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't Hooft bound-state equation: A view from two gauges*

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Two-dimensional U(N)-invariant chromodynamics is canonically quantized in both the light-cone and the axial gauges. A principal-value infrared cutoff is adopted. A direct Hamiltonian method leads to two different meson bound-state equations in the limit of $N\to\infty$, g^2N kept fixed. In the light-cone gauge, 't Hooft's equation is obtained; in the axial gauge, the corresponding equation suffers from covariance problems rooted in the severe infrared divergences of the theory. The Bose form of the model is also presented.

I. INTRODUCTION

It is widely believed that a color gauge theory of quarks and gluons, quantum chromodynamics (QCD),[1] provides a not manifestly wrong and economical foundation for the dynamics of strong interactions. While much effort, both conceptual and computational,[2] has been spent to uncover the true infrared structure of QCD, little has yet emerged to warrant confrontation with our experimental knowledge of hadrons. However, if one is willing to contemplate an unphysical world in $1+1$ dimensions,[3] the theoretical outlook is brighter. QED and QCD[4,5] manifestly confine since the attractive Coulomb potential between a fermion-antifermion pair rises linearly with distance. Of particular interest are the works of 't Hooft,[6,7] who studied the large-N limit keeping g^2N fixed [N refers to the local SU(N) group of color and g the associated group charge]. For two-dimensional QCD, referred to hereafter as TDQCD, he showed it to be solvable in this limit, and by summing the set of all planar Feynman graphs for a given channel he obtained in the light-cone gauge a covariant bound-state equation for the mesons. This equation reveals an infinite number of finite-mass color-singlet bound states, equally spaced for large masses. This spectrum concurs with one's potential-theoretic intuition in this instance of a shallow well in the weak-coupling regime.

More recently, there has been a resurgence of activities concerning the 't Hooft solution.[8] Restricting themselves to the light-cone gauge, where the model looks particularly simple, Callan *et al.* and Einhorn aim to test the consistency of this solution. Bjorken scaling, the Drell-Yan-West relation, and the Bloom-Gilman relation are seen to be satisfied. All heralds well for the four-dimensional counterpart of the 't Hooft solution to QCD.[9] We note that the identity in the topological structure between the $1/N$ expansion of QCD and the perturbative graphs of dual-resonance models suggests that a transverse-momentum cutoff must be operative in the former as it is in dual models. If this is indeed so, one may end up with having the Galilean subdynamics of the four-dimensional theory reduced to an effective TDQCD in the infinite-momentum frame, or more precisely on the null plane. From the viewpoint of string theories TDQCD also represents the common limiting case of the quark-confining string[10] and a theory with quarks constrained at the end points of Nambu strings.[11] In short, 't Hooft TDQCD is a rich testing laboratory for bound-state problems in color gauge theories.

In this work, we wish to carry into a different direction this consistency study of the 't Hooft solution. We report on a Hamiltonian approach to TDQCD as an alternative $1/N$ expansion scheme to the usual diagrammatic method.

Our intentions are twofold. We formulate TDQCD in two different gauges, the light-cone and the axial gauges. These choices follow, respectively, from the front and the instant forms of dynamics. Postulating in both instances the standard canonical free-field commutation relations for the independent fields and a principal-value infrared cutoff, we attempt to derive the corresponding bound-state equation for quark-antiquark pairs in the $N\to\infty$ limit, g^2N fixed. While the two forms of dynamics share the same Lorentz-invariant action, there is no *a priori* reason for them to be the same since they are not simply connected by a unitary transformation.[12] Moreover, the infrared divergences inherent to the model have varying effects depending on the gauges chosen. Our interest in the axial-gauge version of TDQCD was triggered by the work of Frishman *et al*,[13] who fail to obtain a covariant 't Hooft equation in ghost-free gauges other than the light-cone gauge.

In our work, the $1/N$ expansion is formulated as an old-fashioned Rayleigh-Schrödinger perturbative series in fixed g^2N, where the perturbing potential is the particle-number-changing piece of our Hamiltonian. In the leading order in N, we recover in the light-cone gauge 't Hooft's covariant equation. However, a similar calculation in the axial gauge leads to spinorial complications and a noncovariant equation. We attribute this negative result to the inadequacy of the naive principal-value cutoff in the handling of the particularly severe infrared divergences in the Coulomb gauge. Possible cures for this problem are discussed.

Finally, we present the Bose-form equivalent of TDQCD, a form which generalizes the Bose form of the massive Schwinger model. This dual form of TDQCD will be useful in the strong-coupling regime.

Our paper is organized as follows: In Sec. II we define our notation, the null-plane quantization of TDQCD is performed, and a Hamiltonian method to get the 't Hooft equation is given. In Sec. III a similar analysis is done in the axial gauge. In Sec. IV we close with writing down the Bose form of TDQCD and discuss our results.

II. TDQCD IN THE LIGHT-CONE GAUGE

For definiteness, we consider the standard locally U(N)-invariant Lagrangian density

$$\mathcal{L}=\bar{\psi}(i\gamma^{\mu}D_{\mu}-m)\psi-\tfrac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\,, \tag{2.1}$$

where

$$D_{\mu}\psi=(\partial_{\mu}-igA_{\mu})\psi\,, \tag{2.2}$$

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}+g[A_{\mu},A_{\nu}]\,. \tag{2.3}$$

The matrix notation, e.g., $A_{\mu}=A_{\mu}^{\alpha}\,T^{\alpha}$, is used throughout. The T^{α} are the matrix representations of the generators of U(N).

$$\psi=\begin{pmatrix}\psi_1^a\\ \psi_2^a\end{pmatrix}$$

denotes a Dirac 2-spinor which is an N-component vector in the color space. A_{μ} and $F_{\mu\nu}$ are the color gauge potentials and the covariant Yang-Mills fields, respectively. Flavor indices have been deleted since only the dynamics of color is of interest here.

Following 't Hooft,[7] we choose to work with the group U(N) instead of SU(N); the difference is the singlet A_a^a which decouples and is a free field. To leading order in $1/N$, either group leads to identical results. As is apparent in (2.1) we limit our treatment to the equal-quark-mass case; the general situation is a trivial extension.

