Confinement, form factors, and deep-inelastic scattering in two-dimensional quantum chromodynamics

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We clarify the gauge invariance, infrared structure, and completeness of 't Hooft's solution for the meson sector of two-dimensional quantum chromodynamics. Electromagnetic form factors of mesons are then shown to obey an asymptotic power law, whose power is *dynamically* determined and is not related to the short-distance behavior of the theory. Following a review of the total annihilation cross section for producing hadrons, we discuss deep-inelastic lepton scattering. As expected, Bjorken scaling is obtained, but we show how the sum over hadronic final states reproduces the parton model precisely. The Drell-Yan-West relation and Bloom-Gilman duality are fulfilled for the relation between the scaling function and form factors. We conclude by speculating on the applicability of our picture of form factors to the real, four-dimensional world. We argue that this is a viable alternative to dimensional scaling and, phenomenologically, the differences between our predictions and the dimensional counting rules are slight for light quarks. Finally, we attempt to abstract those features of the model which may guide us toward a solution to the four-dimensional problem.

INTRODUCTION

Quantum chromodynamics,[1] the theory of colored quarks interacting via colored gluons in a locally gauge-invariant manner, is a promising candidate for the theory of strong interactions. Its consistency (up to logarithms) with free-field short-distance structure having been established,[2] an army of theorists are attempting to discover what are its spectrum and long-range forces. It is hoped that the ill-defined infrared structure will somehow lead to a confining potential between quarks, antiquarks, and gluons in color-singlet, gauge-invariant channels, so that the color-singlet bound states (hadrons) may form a complete set of asymptotic states, and neither quarks nor gluons will ever appear as physical states. It seems certain that some approximation scheme must be found which renders the infrared behavior tractable.

One promising idea[3] is to consider an expansion in N, the number of colors, viz. $N\to\infty$ for fixed g^2N (g the coupling constant of QCD). The expansion is suggestive, for it is in one-to-one correspondence to the dual perturbation theory.

So far, little further progress has been made in this approach in four-dimensional space-time, but 't Hooft has pointed out[4] that in two dimensions, the structure of the theory *can* be ascertained. Although in four dimensions no one yet knows whether a confining force arises, in two dimensions it obviously does. Even in lowest order, the exchange of a massless gluon corresponds to a linearly rising attractive potential between a quark-antiquark pair in a singlet channel. By restricting our attention to two dimensions, we remove any possibility of understanding the confinement mechanism in four. Nevertheless, the two-dimensional model remains of interest as a nontrivial, solvable quantum field theory of confinement. Since we are not so familiar with such theories, we may employ the model as a theorectical laboratory illustrating the phenomena conjectured to occur in the four-dimensional theory.

Recently the model has been further elaborated[5,6] and such questions as hadron scattering, unitarity, short-distance behavior, higher-order corrections in $1/N$, and "charmonium[7]" have been discussed. The purposes of this paper are twofold: First, we clarify further some of the mathematical features of the model, paying attention to questions of the infrared cutoff, gauge invariance, completeness, and the breakdown of the cluster decomposition. Secondly, we calculate mesonic form factors with particular interest in their asymptotic behavior. Thirdly, we investigate deep-inelastic lepton scattering in the Bjorken limit. Since our theory is asymptotically free, we must find scaling, but we are interested in how this is reproduced by the hadronic final states. From their inception, parton models[8] have been vague about hadronic final states, maintaining an uneasy coexistence between Bjorken scaling and the absence of quark final states. Since the model discussed here is precise and simultaneously satisfies scaling and quark confinement, many of these worries can be laid to rest. We shall see how previous attempts to marry Feynman's qualitative description with field theory fail and we shall indicate the manner in which a new parton model may be formulated to be consistent with our experience here. Such a reformulation is potentially of great importance phenomenologically, since one could begin to discuss the subject of hadronic final states in the deep-inelastic region with some confidence.

The outline of this paper is as follows: In Sec. I, we take up the mathematical structure of the model. We will argue, in particular, that 't Hooft's cutoff parameter λ can be thought of as a gauge parameter (as $\lambda \to 0$) and has nothing really to do with confinement. We show that the bound-state wave function $\phi_n(x)$ is gauge invariant. We discuss completeness and crossing symmetry. Finally, we derive a useful scaling relation obeyed by the bound-state wave functions. In Sec. II, we discuss the quark "form factor" for an arbitrary local source. In Sec. III, we review $e^-e^+ \to$ hadrons[6] as well as the production from a scalar source. In Sec. IV, we discuss hadron electromagnetic form factors. In this model, the Brodsky-Farrar[9] picture fails; nevertheless, the form factors fall like a power $(q^2)^{-1-\beta}$, where β is a dynamically determined parameter. In Sec. V, we discuss deep-inelastic lepton scattering, $e^-h \to e^-X$, showing that, in the Bjorken limit, the asymptotic behavior is the same as the "handbag" diagram even though there are no quark final states. The Drell-Yan-West[10] relation and Bloom-Gilman[11] duality are satisfied. Finally, in Sec. VI, we summarize the lessons learned, outline future applications of the model, and offer some conjectures about four-dimensional theory. In subsequent papers, we shall discuss other inelastic processes, such as inclusive annihilation, $e^-e^+ \to hX$, and the production of a massive photon, $hh \to e^-e^+X$. Even in this deep-inelastic region, these processes are not simply related to the short-distance structure of the theory, and it is interesting to determine whether, in this theory, they agree with the parton-model predictions.

I. MATHEMATICAL STRUCTURE OF THE MODEL

In this section, we discuss the mathematical structure of the model. We shall assume that the reader is already familiar with Refs. 4 and 6, and, except as noted, we shall generally follow the notation of these authors. The following discussion will serve to remind the reader of properties of the theory. QCD is defined by the SU(N) locally gauge-invariant Lagrangian

$$\mathcal{L} = \tfrac{1}{4} G_{\mu\nu,}{}_i^j G^{\mu\nu,}{}_j^i + \bar{q}^{a,i}(i\not{D}_i^j - m_a \delta_i^j) q_j^a ,$$

where

$$G_{\mu\nu,}{}_i^j = \partial_\mu A_{\nu,}{}_i^j - \partial_\nu A_{\mu,}{}_i^j + g[A_\mu, A_\nu]_i^j ,$$

$$D_{\mu,}{}_i^j = \partial_\mu \delta_i^j + g\tilde{A}_{\mu,}{}_i^j .$$

Here q_j^a is the quark field corresponding to color i ($i=1,\ldots,N$) and flavor a, and $\tilde{A}_{\mu,}{}_i^j$ is the anti-Hermitian, traceless gluon field.[12] It is related to the auxiliary U(N) field A_i^j by $A_i^j = A_i^j - (1/N)\delta_i^j A_k^k$. [The singlet trace, A_k^k, is simply a free field, but it is simpler to write the Feynman rules in terms of the U(N) field.]

