrix}R_+\\R_-\end{pmatrix}. \tag{4.31}$$

Since any constant vector R will give a solution, this substitution will give a first-order system for the derivatives $R_\pm$. The resulting system is slightly simpler when expressed in terms of variables $S_\pm$,

$$\begin{pmatrix}R_+'\\R_-'\end{pmatrix}=\begin{pmatrix}x^{j+2} & 0\\ 0 & x^{-j+1}\end{pmatrix}\begin{pmatrix}S_+\\S_-\end{pmatrix}, \tag{4.32}$$

$$\left[x\frac{d}{dx}+\begin{pmatrix}-j & 0\\ 0 & j+1\end{pmatrix}-\frac{2}{x-1}\begin{pmatrix}a_+^2 & a_+a_-\\ a_+a_- & a_-^2\end{pmatrix}\right]\begin{pmatrix}S_+\\S_-\end{pmatrix}=0\,. \tag{4.33}$$

Making a final redefinition of variables,

$$\begin{pmatrix}T_+\\T_-\end{pmatrix}=\begin{pmatrix}a_+ & a_-\\ a_- & -a_+\end{pmatrix}\begin{pmatrix}S_+\\S_-\end{pmatrix}, \tag{4.34}$$

$$\left[x\frac{d}{dx}+\begin{pmatrix}0 & [j(j+1)]^{1/2}\\ [j(j+1)]^{1/2} & 0\end{pmatrix}+\begin{pmatrix}2/(1-x) & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}T_+\\T_-\end{pmatrix}=0\,, \tag{4.35}$$

and eliminating T_+, gives

$$\left\{\frac{d^2}{dx^2}+\frac{2}{x}\frac{x-2}{x-1}\frac{d}{dx}-\frac{1}{x^2}\left[\frac{2}{x-1}+j(j+1)\right]\right\}T_-=0\,. \tag{4.36}$$

This is related to a highly degenerate form of the hypergeometric equation whose solutions are rational functions. The substitution $T_-=x^{j-1}\overline{T}$ puts it into standard form, and gives for the well-behaved solution

$$\overline{T}=j+1-jx \tag{4.37}$$

or

$$T_-=(j+1)x^{j-1}-jx^j\,. \tag{4.38}$$

Using the second component of Eq. (4.35) to get T_+ and undoing all the variable changes gives the simple result

$$\begin{pmatrix}P_+\\P_-\end{pmatrix}=\begin{pmatrix}a_-\\-a_+\end{pmatrix}x^j-\frac{2j+1}{2j+3}a_-\begin{pmatrix}1\\0\end{pmatrix}x^{j+1}\,. \tag{4.39}$$

This completes the analysis of the $q\to 0$, $r\to 0$ nonuniformity. We have shown that the $q\to 0$ limits of the two solutions of Eq. (4.29) which are well behaved at $r=0$ $(x\to 0)$,

$$\begin{pmatrix}P_+\\P_-\end{pmatrix}_{1,2}\sim\begin{pmatrix}a_-\\-a_+\end{pmatrix}r^j\,,\quad\begin{pmatrix}0\\1\end{pmatrix}r^{j-1}\,, \tag{4.40}$$

have the $r\to 0$ limits $(x\to\infty)$

$$\begin{pmatrix}P_+\\P_-\end{pmatrix}_{1,2}\sim\begin{pmatrix}1\\0\end{pmatrix}r^{j+1}\,,\quad\begin{pmatrix}0\\1\end{pmatrix}r^{j-1}\,. \tag{4.41}$$

Both of these solutions are ordinary spherical Bessel functions which, for large r, contain equal admixtures of $e^{i\omega r}$ and $e^{-i\omega r}$ parts. Since exponentially divergent behavior is excluded, only real eigenvalues ω are allowed for $q=0$.

For finite q, the $r\to\infty$ behavior of any linear combination of solutions which are well behaved at the origin will be $(c_1{}^2+c_2{}^2=1)$

$$\begin{pmatrix}P_+\\P_-\end{pmatrix}\sim\left[c_1\begin{pmatrix}U_1^+(q)\\U_1^-(q)\end{pmatrix}+c_2\begin{pmatrix}U_2^+(q)\\U_2^-(q)\end{pmatrix}\right]e^{i\omega r}+\left[c_1\begin{pmatrix}V_1^+(q)\\V_1^-(q)\end{pmatrix}+c_2\begin{pmatrix}V_2^+(q)\\V_2^-(q)\end{pmatrix}\right]e^{-i\omega r}\,. \tag{4.42}$$

The condition necessary to eliminate either term, and so permit complex eigenvalues, is that either of the two relations

$$\begin{aligned}U(q)&=U_1^+U_2^--U_2^+U_1^-=0\,,\\V(q)&=V_1^+V_2^--V_2^+V_1^-=0\end{aligned} \tag{4.43}$$

must hold. However, both U and V are analytic functions of q (and independent of the dimensional eigenvalue ω), and we have explicitly calculated

$$|U(0)|=|V(0)|\,, \tag{4.44}$$

and so there is a finite range of coupling q within which neither function vanishes relative to the other. For this range of q, the eigenvalues ω must all be real, and so the Coulomb-type classical Yang-Mills potential is classically stable.

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