Variation of the fields ψ and A_{μ} yields the coupled set of Dirac and Yang-Mills equations of motion

$$(i\gamma^{\mu}D_{\mu}-m)\psi=0\,, \tag{2.4}$$

$$\partial^{\nu}F_{\nu\mu}=-g(j_{\mu}-i[F_{\mu\nu},A^{\nu}])\,, \tag{2.5}$$

where the color current is $J_{\mu}=\bar{\psi}\gamma_{\mu}\vec{T}\psi\cdot\vec{T}$.

We shall also need the conserved energy-momentum tensor

$$T_{\mu\nu}=-g_{\mu\nu}\mathcal{L}+i\bar{\psi}\gamma_{\mu}\partial_{\nu}\psi \tag{2.6}$$

given by Poincaré invariance through Noether's theorem.

We recall that TDQCD is superrenormalizable, as the group charge g has the dimension of a mass. Both mass and coupling-constant renormalizations are finite. Owing to the two-dimensionality of the system, there is the added bonus in any ghost-free gauge $n\cdot A=0$ that there are no nonlinear interactions among the gluons since $[A_{\mu},A_{\nu}]=0$. For this reason, we shall consider two such gauges, the light-cone and axial gauges, respectively.

It is well known that dynamics at infinite momentum, or the front form of dynamics, present definite computational advantages in bound-state problems in relativistic theories.[14] The key reason lies in the vanishing of the usually troublesome vacuum fluctuation and topologically complex graphs which are stumbling blocks in the derivation of useful integral equations for bound states in an ordinary Lorentz frame. Therefore we begin by analyzing in some detail the null-plane dynamics of system (2.1) in the light-cone gauge.

Our metric tensor components are $g_{++}=g_{--}=0$, $g_{+-}=g_{-+}=1$, with the coordinates and the γ matrices defined as

$$\begin{aligned}&x^{\pm}=x_{\mp}=(x^0\pm x^1)/\sqrt{2}\,,\\ &\gamma^{\pm}=(\gamma^0\pm\gamma^1)/\sqrt{2}\,,\quad \gamma^{+2}=\gamma^{-2}=0\,,\\ &\{\gamma^+,\gamma^-\}=2\,,\quad \gamma^5=\tfrac{1}{2}[\gamma^-,\gamma^+]\,.\end{aligned} \tag{2.7}$$

We use the Weyl representation

$$\begin{aligned}&\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix}\equiv\sigma_1\,,\quad \gamma^1=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\equiv i\sigma_2\,,\\ &\gamma^5=\gamma^0\gamma^1=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\equiv-\sigma_3\,,\end{aligned} \tag{2.8}$$

where the free spinor fields are such that $u(0)=\binom{1}{1}$ and $v(0)=\binom{1}{-1}$. Usually used for massless fermions, this representation is the most natural choice in the null-plane quantization which does *not* know about masses.[12] We also make use of the Hermitian projection operators

$$\begin{aligned}&P_+=\tfrac{1}{2}\gamma^-\gamma^+=\tfrac{1}{2}\gamma_+\gamma_-=\begin{pmatrix}0&0\\0&1\end{pmatrix},\\ &P_-=\tfrac{1}{2}\gamma^+\gamma^-=\tfrac{1}{2}\gamma_-\gamma_+=\begin{pmatrix}1&0\\0&0\end{pmatrix},\end{aligned} \tag{2.9}$$

fulfilling the properties

$$\begin{aligned}&P_++P_-=I\,,\quad P_+^2=P_+\,,\quad P_-^2=P_-\,,\\ &P_-\gamma^+P_+=\gamma^+\,,\quad P_+\gamma^-P_-=\gamma^-\,,\end{aligned} \tag{2.10}$$

and

$$P_+\psi^a = \begin{pmatrix} 0 \\ \psi_2^a \end{pmatrix} \equiv \psi_+^a, \quad P_-\psi^a = \begin{pmatrix} \psi_1^a \\ 0 \end{pmatrix} \equiv \psi_-^a . \tag{2.11}$$

By way of these γ-matrix identities in the light-cone gauge $A_- = 0$, the Dirac equation for ψ splits into

$$2iD_+\psi_+^a = m\gamma_+\psi_-^a , \tag{2.12}$$

$$2i\partial_-\psi_-^a = m\gamma_-\psi_+^a . \tag{2.13}$$

With ∂_+ playing the role of a "time" derivative in the null-plane dynamics, ψ_+^a is seen as the independent variable of our problem. Equation (2.13) plays the role of a constraint for ψ_-^a and can be written as

$$\psi_-^a(\tau, z) = \frac{m}{4i}\int_{-\infty}^{\infty} dz'\epsilon(z - z')\gamma_-\psi_+^a(\tau, z') , \tag{2.14}$$

with

$$\epsilon(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} ,$$

$$\tau \equiv x^+ ,$$

$$z \equiv x^- .$$

Consequently (2.12) becomes

$$(\partial_+ + \tfrac{1}{2}m^2\partial_-^{-1})\psi_+^a = -igA_+\psi_+^a . \tag{2.15}$$

Since there is no physical transverse degree of freedom associated with the field in two dimensions, the Yang-Mills equations (2.5)

$$\partial_-^2 A_+(\tau, z) = -gj_-(\tau, z) \tag{2.16}$$

yield another constraint leading to a nonlocal Coulomb interaction between the quarks.

Here we recover the remarkable feature of null-plane dynamics; the number of independent canonical variables is reduced to half the number present in equal-time dynamics. ψ_+^a is the only variable from which all the other operators of the theory can be built.

From (2.6), the null-plane Hamiltonian is derived:

$$H = \int T_+^+ dz = H_f + H_I , \tag{2.17}$$

where a separation is made of the free part H_f

$$H_f = \int dz\, \overline{\psi}(m - i\gamma^-\partial_-)\psi \tag{2.18}$$

and the interaction part H_I

$$H_I = \int dz\, \mathrm{Tr}[gJ_-A_+ - \tfrac{1}{2}(\partial_-A_+)^2] . \tag{2.19}$$

Eliminating ψ_- by means of the constraint (2.14), we get the expected form

$$H_f = \frac{im^2}{4}\iint dz\, dz'\psi_+^\dagger(z')_a\epsilon(z - z')\psi_+(z)^a . \tag{2.20}$$

Similarly, solving for A_+ by use of (2.16) we get the Coulomb term

$$H_I = -\frac{g^2}{4}\iint dz\, dz' J_-(z)_a^b |z - z'| J_-(z)_b^a , \tag{2.21}$$

where

$$J_-(\tau, z)_a^b = \sqrt{2}\,\psi_+^\dagger(\tau, z)_a\psi_+(\tau, z)^b . \tag{2.22}$$