In two dimensions, the coupling constant g has the dimensions of mass. The theory is super-renormalizable, with finite mass and coupling-constant renormalizations. Asymptotic freedom is trivial, simply demonstrable by power counting. From the point of view of confinement, however, the essential simplification of two dimensions, however, is that one can choose gauges such that the commutator $[A_\mu, A_\nu]$ vanishes, and, consequently, the self-coupling of the gluons disappears. This makes it possible to understand easily the infrared behavior and to solve the theory nonperturbatively. The theory appears to be simplest when viewed in the infinite-momentum frame. Correspondingly, we introduce "light-cone" coordinates and Dirac matrices (the metric tensor is $g_{+-} = g_{-+} = 1, g_{++} = g_{--} = 0$):

$$p^{\pm} = p_{\mp} = \frac{1}{\sqrt{2}} (p^0 \pm p^1) ,$$

$$p \cdot q = p_- q_+ + p_+ q_- ,$$

$$\gamma_+{}^2 = \gamma_-{}^2 = 0 , \quad \{\gamma_+, \gamma_-\} = 2 , \quad [\gamma_+, \gamma_-] \equiv 2\gamma_5 .$$

We choose the gauge $A_- = 0$ and quantize on a null plane (line). We will think of x_- as the "time" and x_+ as the "space" coordinate. The two components of the quark field $q_{R,L} = \frac{1}{2}(1 \pm \gamma_5) q$ are not independent dynamical variables, so one must modify the canonical quantization procedure.[13] One chooses

$$\{q_R(x), q_R^\dagger(y)\} = \delta(x_+ - y_+) \frac{1+\gamma_5}{2} . \tag{1}$$

Then

$$q_L = \frac{-im}{2} \frac{\gamma_-}{\partial_-} q_R . \tag{2}$$

As in Ref. 13, we define the integral operator ∂_-^{-1} to be the principal value

$$\frac{1}{\partial_-} = \tfrac{1}{2} \int d^2y\, \epsilon(x_+ - y_+) \delta(x_- - y_-) , \tag{3a}$$

or in momentum space

$$\frac{1}{k_-} \equiv \frac{1}{2} \left(\frac{1}{k_- + i\epsilon} + \frac{1}{k_- - i\epsilon} \right) . \tag{3b}$$

A. Infrared structure

Another nondynamical equation of motion is

$$\partial_-{}^2 A_+ = - J_- = - \sqrt{2}\, g q_R^\dagger q_R . \tag{4}$$

The general solution for a point source $\delta^2(x)$ is

$$A_+ = \delta(x_-)\left(\tfrac{1}{2}\,|x_+| + Bx_+ - A\right). \qquad (5)$$

The first term is a linear potential, the Coulomb potential in two dimensions. This is responsible for the binding force between qq pairs in singlet channels. The second term corresponds to a constant background colored "electric" field. Without loss of generality,[14] we may take $B=0$. The constant term A contains no physics and can be gauged away, even within the class of gauges having $A_-=0$. So, without loss of generality, we may also choose $A=0$ if we wish.[15] So we may take

$$\partial_-^{-2} = \tfrac{1}{2}\int d^2y\, |x_+ - y_+|\,\delta(x_- - y_-)\,, \qquad (6a)$$

or in momentum space

$$\frac{1}{k_-^2} \equiv \frac{1}{2}\left[\frac{1}{(k_- - i\epsilon)^2} + \frac{1}{(k_- + i\epsilon)^2}\right], \qquad (6b)$$

a definition which has been called[6] the "regular cutoff" of the infrared singularity at $k_-=0$. A different procedure was employed by 't Hooft, which has led to some confusion of interpretation. Faced with ill-defined momentum integrals, he simply cut away the small momenta, $|k_-|>\lambda$. In coordinate space, this looks rather complicated,[16] but as $\lambda\to 0$, we find

$$\partial_-^{-2} \approx \frac{1}{2}\left(|x_+| - \frac{2}{\pi\lambda}\right)\delta(x_-)\,. \qquad (7)$$

Thus, as $\lambda\to 0$, we may think of λ^{-1} as the *gauge parameter* A. Consequently, we are assured that, as $\lambda\to 0$, it will simply cancel out of any gauge-invariant quantity. This is why, as we shall discuss further below, 't Hooft found that the terms dependent on λ cancel out of the hadronic bound-state equation and why, in Ref. 6, all gauge-invariant quantities are finite and independent of λ as $\lambda\to 0$. It is *not* so much a miraculous cancellation of infrared divergences which is operating here as it is simply an expression of gauge invariance.[17] Instead, as we shall elaborate below, the dependence on λ has nothing whatever to do with the confinement mechanism. Except when otherwise stated, we shall employ the regular cutoff in this paper.

In summary then, we obtain the Feynman rules as in Ref. 4, except that the free quark propagator is

$$S_0 = \frac{p_-\gamma_+ + (m^2/2p_-)\gamma_- + m}{p^2 - m^2} \qquad (8)$$

instead of $\tilde{S}_0 = 1/(\not p - m)$.

To leading order in $1/N$, the dressed quark propagator is the sum of "rainbow" graphs and is

FIG. 1. Fermions are produced independently of the presence of the "gauge-dependent" interaction.

given (for arbitrary A) by

$$S = \frac{p_-\gamma_+(m^2/2p_-)\gamma_- + m}{p^2 - m^2 + g^2N/\pi - g^2NA\,|p_-|}\,. \qquad (9)$$

It has been suggested[4] that, because the pole in p_+ moves off to infinity as $A\to\infty$, this is the reason quarks are confined. To see that this is *not* true, let us switch off the Coulomb potential but retain the constant, gauge-dependent term in A.[18] This is equivalent to adding a term to the free-quark Hamiltonian of the form $A\sum_{i,j} Q_{-j}^{\,i} Q_{-i}^{\,j}$, where $Q_{-i}^{\,j}$ is the "null-plane" color charge:

$$Q_{-i}^{\,j} = \sqrt{2}\int dx_+ q_R^{j\dagger} q_{iR}\,.$$

The denominator of the dressed quark propagator is simply $p^2 - m^2 - A\,|p_-|$. However, if one considers a gauge invariant such as the two-point function of a color-singlet, local source as $J_\mu = \bar q\gamma_\mu q$, one finds that the interaction between $q\bar q$ pairs precisely cancels the term in the self-energy, so that free quarks of mass m *are* produced (see Fig. 1).