The main steps leading up to (2.21) are as follows. The general solution of the boundary-value problem (2.16) is given as

$$A_+ = -g\partial_-^{-2}j_- - Ez - G , \tag{2.23}$$

where

$$\partial_-^{-2}j_-(z) = \tfrac{1}{2}\int dz' |z - z'| j_-(z') . \tag{2.24}$$

The constant matrix G can be gauged to zero. The E correspond to the N constant-color background fields.[5] They cannot affect the color-singlet sector of the theory since the quark-antiquark bound states carry no dipole moments.[8] Hence the E can be set equal to zero in this singlet sector. By direct insertion of (2.23) and (2.24) in (2.19) and by use of the identity

$$|z_1 - z_2| = \tfrac{1}{2}\int dz\, \epsilon(z_1 - z)\epsilon(z - z_2) ,$$

(2.21) results.

Now the null-plane classical initial-value problem for ψ_- obeying (2.15) requires the additional assumption $\psi_+(\pm\infty, \tau) = 0$ with τ held fixed.[15] This condition means that the physical system is local and is anyhow required for the existence of such generators as the Hamiltonian H (2.17), the momentum operator $P = \int dz T_+^-$, and the total matrix charge $Q = \int_{-\infty}^{+\infty} j_-(z, 0)dz$. Regarding the total charge, carrying out in the light-cone gauge an easy calculation analogous to that of Zumino,[16] we can obtain

$$\left(\partial_+^2 + \frac{g^2}{\pi}\right) Q = 0 \tag{2.25}$$

as a consequence of the anomaly in the current $\partial_\mu j^\mu = -(g/4\pi)\epsilon^{\mu\nu}F_{\mu\nu}$. Since Q cannot change in "time," (2.25) implies the constancy of the charge Q, which is true only if $Q = 0$. Similarly, Q is independent of the Lorentz frame only if it vanishes. However, as a quantized operator such that $[Q, \psi] = -e\psi$, Q can only vanish *weakly*, i.e., the null-plane quantization of TDQCD is only *covariant* in the singlet sector of the Hilbert space of states. In this work we shall confine ourselves to this bound-state sector of the theory.

Having eliminated all dependent fields we can proceed to quantize the theory canonically. It is natural to work in the Schrödinger picture by postulating, as is usually done in null-plane quantization, a *free* field expansion for ψ_+ at fixed time $\tau=0$:

$$\psi_+(z,\tau=0)=\frac{1}{\sqrt{2\pi}}\int d\eta e^{-i\eta z}a(\eta)\,, \tag{2.26}$$

where

$$\{a(\eta),a^\dagger(\eta')\}=\delta(\eta-\eta')\,,\quad \{a(\eta),a(\eta')\}=0 \tag{2.27}$$

are the covariant anticommutation relations. Equivalently in the space-coordinate representation we have

$$\{\psi_+(\tau,z),\psi^\dagger(\tau,z')\}=(P_+/\sqrt{2})\,\delta(z-z')\,,\quad \{\psi_+(\tau,z),\psi_+(\tau,z')\}=0\,. \tag{2.28}$$

The current $J_-(z,0)$ (2.22) then takes the form

$$J^b_{-a}=\frac{1}{\sqrt{2\pi}}\iint d\eta\,d\eta' e^{-i(\eta-\eta')z}\!:a^\dagger(\eta')_a a(\eta)^b:\,. \tag{2.29a}$$

Letting $\eta-\eta'=m$ and defining the density operators

$$\rho^b_a(m)=\int d\eta'\,:a^\dagger(\eta')_a a(\eta'+m)^b: \tag{2.29b}$$

familiar in dual theories[17] and in solvable models of a one-dimensional electron gas,[18] we have

$$J^b_a(z)=\frac{1}{\sqrt{2\pi}}\int dm\,\rho^b_a(m)e^{-imz}=\frac{1}{\sqrt{2\pi}}\int_{m>0}dm[\rho^b_a(m)e^{-imz}+\rho^b_a(m)^\dagger e^{imz}]\,. \tag{2.30}$$

Making the identification through (2.26) of

$$a(\eta)=c(\eta)\,,\quad a(-\eta)=d(\eta)^\dagger\,,\qquad \eta=1,2,\ldots\,, \tag{2.31}$$

where $c^\dagger$ and $d^\dagger$ are the quark and antiquark creation operators, respectively, we obtain with a little algebra

$$\rho^b_a(m)=\int_0^\infty d\eta[c^\dagger(\eta)_a c(\eta+m)^b-d^\dagger(\eta)_a d(\eta+m)^b+d(m-\eta)_a b(\eta)^b] \tag{2.32}$$

and

$$\rho^b_a(m)^\dagger=\rho^b_a(-m)\,. \tag{2.33}$$

Substituting these in the interaction Hamiltonian we obtain

$$H_\mathrm{I}=\frac{g^2}{2\pi}\int_{0+}^\infty\frac{d\eta}{\eta^2}\,[\rho^b_a(\eta)^\dagger\rho^b_a(\eta)+\mathrm{H.c.}]\,. \tag{2.34}$$

Similarly, the free Hamiltonian is given as

$$H_\mathrm{f}=\int_{0^+}^\infty d\eta\frac{m^2}{\eta^2}[c^\dagger(\eta)_a c(\eta)^a+d^\dagger(\eta)_a d(\eta)^a]\,. \tag{2.35}$$

Upon introduction of the fermion vacuum, one constructs the Hilbert space by cyclic action of the creation operators on this vacuum. When sandwiched between any vector in this Hilbert space the density operators $\rho^b_a(m)$ satisfy Bose-type commutation relations

$$[\rho(m),\rho^\dagger(n)]=\delta(m-n)\,. \tag{2.36}$$

(2.36) is then to be understood in the weak sense of Dirac.[19]

Having thus set up the above machinery, we now define our eigenvalue problem for the quark-antiquark bound states. Let

$$|Q\overline{Q}\rangle_P=\int_0^P dk\,\phi(P,k)c^\dagger(k)_a d^\dagger(P-k)^a|0\rangle \tag{2.37}$$

denote the quark-antiquark bound-state ket vector in the infinite-momentum frame; $\phi(P,k)$ is the amplitude for finding a quark with momentum k and an antiquark with momentum $(P-k)$ (there is no spin in two dimensions). P is the total momentum of the hadron.