B. Gauge invariance

Next let us discuss the bound-state equation from the point of view of gauge invariance. t' Hooft begins with the proper vertex $\Gamma_n^{a,b}(p,r)$ for finding quark a and antiquark $\bar b$ in an on-mass-shell, bound state n. It satisfies a simple Bethe-Salpeter equation (Fig. 2)

$$\Gamma_n^{a,b}(p,r) = \frac{ig^2N}{\pi^2}\int \frac{d^2k}{(k_- - p_-)^2}\,\psi_n^{ab}(k,r)\,, \qquad (10)$$

where we have defined the wave function

$$\psi_n^{ab}(p,r) = S_E^a(p)\Gamma_n^{ab}(p,r)S_E^b(p-r)\,. \qquad (11)$$

[To simplify notation, we have suppressed the γ matrices. Following Ref. 6, we have defined $\gamma_- S(p)\gamma_- \equiv 2S_E(p)\gamma_-$.] We remark that all the quantities defined above, propagators and vertices, are gauge dependent. Define

$$\phi_n^{ab}(x) = \frac{i}{\pi}\int dp_+\,\psi_n^{ab}(p,r) \qquad (x = p_-/r_-). \qquad (12)$$

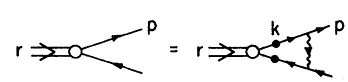

FIG. 2. Bethe-Salpeter equation for mesons.

The variable x may be identified with the momentum fraction carried by the quark in a right-moving infinite-momentum frame. This satisfies the simple equation[4]

$$\mu_n^2\phi_n(x) = H\phi_n(x)\,, \tag{13}$$

where the "Hamiltonian" H is given by $H = H_0 + V$,

$$H_0\phi(x) = \left(\frac{\gamma_a - 1}{x} + \frac{\gamma_b - 1}{1-x}\right)\phi(x)\,, \tag{14a}$$

$$V\phi(x) = -⨍_0^1 \frac{\phi(y)dy}{(y-x)^2} \tag{14b}$$

$[\gamma_a = (\pi/g^2N)m_a^2]$, where ⨍ denotes the principal value defined by Eq. (6b). It has been pointed out by W. Bardeen that the equation may be regarded as a boundary-value problem in potential theory. The argument is presented in Appendix A.

The bound-state equation (13) is, as 't Hooft showed, independent of λ, which we have reinterpreted as the gauge parameter A^{-1}. For this reason, as well as a desire for the bound-state spectrum (mesons) to be physically significant, one suspects that μ_n^2 and $\phi_n(x)$ are gauge invariants. That this is correct may be seen as follows: Assuming there exists a physical meson $|n\rangle$, we *define* a gauge-invariant wave function[19]

$$\Psi_n(x;y) \equiv T\left\langle n\left|\bar\psi(x)\exp\left[g\int_y^x d\xi_\mu A^\mu(\xi)\right]\psi(y)\right|0\right\rangle. \tag{15}$$

[The integration is along the straight path $\xi_\mu = y_\mu + \xi(x_\mu - y_\mu)$, $0 \le \xi \le 1$. We have suppressed color indices, the color-singlet bilocal operator is to be understood. The time-ordering precedes the contraction of the indices.] To obtain a gauge invariant, we must allow for the possibility of an infinity of gluon emissions between x and y. To leading order in $1/N$, the wave function may be depicted as in Fig. 3. What does this look like in the light-cone gauge $A_- = 0$? The exponent is simply

$$(x_- - y_-)\int_0^1 A_+(\xi)d\xi\,.$$

Consequently, for $x_- = y_-$, the exponent vanishes so that the wave function reduces to

$$\Psi_n(x;y) = T\langle n|\bar\psi(x_-,x_+)\psi(x_-,y_+)|0\rangle\,. \tag{16}$$

Consequently, the gauge-variant wave function, $\psi_n(x,y) = T\langle n|\bar\psi(x)\psi(y)|0\rangle$, is, for $x_- = y_-$, equal to the gauge-invariant wave function Ψ_n. In Fourier-transform space, the condition $x_- = y_-$ is realized by integrating over the conjugate momentum p_+. The function $\phi_n(p_-/r_-)$ is simply the Fourier transform

$$\phi_n\left(\frac{p_-}{r_-}\right) \propto \int dx_+\, e^{-ip_-x_+}\Psi_n(0,x_+;0)\,. \tag{17}$$

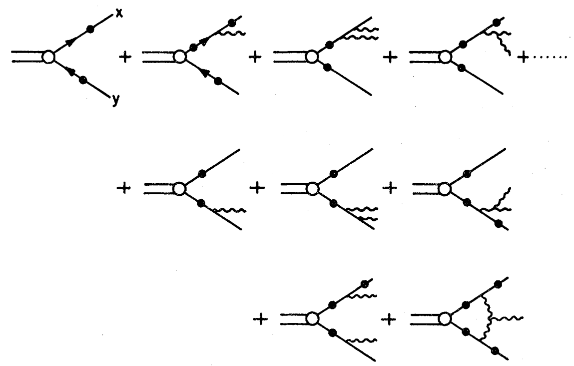

FIG. 3. Diagrammatic form of the gauge-invariant wave function.

In conclusion, $\phi_n(x)$ is a gauge-invariant quantity. In the $A_- = 0$ gauge, it may be interpreted as the probability amplitude to find a quark with momentum fraction x in the right-moving infinite-momentum frame.[20] [Indeed, $|\phi_n(x)|$ turns out to be, in principle, measurable in deep-inelastic lepton scattering.]

Recall that $\phi_n^{ab}(x)$ is nonzero only for $0 \le x \le 1$; in the infinite-momentum frame, the wave function scales only for both constituents $a\bar b$ moving in the same direction as the hadron. However, the proper vertex $\Gamma_n^{ab}(p,r)$ obeys crossing and represents not only the amplitude to find a quark-antiquark pair in the meson, but also the amplitude for a quark or antiquark to emit a meson (Fig. 4). From Eq. (10), we see that Γ_n^{ab} in the regions $x<0$ and $x>1$ is uniquely determined by the values inside the interval $0 \le x \le 1$, since (Fig. 5)

$$\Gamma_n^{ab}(x,r) = \frac{g^2N}{\pi}\int_0^{r_-}\frac{dk_-}{(k_- - p_-)^2}\,\phi_n^{ab}\left(\frac{k_-}{r_-}\right) \equiv \frac{1}{r_-}\Gamma_n^{ab}(x)\,. \tag{18}$$

It is sometimes useful to extend the *definition* of $\phi_n(x)$ outside $(0,1)$ so that

$$\Gamma_n^{ab}(x) = \frac{-g^2N}{\pi}\left(\mu_n^2 - \frac{\gamma_a - 1}{x} - \frac{\gamma_b - 1}{1-x}\right)\phi_n^{ab}(x) \tag{19}$$

holds everywhere.