The Schrödinger equation for a bound state of invariant mass μ^2 is then

$$H|Q\overline{Q}\rangle_P=\frac{\mu^2}{2P}\,|Q\overline{Q}\rangle_P\,. \tag{2.38}$$

Before proceeding further we first normal-order the Hamiltonian, making use of the trace identity $\mathrm{Tr}(\delta^{ab})=N$ and (2.27). In this manner, the mass renormalization contributions are separated from the rest of the interaction; they are quadratic in $c^\dagger c$ and $d^\dagger d$. We obtain

$$H_\mathrm{I}=\frac{g^2}{2\pi}\int_{k>0}^\infty\frac{dk}{k^2}[c^\dagger(k)_a c(k)^a+d^\dagger(k)_a d(k)^a]+\frac{Ng^2}{\pi}\int_{k>0}^\infty\frac{dk}{k^2}\int dk'[c^\dagger_a(k'+k)c^a(k'+k)+d^\dagger_a(k'+k)d^a(k'+k)]+\frac{g^2}{2\pi}\int_{k>0}^\infty\frac{dk}{k^2}:[\rho^b_a(k)\rho^b_a(k)+\mathrm{H.c.}]:\,. \tag{2.39}$$

The $(k>0)$ lowest limit of integration means the exclusion of the zero mode to be specified by a principal-value cutoff. The first term yields a constant and can be dropped. The second term gives rise to mass renormalizations. The last term constitutes the actual interactions.

Regrouping again the various terms, the Hamiltonian is written as

$$H = H_t + H_{\rm MR} + H' , \tag{2.40}$$

where H' is split into a particle-number-conserving piece $H_{\rm PC}$ and the rest R. We have

$$\begin{aligned} H_{\rm PC} = &\frac{g^2}{2\pi}\int\frac{dk}{k^2}\int\!\!\int dk'dk''[c^\dagger(k')_a d^\dagger(k''+k)^b c(k'+k)_b d(k')^a + d^\dagger(k''+k)_a c^\dagger(k')^b d(k'')_b c(k'+k)^a] \\ &+\frac{g^2}{2\pi}\int\frac{dk}{k^2}\int\!\!\int dk'dk''[d^\dagger(k')_a c^\dagger(k''+k)^b d(k'+k)_b c(k'')^a + c^\dagger(k''+k)_b d^\dagger(k')^a c(k'')_a d(k'+k)^b] \\ &+\frac{g^2}{2\pi}\int\frac{dk}{k^2}\int\!\!\int dk''dk'[c^\dagger(k'')^b d^\dagger(k-k')_a d(k-k'')^a c(k')_b + c^\dagger(k'')_a d^\dagger(k-k'')^b d(k-k')_b c(k)^a] . \end{aligned} \tag{2.41}$$

We observe that the first two terms are Coulomb-exchange interactions; the last term corresponds to annihilation processes which are suppressed on the light cone.[12,14] Thus our bound-state-equation approach consists then in taking into account, in the leading order in $1/N$, the contribution in H due only to the mass renormalization and Coulomb exchanges. It follows that

$$\frac{\mu^2}{2P}\phi(P,k) = \left(\frac{m^2 - g^2N/\pi}{|k|} + \frac{m^2-g^2N/\pi}{|P-k|}\right)\phi(P,k) - \frac{Ng^2}{\pi}\,\mathrm{P}\int_0^P dq\,\frac{\phi(P,q)}{|k-q|^2} . \tag{2.42}$$

The symbol P denotes a principal-value integral. Alternatively, making use of the Feynman variable $k = xP$ and the identity

$$\int_0^1 \frac{\tilde\phi(x)}{|x-y|^2}dy = -\tilde\phi(x)\left(\frac{1}{x}+\frac{1}{1+x}\right)$$

and defining $\phi(P,x) = \tilde\phi(x)$, we have the 't Hooft equation

$$\mu_t^{\,2}\tilde\phi = \left[\frac{(\alpha+1)}{x} + \frac{(\alpha+1)}{1+x}\right]\phi(x) - \mathrm{P}\int_0^1 dy\,\frac{\phi(y)-\phi(x)}{(x-y)^2} , \tag{2.43}$$

where $\mu_t^{\,2} = \mu^2(g^2N/\pi)^{-1}$ is dimensionless and $\alpha = (m^2 - g^2N/\pi)(g^2N/\pi)^{-1}$. In the above Schrödinger-equation approach, higher-order $1/N$ corrections are in principle calculable by way of old-fashioned perturbation theory in the particle-number-changing potential R. It can be verified without calculation that a second-order perturbation theory in R to the energy eigenvalues in (2.43) is proportional to $(g^2N)^2/N$ on dimensional grounds.

We shall not go into any of the details regarding the solutions to Eq. (2.43). They have been treated by 't Hooft and will be the topics of a forthcoming work of Hanson *et al.*,[20] who apply a powerful method of numerical analysis to several one-dimensional bound-state equations of QCD and string theory in different coupling regimes.

III. TDQCD IN THE AXIAL GAUGE

We now proceed to the quantization of TDQCD in the axial gauge in a manner entirely analogous to that for the light-cone gauge. From (2.1) the Hamiltonian density is

$$H = \bar\psi(m + i\gamma^1\partial_1)\psi + gj_0A_0 - \tfrac{1}{2}(\partial_1 A_0)^2 , \tag{3.1}$$

with $A_1 = 0$; the Euler-Lagrangian equations for the gauge fields are

$$\partial_1^{\,2}A_0 = -gj_0 . \tag{3.2}$$

They have the nature of a constraint. The general solution of (3.2) is

$$A_0 = -g\partial_1^{\,2}j_0 - Ex - G . \tag{3.3}$$

For the same reasons given in Sec. II, $G=0$. The background color fields E can be set equal to zero in the color-singlet sector.

By direct substitution of (3.3) in H, we get

$$H = \int dx\,\bar\psi(m + i\gamma^1\partial_1)\psi - \frac{g^2}{4}\int\!\!\int dx\,dx' j_0(x)\,|x-x'|\,j_0(x) . \tag{3.4}$$

Just as in Sec. II we assume a free-field expansion for the 2-spinor

$$\psi = \frac{1}{\sqrt{2\pi}}\int dp[A(p)u(p)e^{ipx} + B^\dagger(p)v(p)e^{-ipx}] . \tag{3.5}$$

The spinors satisfy

$$(\alpha p + m\beta)u = E_p u , \quad (-\alpha p + m\beta)v = -E_p v ,$$

where

$$E_p = (p^2+m^2)^{1/2} ,$$

and

$$\{A(p), A^\dagger(p')\}_+ = \{B(p), B^\dagger(p')\}_+ = \delta(p-p') \tag{3.6}$$

are the postulated equal-time canonical commutation relations. For the γ-matrix representation, we choose

$$\gamma_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \gamma_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$
$$\gamma_5=\gamma_0\gamma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{3.7}$$

In terms of the quark $(A,A^\dagger)$ and antiquark $(B,B^\dagger)$ creation and annihilation operators we can rewrite

$$H=H_0+H_I,$$
$$H_0=\int dp E_p[A^\dagger(p)^a A(p)^a+B^\dagger(p)^a B(p)^a]. \tag{3.8}$$

In order to reexpress H_I in terms of these oscillators we must replace J_0 in (3.4) by its normal-ordered current[21] [omitting the U(n) matrix label]

$$:J_0:=\frac{1}{2\pi}\int dp\,dp'[A^\dagger(p')A(p)u^\dagger(p')u(p)e^{-i(p'-p)x}+B^\dagger(p)B(p')v^\dagger(p')v(p)e^{i(p'-p)x}$$
$$+A^\dagger(p')B^\dagger(p)u^\dagger(p')v(p)e^{-i(p'+p)x}+B(p')A(p)v^\dagger(p')u(p)e^{i(p'+p)x}]. \tag{3.9}$$

Just as in the light-cone case, we set up the eigenvalue equation for the meson bound states

$$H|Q\overline{Q}\rangle_P=(\mu^2+P^2)^{1/2}|Q\overline{Q}\rangle_P. \tag{3.10}$$

Similarly, we shall only take into account in H terms which do *not* change the number of particles. After normal ordering and dropping the particle-number-changing terms we are left with

$$H=H_0+H_{MR}+H_c+H_a, \tag{3.11}$$

where the mass-renormalization piece is (P stands for principal value)

$$H_{MR}=\frac{g^2N}{4\pi}\iint dp\,dq\,\mathrm{P}\frac{1}{(p-q)^2}[u^\dagger(p)u(q)u^\dagger(q)u(p)A^{\dagger a}(p)A^a(p)+v^\dagger(p)v(q)v^\dagger(q)v(p)B^{\dagger a}(q)B^a(q)]$$
$$-\frac{g^2}{4\pi}\int dp\,dp'\,dq\,dq'\,\mathrm{P}\frac{1}{(p-p')^2}\delta(p-p'+q-q')[u^\dagger(p)u(p')u^\dagger(q)u(q')A^{\dagger a}(p)A^{\dagger b}(q)A^b(p')A^a(q')$$
$$+v^\dagger(p)v(p')v^\dagger(q)v(q')B^{\dagger b}(p)B^{\dagger a}(q)B^a(p')B^b(q')], \tag{3.12}$$

the Coulomb interaction piece is

$$H_c=\frac{-g^2}{4\pi}\int dp\,dp'\,dq\,dq'\,\mathrm{P}\frac{1}{(p-p')^2}\delta(p-p'+q'-q)[u^\dagger(p)u(p')v^\dagger(q)v(q')A^{\dagger a}(p)B^{\dagger a}(q')A^b(p')B^b(q)$$
$$+v^\dagger(p)v(p')u^\dagger(q)u(q')B^{\dagger b}(p')A^{\dagger b}(q)B^a(p)A^a(q')], \tag{3.13}$$

and the annihilation piece is

$$H_a=\frac{-g^2}{4\pi}\int dp\,dp'\,dq\,dq'\,\delta(p+p'-q-q')\mathrm{P}\frac{1}{(p+p')^2}$$
$$\times[u^\dagger(p)v(p')v^\dagger(q)u(q')A^{\dagger a}(p)B^{\dagger b}(p)B^b(q)A^a(q')+v^\dagger(p)u(p')u^\dagger(q)v(q')A^{\dagger b}(q)B^{\dagger a}(q')B^a(p)A^b(p')]$$
$$-\frac{Ng^2}{4\pi}\iint dp\,dq\,\mathrm{P}\frac{1}{(p+q)^2}v^\dagger(p)u(q)u^\dagger(q)v(p)[B^{\dagger a}(p)B^a(p)+A^{\dagger a}(q)A^a(q)]. \tag{3.14}$$

By further inspection we observe that the second group of terms in (3.12) is down by a factor of N compared to the first group, and therefore can be neglected. Regrouping our interaction Hamiltonian again, we get

$$H_{MR}=\frac{Ng^2}{4\pi}\iint dp\,dq\,\mathrm{P}\frac{1}{(p-q)^2}\{[u^\dagger(p)u(q)u^\dagger(q)u(p)]A^{\dagger a}(p)A^a(p)+[v^\dagger(p)v(q)v^\dagger(q)v(p)]B^{\dagger a}(q)B^a(q)\}$$
$$-\frac{Ng^2}{4\pi}\iint dp\,dq\,\mathrm{P}\frac{1}{(p+q)^2}v^\dagger(p)u(q)u^\dagger(q)v(p)[A^{\dagger a}(q)A^a(q)+B^{\dagger a}(p)B^a(p)] \tag{3.15}$$

and

$$H_a=-\frac{g^2}{4\pi}\int dp\,dp'\,dq\,dq'\,\delta(p+p'-q-q')\mathrm{P}\frac{1}{(p+p')^2}$$
$$\times[u^\dagger(p)v(p')v^\dagger(q)u(q')A^{\dagger a}(p)B^{\dagger b}(p')B^b(q)A^a(q')+v^\dagger(p)u(p')u^\dagger(q)v(q')A^{\dagger b}(q)B^{\dagger a}(q)B^a(p)A^b(p')]. \tag{3.16}$$