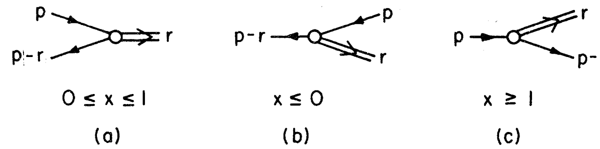

FIG. 4. Three processes related by crossing symmetry: (a) $\bar q q \to h$, (b) $\bar q \to \bar q h$, and (c) $q \to hq$.

C. Confinement

Next, we wish to discuss completeness of the meson spectrum and confinement. Consider the Hilbert space $L^2[0,1]$ of square integrable functions on the interval (0,1) with the usual inner product. The Hamiltonian H is not well defined on this space since, even if $\phi \in L^2$, $H\phi$ may not be. To define H, we restrict its domain to functions which vanish at the boundary like $x^{\beta a}[(1-x)^{\beta b}]$ as $x \to O(1)$. (This set is dense in the Hilbert space.) Then, as 't Hooft showed, H is Hermitian:

$$(\psi, H\phi) = (H\psi, \phi) .$$

Although H is Hermitian, it is not self-adjoint. If we showed that H possesses a self-adjoint extension, then we could conclude that its spectrum is complete. If H were *essentially self-adjoint*,[21] then it would have a unique self-adjoint extension and its spectrum and eigenfunctions would be unique. Since we are examining square-integrable functions in an infinite potential well, it seems likely the spectrum is purely discrete. If so, we could finally conclude that the eigenfunctions $\{\phi_n(x)\}$ were unique and complete on L^2.

Even if all this were proved, this would *not* settle the confinement issue. For example, if one were solving the nonrelativistic hydrogen atom and restricted oneself to square integrable functions, one would find the bound-state solutions but would miss the ionization states, which are not normalizable. In our case, since the meson spectrum extends to infinity, we suspect that a meson cannot ionize to a quark-antiquark pair, but it is worth asking how the infrared behavior prevents this. Let us write the bound-state equation as

$$\phi_n = \frac{1}{\mu_n^2 - H_0} V\phi_n .$$

Normally what would happen is that at those momenta $x_\pm$ where the quarks would go on-shell [where $\mu_n^2 = (\gamma_a - 1)/x_\pm + (\gamma_b - 1)/(1-x_\pm)$], the energy denominator would vanish, so that $\phi_n(x)$ would develop a pole, corresponding to the decay. Confinement requires that $\Gamma_n(x_\pm, r) = V\phi_n = 0$ at those two values of x. In fact, it is easy to see that this must be the case, since if we supposed ϕ_n had a pole at some value x_0, then $V\phi_n$ would be even more singular there. More generally, self-consistency of Eq. (13) requires that $\phi_n(x)$ not be singular in (0,1). Consequently $\Gamma_n(x)$ must vanish at least linearly at $x_\pm$.

Note that if the potential were cut off, e.g., in a Yukawa fashion so that

$$V'\phi = -\int \frac{\phi(y)dy}{(y-x)^2 + \kappa^2} ,$$

then $V'\phi_n$ would remain finite even if ϕ had a pole

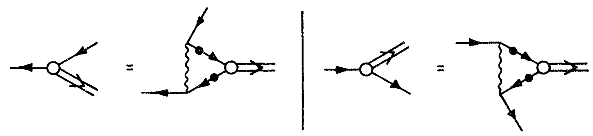

FIG. 5. Hadron emission is determined by the probability amplitude $\phi_n(x)$ to find constituents.

and, as expected, μ_n^2 would become complex, so there would be no bound states at all. The confinement condition

$$\int_0^1 \frac{dy}{(y-x_\pm)^2}\, \phi_n(x) = 0$$

is an amusing property of these wave functions.

To summarize these results, we find that, in the $N \to \infty$ limit, the color-singlet meson sector contains an infinity of stable meson states, i.e., in first approximation, this is a free particle theory.

The quark-antiquark continuum has disappeared in this nonperturbative approximation to the field theory and, it is hoped, will not appear in higher order in $1/N$. This seems likely, unless polarization effects somehow cancel the Coulomb potential.

D. A scaling relation for the mesons

In preparation for later applications, we wish to point out a simple scaling law for the wave functions. Consider $\phi_n^{ab}(x)$ and let $x = \xi/\mu_n^2$. Then it can easily be shown from the bound-state equation that[22]

$$\lim_{n\to\infty} \phi_n^{ab}\left(\frac{\xi}{\mu_n^2}\right) = \phi^a(\xi) , \tag{20}$$

where the scaling function obeys the equation (for $\xi > 0$)

$$\left(1 - \frac{\gamma_a - 1}{\xi}\right)\phi^a(\xi) = -\,⨍_0^\infty \frac{d\eta\, \phi^a(\eta)}{(\eta-\xi)^2} . \tag{21}$$

Some properties of the scaling function are

$$\phi^a(\xi) \to \xi^{\beta a} \quad \text{as } \xi \to 0 , \tag{22a}$$

$$\phi^a(\xi) \to \sin(\xi/\pi) \quad \text{as } \xi \to \infty . \tag{22b}$$

Similarly, one may define

$$\phi^b(\xi) = \lim_{n\to\infty} (-)^n \phi_n^{ab}\left(1 - \frac{\xi}{\mu_n^2}\right) . \tag{23}$$

Another useful property for future work is derivable from the parity relation,[6]

$$m_a \int_0^1 \frac{dx}{x}\, \phi_n(x) = m_b \int_0^1 \frac{dx}{1-x}\, \phi_n(x) . \tag{24}$$

Then

$$m_a \int_0^\infty \frac{d\xi}{\xi}\, \phi^a(\xi) = m_b \int_0^\infty \frac{d\xi\, \phi^b(\xi)}{\xi}\,. \tag{25}$$

Since the left-hand side depends only on m_a, while the right-hand side depends only on m_b, they must be independent of the quark mass. In Sec. III, we will show that agreement with the short-distance behavior of the theory requires that

$$\frac{(2\gamma_a)^{1/2}}{\pi} \int_0^\infty \frac{d\xi}{\xi}\, \phi^a(\xi) = 1\,, \tag{26}$$

but have not found a direct demonstration of the remarkable relation. Another relation which follows from this is

$$\lim_{n\to\infty} \mu_n^2 \int_0^1 dx\, \phi_n(x) = \frac{\pi}{\sqrt{2}}(\sqrt{\gamma_a} \mp \sqrt{\gamma_b})\,. \tag{27}$$

[Choose − (+) for even (odd) parity.]

Although the integral

$$\int_0^\infty d\xi\, \phi^a(\xi)\,.$$

formally diverges, we may *define* it from the integral equation to be

$$\int_0^\infty d\xi\, \phi^a(\xi) = \gamma_a \int^\infty \frac{d\xi}{\xi}\, \phi^a(\xi) = \pi\left(\frac{\gamma_a}{2}\right)^{1/2}. \tag{28}$$

These relations will also be very useful in later applications.

Define the Green's function $G(x,y;\mu^2)$ by

$$(\mu^2 - H)G(x,y;\mu^2) = \delta(x-y)\,, \quad x,y \in (0,1)\,. \tag{29}$$

Of course, the solution to this equation is

$$G(x,y;\mu^2) = \sum_n \frac{\phi_n^{ab}(x)\phi_n^{ab}(y)}{\mu^2 - \mu_n^2 + i\epsilon}\,. \tag{30}$$

Now consider $G(x;\mu^2) \equiv \int_0^1 dy\, G(x,y;\mu^2)$, satisfying

$$(\mu^2 - H)G(x;\mu^2) = 1 \quad (0<x<1)\,. \tag{31}$$

[It is sufficient for subsequent applications to suppose $m_a = m_b$ here, so $G(x;\mu^2) = G(1-x;\mu^2)$.] Suppose $\mu^2 > 0$ and let $x = \xi/\mu^2$. Then one can show that

$$\lim_{\mu^2\to+\infty} \mu^2 G\left(\frac{\xi}{\mu^2};\mu^2\right) \equiv h_+^a(\xi)\,, \tag{32}$$

where the scaling function $h_+{}^a$ satisfies

$$\left(1 - \frac{\gamma_a - 1}{\xi}\right) h_+^a(\xi) + \int_0^\infty \frac{d\eta}{(\eta-\xi)^2} h_+^a(\eta) = 1\,. \tag{33}$$

$h_+^a(\xi)$ has both a real and imaginary part, although the imaginary part is rather ill-defined because ImG is a series of δ functions. Using the smoothing procedure discussed more fully in Sec. III, we find a simple relation between $h_+{}^a$ and ϕ^a:

$$h_+^a(\xi) = \frac{(2\gamma_a)^{1/2}}{\pi}\, \xi \int_0^\infty \frac{du\, \phi^a(u)}{u(\xi - u + i\epsilon)} = 1 - \frac{(2\gamma_a)^{1/2}}{\pi} \int_0^\infty \frac{du\, \phi^a(u)}{u - \xi - i\epsilon}\,. \tag{34}$$

Consequently, $h_+^a(\xi)$ is the boundary value of an analytic function $h^a(\xi)$, analytic in the complex plane cut along the positive real axis. Note

$$\mathrm{Im} h_+^a(\xi) = -(2\gamma_a)^{1/2}\phi^a(\xi)\,. \tag{35}$$

Some other useful properties are

$$h^a(\xi) \sim \xi^{\beta a} \quad \text{as } \xi \to 0\,, \tag{36a}$$

$$h^a(\xi) \sim 1 + \gamma_a/\xi \quad \text{as } \xi \to \infty \text{ (except along the positive real axis)}. \tag{36b}$$

For a massless quark ($m_a = 0$), $h^a(\xi) = 1$ is the solution. We may also define a spacelike scaling function

$$h_-^a(\xi) = \lim_{\mu^2\to-\infty} \mu^2 G\left(\frac{\xi}{-\mu^2};\mu^2\right) \quad (\xi > 0)\,. \tag{37a}$$

One can show that

$$h_-^a(\xi) = h^a(-\xi)\,. \tag{37b}$$

More generally, if we take $\mu^2 \to \infty$ along any ray, we may define

$$h^a(\xi) = \lim_{\mu^2\to\infty} \mu^2 G\left(\frac{\xi}{\mu^2};\mu^2\right),$$

where ξ is assigned the phase of μ^2.

Although it is unnecessary for the applications in this paper, one can also show that the Green's function itself scales:

$$\lim_{\mu^2\to\infty} G\left(\frac{\xi}{\mu^2}, \frac{\xi'}{\mu^2};\mu^2\right) = h_+(\xi,\xi') \tag{38}$$

where the scaling function is given by

$$h_+(\xi,\xi') = \frac{1}{\pi^2} \int_0^\infty \frac{d\lambda}{\lambda - 1 + i\epsilon}\, \phi^a(\lambda\xi)\phi^a(\lambda\xi')\,. \tag{39}$$

E. Cluster properties

In a theory of confinement for which, by definition, there are no asymptotic physical states associated with the basic fields but only with composite fields, the cluster decomposition of Green's functions must break down. Let us consider, for example, the "T matrix" for quark-antiquark scattering in a color-singlet channel. This was derived in Ref. 6 and is given by (Fig. 6)

$$T(x,y;r) = \frac{g^2}{r_-^2}\frac{1}{(x-y)^2} - \frac{\pi}{Nr_-^2}\left(\frac{\pi}{g^2 N}\right) \sum_n \frac{1}{\mu^2 - \mu_n^2}\, \Gamma_n^{ab}(x)\Gamma_n^{ab}(y)\,, \tag{40}$$

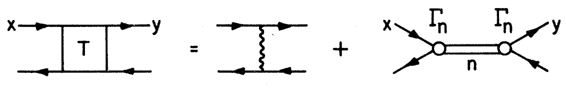

FIG. 6. $q\bar{q}$ scattering in the color-singlet channel.