So the truncated Hamiltonian is $H_t = H_0 + H_{MR} + H_c + H_a$. We then compute $H|Q\bar{Q}\rangle_P = E_P|Q\bar{Q}\rangle_P$ taking as our bound-state representation

$$|Q\bar{Q}\rangle_P = \int_0^\infty dk\,\phi(P,k)A^{\dagger a}(k)B^{\dagger a}(P-k)|0\rangle ,$$

which is consistent with our ansatz of a free-spinor expansion; here ϕ is chosen to be an even function of k. Calculating each term separately, the results are

$$H_0|Q\bar{Q}\rangle_P = \int_0^\infty dk(E_k + E_{P-k})\phi(P,k)A^{\dagger a}(k)B^{\dagger a}(P-k)|0\rangle , \tag{3.17}$$

$$H_{MR}|Q\bar{Q}\rangle_P = \frac{Ng^2}{4\pi}\int_0^\infty dk\,\phi(P,k)\mathrm{P}\int_0^\infty dq\left[\frac{U(k,q)}{(k-q)^2} + \frac{V(q,P-k)}{(k+q-P)^2} - \frac{W(q,P-k)}{(k+q)^2} - \frac{W(q,P-k)}{(q-k+P)^2}\right]A^{\dagger a}(k)B^{\dagger a}(P-k)|0\rangle , \tag{3.18}$$

where

$$\begin{aligned} U(p,q) &= u^\dagger(p)u(q)u^\dagger(q)u(p) , \\ V(p,q) &= v^\dagger(p)v(q)v^\dagger(q)v(p) , \\ W(p,q) &= v^\dagger(p)u(q)u^\dagger(q)v(p) . \end{aligned} \tag{3.19}$$

Notice that in the representation in which we are working the quark propagator, which appears as the middle term above, is

$$\begin{aligned} u(q)u^\dagger(q) &= \frac{1}{E_q}(E_q\gamma_0 - q\gamma_1 + m)\gamma^0 \\ &= \frac{1}{E_q}(E_q + q\gamma_5 + m\gamma_0) . \end{aligned} \tag{3.20}$$

Thus the first term in (3.18) is

$$\frac{Ng^2}{4\pi}\int_0^\infty dk\,\phi(P,k)\mathbf{P}\int_0^\infty dq\,u^\dagger(k)\left(\frac{E_q + q\gamma_5 + m\gamma_0}{E_q}\right)u(k)\frac{1}{(k-q)^2} . \tag{3.21}$$

Using the explicit representation for the free spinors

$$u(k) = \frac{1}{[2E(E+m)]^{1/2}}\begin{pmatrix} E+m \\ k \end{pmatrix} \tag{3.22}$$

this becomes

$$\frac{Ng^2}{4\pi}\int_0^\infty dk\,\phi(P,k)\int_0^\infty dq\left(\frac{E_k E_q + kq + m^2}{E_k E_q}\right)\mathbf{P}\frac{1}{(k-q)^2} . \tag{3.21'}$$

The sign of the first term in the integrand may be changed freely because its contribution to the integral is zero. So the numerator vanishes quadratically when $p=q$; therefore the principal-value symbol can be dropped. Thus (3.21′) becomes

$$\frac{Ng^2}{4\pi}\int_0^\infty dk\,\frac{\phi(P,k)}{E_k}\int_0^\infty dq\frac{1}{E_q}(m^2 + kq - E_k E_q)\frac{1}{(k-q)^2} . \tag{3.21''}$$

Introducing the new variables, $q = m\sinh x$ and $k = m\sinh c$, the q integral is evaluated easily, and the result is

$$-\frac{Ng^2}{2\pi}\int_0^\infty dk\frac{\phi(P,k)}{E_k} . \tag{3.23}$$

Using the same integrations for the other three terms in (3.18) we get zero for W terms, and a similar result for the second term. So finally,

$$H_{MR}|Q\bar{Q}\rangle_P = -\frac{Ng^2}{2\pi}\int_0^\infty dk\left(\frac{1}{E_k} + \frac{1}{E_{P-k}}\right)\phi(P,k)A^{\dagger a}(k)B^{\dagger a}(P-k)|0\rangle . \tag{3.24}$$

Moving on to the Coulomb term in (3.13), we have

$$H_c|Q\bar{Q}\rangle = \frac{Ng^2}{4\pi}\int_0^\infty dp\,dp'\,dq\,dq'\,\mathbf{P}\frac{1}{(p-p')^2}$$
$$\times\Big[\int_0^\infty dk\,\delta(p-p'+q'-q)\delta(p'-k)\delta(q+k-P)u^\dagger(p)u(p')v^\dagger(q)v(q')\phi(k)A^{\dagger a}(p)B^{\dagger a}(q')\,|0\rangle$$
$$-\int_0^\infty dk\,\delta(p-p'+q'-q)\delta(q'-k)\delta(p+k-P)v^\dagger(p)v(p')u^\dagger(q)u(q')\phi(k)B^{\dagger a}(p')A^{\dagger a}(q)\,|0\rangle\Big]\,, \tag{3.25}$$

which simplifies to

$$H_c|Q\bar{Q}\rangle_P = \frac{Ng^2}{2\pi}\int_0^\infty dp\int_0^\infty dk\,\mathbf{P}\frac{1}{(p-k)^2}\phi(P,k)[u^\dagger(p)u(k)v^\dagger(P-k)v(P-p)]A^{\dagger a}(p)B^{\dagger a}(P-p)\,|0\rangle\,. \tag{3.26}$$

We get after a change of variable the form

$$H_c|Q\bar{Q}\rangle_P = \frac{Ng^2}{2\pi}\int_0^\infty dk\,\mathbf{P}\int_0^\infty dp\frac{K(P,k,p)}{(p-k)^2}\phi(P,k)A^{\dagger a}(k)B^{\dagger a}(P-k)\,|0\rangle\,, \tag{3.27}$$

with

$$K(P,k,p) = u^\dagger(p)u(k)v^\dagger(P-k)v(P-p)\,.$$

While the annihilation term is down by a factor of $1/N$ compared to the other terms, we give for the sake of completeness the corresponding equation

$$H_a|Q\bar{Q}\rangle = \frac{g^2}{4\pi}\int_0^\infty dp\int_0^\infty dk\mathbf{P}\frac{1}{(p+k)^2}u^\dagger(p)v(P-p)v^\dagger(P-k)u(k)\phi(P,k)A^{\dagger a}(p)B^{\dagger a}(P-p)\,|0\rangle$$
$$-\frac{g^2}{4\pi}\int_0^\infty dq\int_0^\infty dk\mathbf{P}\frac{1}{(q+k)^2}v^\dagger(P-k)u(k)u^\dagger(q)v(P-q)A^{\dagger a}(q)B^{\dagger a}(P-q)\phi(P,k)\,|0\rangle\,. \tag{3.28}$$

In the event that the group is U(1), so that (1) become the massive Schwinger model, H_a can be neglected on the grounds that it is of order $\hbar^2$. In this case, the weak-coupling limit has been studied by Coleman,[22] who made use of semiclassical approximations while preserving relativistic kinematics.