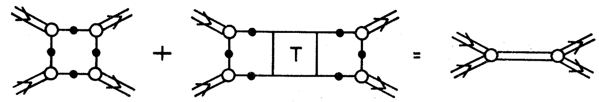

FIG. 7. Hadron-hadron scattering.

where $\Gamma_n(x)$ is the proper vertex function defined earlier, Eq. (18), $x=p_-/r_-$, $y=p'_-/r_-$, and $\mu^2=\pi r^2/g^2N$. For $\mu^2\to-\infty$ for fixed x, this is a light-cone limit for which the Born series converges, each term of which is suppressed by $g^2N/\pi r^2=1/\mu^2$ with respect to the preceding term. However, for $\mu^2\to+\infty$, the timelike region, the Born series diverges. The imaginary part of T is given by

$$\mathrm{Im}T=\frac{\pi^2}{r_-^{\,2}N}\left(\frac{\pi}{g^2N}\right)\sum_n \delta(\mu^2-\mu_n^{\,2})\Gamma_n^{ab}(x)\Gamma_n^{ab}(y)\,, \tag{41}$$

and is either zero or infinite. This divergence of the perturbation series results in the formation of bound states. For the purpose of discussing unitarity, it is useful to rewrite the series as follows:

For $x,y\in(0,1)$, we may use the completeness relation and bound-state equation to manipulate T into the form

$$\begin{aligned}T=&\frac{g^2}{r_-^{\,2}}\left(\mu^2-\frac{\gamma_a-1}{x}-\frac{\gamma_b-1}{x}\right)\delta(x-y)\\&-\frac{g^2}{r_-^{\,2}}\left(\mu^2-\frac{\gamma_a-1}{x}-\frac{\gamma_b-1}{1-x}\right)\\&\times\left(\mu^2-\frac{\gamma_a-1}{y}-\frac{\gamma_b-1}{1-y}\right)G(x,y;\mu^2)\,.\end{aligned} \tag{42}$$

The appearance of the δ function illustrates the breakdown of the cluster decomposition. In any calculation involving $q\bar{q}$ scattering for $x,y\in(0,1)$, the first term cancels the disconnected diagram[23] (see, e.g., Fig. 7). The second term contains two inverse energy denominators, precisely as required to cancel quark singularities, leaving only a sum over meson states. Although Eq. (42) is less convenient than the Born series for obtaining the asymptotic behavior, it is much more suitable to a discussion of unitarity. [We remark that, for either x or y outside the region $(0,1)$, the δ function does not emerge and the disconnected diagram appears to survive.]

One final remark about interpretation: In a gauge theory, the Hilbert space depends on the choice of gauge. The choice $A_-=0$ has the advantage of being ghost-free and manifestly covariant. There is still the freedom to shift A_+ by a constant, and the natural choice of "regular cutoff" [$A=0$ in Eq. (5)] has the advantage of preserving manifest covariance. In this gauge, there *are* quarks of finite mass[24] $(m_a^{\,2}-g^2N/\pi)^{1/2}$. Choosing a gauge with $A\neq0$, although somewhat more difficult to interpret, has the great computational advantage that, when calculating a gauge-invariant quantity, the dependence on A must cancel out. For some purposes, 't Hooft's choice $A\to\infty$ is convenient; for others, $A\to0$ seems more natural.

This concludes our discussion of the mathematical structure of the model. Let us now turn to applications involving local currents. Hereafter, except as indicated otherwise, we shall choose units so that $g^2N/\pi=1$.

II. QUARK "FORM FACTOR"

Consider the three-point function Γ for an arbitrary local source Δ. (See Fig. 8 for notation.) (For simplicity, we shall restrict ourselves to sources independent of p_+.) It is a simple matter to insert the T matrix and obtain the solution

$$\Gamma_\Delta(x,q)=\Delta(x,q)-\frac{1}{4}\int_0^1\int_0^1\frac{dy\,dy'}{(y-x)^2}\,G(y',y;q^2)\left[\gamma_-\left(\gamma_++\frac{m_a}{y'q_-}\right)\Delta(y',q)\left(\gamma_+-\frac{m_b}{(1-y')q_-}\right)\gamma_-\right], \tag{43}$$

where $x\equiv p_-/q_-$. The second term is obviously proportional to γ_-. If we insert the explicit form for the Green's function, we find

$$\Gamma_\Delta(x,q)=\Delta(x,q)-\sum_n\frac{\Gamma_n(x)}{(q^2-\mu_n^{\,2})}\,\gamma_n^\Delta(q)\,, \tag{44a}$$

where

$$\gamma_n^\Delta=\tfrac{1}{4}\int_0^1 dy'\,\phi_n(y')\left[\gamma_-\left(\gamma_++\frac{m_a}{y'q_-}\right)\Delta(y',q)\left(\gamma_+-\frac{m_b}{(1-y')q_-}\right)\gamma_-\right]. \tag{44b}$$

γ_n^Δ is the direct coupling of the source to the meson; Γ_n, the coupling of the meson to the quarks.

For example, for a scalar source, we have

$$\Gamma_S(x,q) = e - \frac{e\gamma_-}{2q_-}\int \frac{dy\,dy'}{(y-x)^2}\,G(y,y';q^2)\left(\frac{m_a}{y'} - \frac{m_b}{1-y'}\right). \tag{45}$$

For a vector source, we have

$$\Gamma_\mu(p,q) = e\gamma_\mu - e\gamma_-\int\int \frac{dy\,dy'}{(y-x)^2}\,G(y,y',q^2)\left[g_{\mu+} - \frac{m_a m_b}{2y'(1-y')q_-}\,g_{\mu-}\right]. \tag{46}$$

From the bound-state equation and the parity relation Eq. (24), it is easy to show that

$$\mu_n^{\,2}\int_0^1 \phi_n(y')dy' = \mp m_a m_b \int_0^1 \frac{dy'\phi_n(y')}{y'(1-y')}\,.$$

Noting that only odd-parity states contribute to the vector current, we find for the divergence

$$q^\mu \Gamma_\mu(p,q) = e\not{q} + \frac{e\gamma_-}{2q_- x(1-x)}\,.$$

In the theory defined by quantization at equal time, it is easy to show that this is the generalized Ward-Takahashi identity. In the light-cone quantization scheme, it is much more laborious to prove but equally true. We omit the proof here.

For future work, it is convenient to note here a simple formula for $\Gamma_-(p,q) \equiv e\gamma_-\Gamma(x,q)$:

$$\Gamma(x,q) = 1 - \int_0^1 \frac{dy}{(y-x)^2}\,G(y;q) \tag{47a}$$

$$= \left(q^2 - \frac{\gamma_a - 1}{x} - \frac{\gamma_b - 1}{1-x}\right)G(x;q), \quad 0<x<1\,. \tag{47b}$$

The first form is convenient for obtaining the asymptotic behavior, the second, for understanding the unitarity structure. Note that, for $x \in (0,1)$, it is, in a sense, fair to say that the bare coupling to quarks has been cancelled. On the other hand, as $q^2 \to \infty$ for fixed x, $\Gamma \to 1$, so it is the same *as if* the coupling were bare in either case.

If the quarks try to go on the mass shell, the inverse energy denominator vanishes. It is amusing to note that a similar thing happens to prevent two mesons from producing a $q\bar{q}$ pair (Fig. 9). Armed with the decomposition of the T matrix, Eq. (42), it is not difficult to show that this amplitude vanishes if we attempt to go on the quark mass shell. Even though this, and the preceding, are gauge-variant statements, they do accurately reflect the mechanism which, as we shall see below, operates in all gauge-invariant calculations.

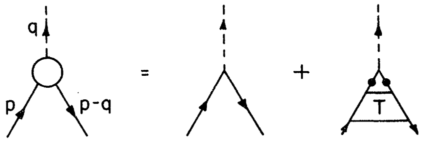

FIG. 8. The quark form factor.

III. TOTAL HADRONIC CROSS SECTIONS

Although this has been discussed in part in Ref. 6, we wish to review it here to establish notation and to emphasize several concepts which are relevant to the discussion of the more complicated processes treated in subsequent sections.

Consider the two-point function for an arbitrary local source[25] Δ. Either using the form factor of the preceding section or the T matrix given earlier, one finds (Fig. 10)

$$\Pi_{\Delta_1\Delta_2}(q) = -\frac{e^2N}{\pi}\int_0^1\int dx\,dy\,g^{\Delta_1}(x,q)G(x,y;q^2) \times g^{\Delta_2}(y,q)\,, \tag{48}$$

where

$$g^\Delta(x,q) \equiv \frac{\mathrm{Tr}(\Delta N_k\gamma - N_{k-q})}{4x(x-1)q_-}\quad \left(x \equiv \frac{k_-}{q_-}\right),$$

and $N_k \equiv k_-\gamma_+ + m^2\gamma_-/2k_- + m$ is the numerator of the quark propagator. We have expressed this in a form which obviously factorizes at the bound-state poles and has no quark singularities. The integral converges at the end points because G vanishes there. Because of this vanishing at the ends, in general, one cannot interchange the limit as $q^2 \to \infty$ with the integration.

If we insert the definition of G, we may write this in the meson-dominated form (see Fig. 11)

$$\Pi_{\Delta_1\Delta_2}(q) = -\frac{e^2N}{\pi}\sum_n \frac{g_n^{\Delta_1}(q)g_n^{\Delta_2}(q)}{q^2 - \mu_n^{\,2}}\,. \tag{49}$$

The coupling g_n^Δ of the bound state to the source is given by[26]

$$g_n^\Delta(q) \equiv \int_0^1 dx\,g^\Delta(x,q)\phi_n(x)\,.$$

Note that the preceding formulas are manifestly gauge invariant.

Consider, for example, a scalar source. Then we find that

$$g^S(x) = \frac{m(1-2x)}{2x(1-x)}\,, \tag{50a}$$

$$g_n^S = \frac{m}{2}\int_0^1 \frac{\phi_n(x)(1-2x)}{x(1-x)}\,dx$$

$$= m\int_0^1 \frac{\phi_n(x)}{x}\,dx \quad \text{(even parity)} \tag{50b}$$

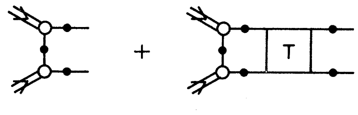

FIG. 9. $hh \to q\bar{q}$.

As $n \to \infty$, g_n^S tends to a constant g_∞. Letting $x = \xi/\mu_n^2$, we have

$$g_n^S \xrightarrow[n\to\infty]{} g_\infty = m \int_0^\infty \frac{d\xi}{\xi}\,\phi_a(\xi)\,. \qquad (51)$$

Since $\mu_n^2 \to n\pi^2$, the sum Eq. (49) diverges (logarithmically) and requires subtraction, which comes as no surprise since the scalar bubble diverges by power counting. The asymptotic behavior may be obtained as follows:

$$\Pi_S(q^2) - \Pi_S(0) = -\frac{e^2N}{\pi}\, q^2 \sum_n \frac{(g_n^S)^2}{\mu_n^2(q^2-\mu_n^2)}\,. \qquad (52)$$

For large q^2, we may write

$$\begin{aligned}\sum_{n=N}^{\infty} \frac{(g_n^S)^2}{\mu_n^2(q^2-\mu_n^2)} &\simeq \frac{1}{\pi^2}\int_\Lambda^\infty \frac{d\mu_n^2 (g_n^S)^2}{\mu_n^2(q^2-\mu_n^2)}\\ &\approx \frac{g_\infty^2}{q^2\pi^2}\int_{\Lambda/-q^2}^\infty \frac{dz}{z(z+1)}\\ &\approx \frac{1}{q^2}\left(\frac{g_\infty}{\pi}\right)^2 \ln(-q^2)\,. \end{aligned} \qquad (53)$$

Hence,

$$\Pi_S(q^2) \sim -\frac{e^2N}{\pi}\left(\frac{g_\infty}{\pi}\right)^2 \ln(-q^2)\,. \qquad (54)$$

Since the theory is asymptotically free, this must agree with the result obtained from the bubble (first term in Fig. 11), which is simply $(-e^2N/2\pi) \times \ln(-q^2)$. Therefore, it must be that

$$\lim_{n\to\infty} \frac{g_n^S}{\pi} = \frac{m}{\pi}\int_0^\infty \frac{d\xi\,\phi_a(\xi)}{\xi} = \frac{1}{\sqrt{2}}\,. \qquad (55)$$

This result is typical of the rather remarkable identities which must be satisfied by the wave functions. Others can be generated by considering nonleading terms in the asymptotic expansion. We have seen in Sec. ID that this integral must be independent of m; unfortunately, a direct proof of the result has eluded us.

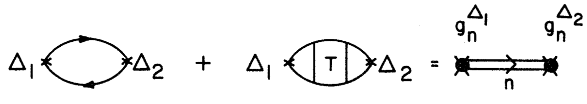

FIG. 10. The total cross section for hadronic production by a local source.

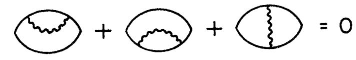

FIG. 11. Corrections to $e^-e^+ \to X$ which cancel to leading order.