Gathering all contributions (3.17), (3.24), and (3.27), we obtain the eigenvalue equation

$$E_P\phi(P,k) = \Big[(E_k+E_{P-k}) - \frac{Ng^2}{2\pi}\Big(\frac{1}{E_k}+\frac{1}{E_{P-k}}\Big)\Big]\phi(P,k) + \frac{Ng^2}{2\pi}\mathbf{P}\int_0^\infty dp\frac{K(P;k,p)}{(p-k)^2}\phi(P,p)\,, \tag{3.29}$$

where

$$E_k = (k^2+m^2)^{1/2}\,,$$
$$K(P;k,p) = u^\dagger(p)u(k)v^\dagger(P-k)v(P-p)\,.$$

By using the equality

$$\Big(E_k - \frac{Ng^2}{\pi}\Big)^{1/2} = E_k - \frac{Ng^2}{2\pi}\frac{1}{E_k} + O((Ng^2)^2) \tag{3.30}$$

which is good in the weak-coupling limit, (3.29) can be cast into the following form:

$$E_P\phi(P,k) = \Big\{\Big[k^2+\Big(m^2-\frac{Ng^2}{\pi}\Big)\Big]^{1/2} + \Big[(P-k)^2+\Big(m^2-\frac{Ng^2}{\pi}\Big)\Big]^{1/2}\Big\}\phi(P,k) + \frac{Ng^2}{2\pi}\mathbf{P}\int_0^\infty dp\frac{K(P;k,p)}{(p-k)^2}\phi(P,p) \tag{3.31}$$

which exposes the "mass renormalization" explicitly.

It is obvious that this equation is not manifestly covariant,[20] owing to the explicit dependence on P, the momentum of the bound state. Of all the frames, the most convenient one is clearly the center-of-mass frame, where $P=0$:

$$\mu\phi(0,k)=2\left[k^2+\left(m^2-\frac{Ng^2}{\pi}\right)\right]^{1/2}\phi(0,k)+\frac{Ng^2}{2\pi}\mathbf{P}\int_0^\infty dp\frac{K(0;p,k)}{(p-k)^2}\phi(0,p)\,, \tag{3.32}$$

where now

$$K(P;p,k)=u^\dagger(p)u(k)v^\dagger(-k)v(-p)$$
$$=\left[\frac{1}{2E_p(E_p+m)2E_k(E_k+m)}\right]^{1/2}[(E_p+m)(E_k+m)+pk]^2\,.$$

The spectrum of this particular equation is being numerically studied by Hanson *et al.*[20]

IV. BOSE FORM AND CONCLUDING REMARKS

What have we learned from the foregoing exercise? First we have seen a key feature of null-plane quantization of TDQCD, namely its lack of sensitivity to the specific form of the infrared cutoff. It has been known that the massive Schwinger model and hence also TDQCD suffer from very severe infrared divergences induced by the bare quark masses.[23] There appears to be a softening of these divergences in the $A_-=0$ gauge reflected in the indifference to cutoff, possibly due to the peculiarities of the null-plane quantization.[12] It does not know about masses and its vacuum is the bare vacuum. These features account for the simplest spinor kinematics, the intuitive picture of constituents in a relativistic bound state.

In the axial (or Coulomb) gauge none of the above properties are available. Thus the interaction kernel (3.29) is plagued with mass-dependent kinematic factors. Nonrelativistic approximations need to be made to obtain a more tractable bound-state equation. No clear picture of a relativistic bound state is available and our assumption of a *free*-field expansion at fixed time is thereby suspect. Indeed the Coulomb-gauge computation performed here should be taken only heuristically. It is known[24] that *no* bona fide quark field operators can be constructed in this gauge, owing to the linearly rising potential. Since a simple principal-value cutoff fails, a more involved technique such as a mass- and coupling-dependent cutoff procedure could be tried. A more illuminating approach, we believe, would be that of Lowenstein and Swieca,[24] who get to the Coulomb gauge via a limiting procedure starting from covariant gauges. This is a difficult problem we are presently studying.

However, irrespective of possible covariance problems, our method of attack of the bound-state problem in the $1/N$ expansion is a straightforward one. The usual apparatus of Schrödinger perturbation theory can be in principle applied to compute higher $1/N$ corrections and to handle the problem of bound-state scattering and the analysis of form factors, etc.[8] In the instance of the massive Schwinger model recently studied by Coleman in the axial gauge, our results in Sec. III carry over provided the coupling is weak, $g\ll m$. The null-plane quantization of the massive Schwinger model gives in our opinion a more tractable as well as atrractive resolution of the weak-coupling structure of the model, when contrasted with the spinor complexities of the axial-gauge formalism.

Finally, another limit of interest to TDQCD is the strong-coupling limit, which is believed to be of genuine relevance to the infrared problem. In two dimensions, a handle on this strong-coupling regime is possible without going to a lattice thanks to a correspondence between, say, the massive Schwinger model and the sine-Gordon theory. The strong-coupling limit in the first model corresponds remarkably to the weak-coupling limit in the second and hence makes it computable. Recent works on the Bose form of the massive SU(N) Thirring models[25,26] allow us to write down the bosonic equivalent of (2.1). We shall be very brief. For more details, the reader is referred to the quoted literature.

We work in the interaction representation and in the axial gauge $A_1^a=0$ of TDQCD.

It is known[25] that the SU(N)-invariant free massive Thirring theory

$$\mathcal{L}=\sum_{i=1}^{N}\bar\psi^i(i\partial\!\!\!/-m)\psi_i \tag{4.1}$$

is equivalent to a theory of N Bose fields ϕ_i:

$$H_e=N_m\sum_{i=1}^{N}\left[\tfrac{1}{2}\pi_i^2+\tfrac{1}{2}(\partial_i\phi_i)^2-cm^2\cos 2\sqrt{\pi}\,\phi_i\right]. \tag{4.2}$$

N_m denotes normal ordering with respect to the mass m, c is a constant of no relevance to our consideration.

The boson forms of the N two-component spinors are[26]

$$\psi^a_{L,R}(x,t)=\frac{1}{\sqrt{\Omega}}:\exp[\mp 2i\sqrt{\pi}\,\Phi^a_{L,R}(x,t)]:\chi^a_{L,R}\,,\qquad a=1,\ldots,N\,. \tag{4.3}$$

Ω is the quantization box length, and the $\chi^a_{L,R}$ are nondynamical anticommuting operators which

Banks *et al.*[26] need to introduce. Their properties are $\chi_a=\chi_a^\dagger$, $\{\chi_a, \chi_\beta\}=2\delta_{\alpha\beta}$, and $P\chi_1P^\dagger=\chi_2$, where P is the parity operator. Then the color current $J_0=J_0^aT^a$ is given

$$J_0^a=\sum_{a\neq b}^{N}\frac{\lambda_{ab}^{(i)}}{2\Omega}\{\chi_1^a\chi_1^b\colon \exp[2i\sqrt{\pi}(\Phi_L^a-\Phi_L^b)]\colon + \chi_2^a\chi_2^b\colon \exp[-2i\sqrt{\pi}(\Phi_R^a-\Phi_R^b)]\}\colon + \frac{1}{\sqrt{\pi}}\sum_{a=1}^{N}\frac{\lambda_{aa}^{(i)}}{2}\partial_x\Phi^a, \quad i=1,\ldots,N^2-1. \tag{4.4}$$

Since in the charge-zero sector the axial-gauge Hamiltonian is

$$\mathcal{H}=\sum_i^N \bar\psi^i(i\gamma_1\partial_1+m)\psi_i+\tfrac{1}{2}(F_{01}^a)^2, \tag{4.5}$$

with

$$F_{01}^a=-g\partial_1^{-1}J_0^a, \tag{4.6}$$

which is the Bose form, TDQCD is readily accomplished using (4.2) and (4.4). For $G=\mathrm{U}(1)$ it reduces to the Bose form of the massive Schwinger model.[22] For the non-Abelian case, the Bose form looks rather intricate. Its study connected with the strong-coupling limit $g^2\gg m^2$ of TDQCD will be the object of another work.

Note. After we completed this work M. K. Prasad kindly informed us of a paper by M. S. Marinov, A. M. Perelomov, and M. V. Terent'ev (Zh. Eksp. Teor. Fiz. Pis'ma Red. 20, 494 (1974) [JETP Lett. 20, 225 (1974)]). These authors proposed the same method as ours to obtain the spectrum of the 't Hooft model in the $A_-=0$ gauge (our Sec. II). However, we believe their resulting bound-state equation is incorrect since it lacks the mass-renormalization contributions present in the 't Hooft equation and in ours.

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[1]H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973).

[2]K. G. Wilson, Phys. Rev. D 10, 2445 (1974); A. M. Polyakov (unpublished); J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975); J. Kogut, D. K. Sinclair, and L. Susskind, CLNS Report No. 336, 1976 (unpublished) and references therein; S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D 14, 1627 (1976).

[3]Such as that of the ignorant Monarch of Lineland in E. A. Abbott, *Flatland* (Dover, New York, 1963).

[4]J. Schwinger, Phys. Rev. 128, 2425 (1962).

[5]A. Casher, J. Kogut, and L. Susskind, Phys. Rev. D 10, 732 (1974); L.-F. Li and J. F. Willemsen, *ibid.* 10, 4087 (1974); S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. (N.Y.) 93, 267 (1975).

[6]G. 't Hooft, Nucl. Phys. B72, 461 (1974). See also G. P. Canning, Phys. Rev. D 12, 2505 (1976).

[7]G. 't Hooft, Nucl. Phys. B75, 461 (1974).

[8]C. G. Callan, N. Coote, and D. J. Gross, Phys. Rev. D 13, 1649 (1976); M. B. Einhorn, preceeding paper, *ibid.* 15, 3451 (1976).

[9]W. A. Bardeen and R. B. Pearson, Phys. Rev. D 14, 547 (1976).

[10]S.-H. H. Tye, Phys. Rev. D 13, 3416 (1976).

[11]I. Bars, Yale Report No. COO-3075-142, 1976 (unpublished) and references therein.

[12]H. Leutwyler, J. R. Klauder, and L. Streit, Nuovo Cimento 47, 536 (1970).

[13]Y. Frishman, C. T. Sachrajda, H. Abarbanel, and R. Blankenbecler, SLAC Report No. SLAC-PUB-1808 (unpublished). We thank Chris Sachrajda for discussing this work with us before publication.

[14]S. Weinberg, Phys. Rev. 150, 1313 (1966).

[15]F. Rohrlich, Acta Phys. Austriaca Suppl. VIII, 277 (1971).

[16]B. Zumino, Phys. Lett. 10B, 224 (1964); L. S. Brown, Nuovo Cimento 29, 617 (1960). See also C. R. Hagen, Nucl. Phys. B95, 477 (1975).

[17]H. Bardakci and M. B. Halpern, Phys. Rev. D 3, 2493 (1971); H. C. Tze, *ibid.* 7, 1675 (1973).

[18]D. C. Mattis and E. Lieb, J. Math. Phys. 6, 304 (1965).

[19]P. A. M. Dirac, Phys. Rev. 73, 1092 (1948).

[20]A. J. Hanson, R. D. Peccei, and M. K. Prasad, SLAC Report No. SLAC-PUB-1816 (unpublished).

[21]P. Federbush, Phys. Rev. 121, 1247 (1961); and A. S. Wightman, *Cargese Lectures 1964* (Gordon and Breach, New York, 1964).

[22]S. Coleman, Ann. Phys. (N.Y.) 101, 239 (1976).

[23]P. L. F. Haberler, Acta Phys. Austriaca 25, 350 (1967).

[24]J. Lowenstein and A. Swieca, Ann. Phys. (N.Y.) 68, 172 (1971); and K. D. Rothe and J. A. Swieca, Phys. Rev. D (to be published).

[25]M. B. Halpern, Phys. Rev. D 12, 1684 (1975).

[26]T. Banks, D. Horn, and H. Neuberger, Nucl. Phys. B108, 119 (1976); H. Neuberger, Tel-Aviv University report, 1976 (unpublished).