As a second example, consider a vector source. Regulating in a way which preserves current conservation, we write[27]

$$\Pi_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi_V(q^2)\,.$$

Π_{--} is simplest to calculate. We find, as in Ref. 6,

$$g_-^V(q) = q_-\,,\quad g_n^V = \int_0^1 dx\,\phi_n(x)\,.$$

Then

$$\begin{aligned}\Pi_V(q^2) &= -\frac{e^2N}{\pi}\int_0^1\!\!\int dx\,dy\,G(x,y;q^2)\\ &= -\frac{e^2N}{\pi}\sum_n \frac{(g_n^V)^2}{q^2-\mu_n^2}\,. \end{aligned} \qquad (56)$$

To discover the asymptotic behavior of g_n^V, note that

$$\begin{aligned}\mu_n^2 g_n^V &= m^2 \int_0^1 \frac{dx\,\phi_n(x)}{x(1-x)}\\ &= 2m^2 \int_0^1 \frac{dx\,\phi_n(x)}{x} \equiv 2m g_n^P \quad \text{(odd parity)}\,. \end{aligned} \qquad (57)$$

(g_n^P can be shown to be the coupling to a pseudoscalar source. As $n\to\infty$, $g_n^P \to g_\infty$.) Hence

$$g_n^V \to \frac{2mg_\infty}{\mu_n^2}\,. \qquad (58)$$

Writing

$$\Pi_V(q^2) = -\frac{e^2N}{\pi q^2}\left[\sum_n (g_n^V)^2 + \sum_n \frac{\mu_n^2 (g_n^V)^2}{q^2-\mu_n^2}\right], \qquad (59)$$

$\sum_n (g_n^V)^2 = 1$ by completeness.[6] Similar to the scalar case, the asymptotic behavior of the second term on the right-hand side must be treated with care since we cannot interchange the limit $q^2\to\infty$ with the sum. We find

$$\Pi_V(q^2) = -\frac{e^2N}{\pi q^2}\left[1 + \left(2m\frac{g_\infty}{\pi}\right)^2 \frac{\ln(-q^2)}{q^2} + \cdots\right]. \qquad (60)$$

A direct calculation of the vector bubble agrees with the above formula, provided $2g_\infty^2/\pi^2 = 1$, the same condition we obtained for agreement in the scalar case. One might have expected terms of order $g^2N/\pi q^2$ to appear from the next terms in the perturbation expansion (Fig. 11). However, as $m\to 0$, these terms must vanish for a vector source, so the coupling constant can only enter in next order as $g^2Nm^2/\pi q^4$.

To summarize the asymptotic behavior, we find that, as expected by superrenormalizability, the limit as $q^2 \to \infty$ (in any direction in the complex plane other than positive q^2) may be calculated perturbatively from graphs involving quarks and gluons. However, for $q^2 > 0$, the timelike region, the perturbation series diverges and the theory must be solved nonperturbatively. For example, for the scalar source,

$$\mathrm{Im}\Pi_S(q^2) = e^2 N \sum_n (g_n^S)^2 \delta(q^2 - \mu_n^2) . \quad (61)$$

Thus the imaginary part is either zero or infinite. However, recalling Eq. (54), we expect in an average sense

$$e^2 N\left(\frac{g_\infty}{\pi}\right)^2 \approx \lim_{q^2 \to \infty} \sum_n (g_n^S)^2 \delta(q^2 - \mu_n^2). \quad (62)$$

To demonstrate this directly, some sort of smoothing procedure must be applied such as averaging over several resonances or averaging the values[28] of $\Pi_S(q^2)$ at $q^2 = \mu_n^2 \pm i\Delta$. The simplest way is to interpolate, using the fact that $\mu_n^2 \to n\pi^2$. Then

$$\begin{aligned}\lim_{q^2 \to \infty} \sum_n (g_n^S)^2 \delta(q^2 - \mu_n^2) &\simeq \lim_{q^2 \to \infty} \int_N^\infty dn (g_n^S)^2 \delta(q^2 - n\pi^2) \\ &\simeq \lim_{q^2 \to \infty} \left(\frac{g_n^S(q^2)}{\pi}\right)^2 \\ &= \left(\frac{g_\infty}{\pi}\right)^2 , \end{aligned} \quad (63)$$

where $n(q^2) = q^2/\pi^2$. It has often been remarked[29] that, in two dimensions, the scalar source is more nearly analogous to $e^- e^+ \to$ hadrons in four dimensions, since a scalar current can create a $q\bar{q}$ pair of massless quarks while a vector current cannot. (This is the reason that the imaginary part of Π_V is proportional to m^2. From the point of view of the operator-product expansion, this requires calculating nonleading terms in singular functions for the vector case.)

Notice that the exact imaginary part has only hadronic intermediate states. However, when properly smoothed,[30] its asymptotic behavior is the same as obtained from the imaginary part of the bubble graph which comes from quark discontinuities. In fact, when the perturbation theory is summed, all quark discontinuities are precisely and completely cancelled and replaced by hadronic discontinuities. This illustrates the inadequacies of all previous attempts to formulate parton models mathematically. The so-called covariant parton model[31] neglects connected diagrams with respect to disconnected graphs, ignoring the fact that, in any theory of confinement, the cluster decomposition must break down. This formulation

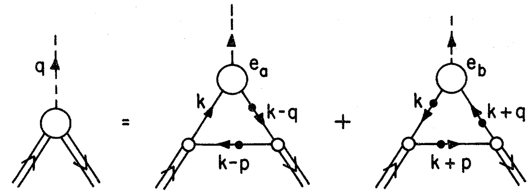

FIG. 12. General form of hadron form factors.

certainly gives scaling but unfortunately also gives quarks. The massive parton model[32] is a step in the right direction except that technically there are many differences. The kinematical limit in which the quark mass becomes infinite bears some resemblance to the $A \to \infty$ gauge, but confinement clearly arises dynamically, and there is really no need for infinitely massive quarks. At this point, it does not seem justified to associate scaling with Pomeron exchange. A somewhat more general formulation[33] comes closest to our experience with this model, although the speculations about the role of duality cannot be investigated in two dimensions.

IV. MESON FORM FACTORS

In this section, we shall discuss the electromagnetic form factors of the meson bound states. Consider the transition form factor from state n to state m (see Fig. 12),

$$\begin{aligned}(F_\mu)_{nm} &= \left(p_\mu + p'_\mu + \frac{(\mu_m^2 - \mu_n^2)}{q^2} q_\mu\right) F_{nm}(q^2) \\ &= \langle n | J_\mu | m \rangle . \end{aligned} \quad (64)$$

Here we have $p^2 = \mu_n^2$, $p'^2 = \mu_m^2$, $q = p - p'$. To leading order in $1/N$, the form factor is given by the diagrams depicted in Fig. 13. From the Ward identity noted earlier, it is a simple matter to show that these diagrams satisfy current conservation. It is simplest to calculate F_-, for which we find[34]

$$\begin{aligned}F_{-\,nm}^{\;a} = \frac{-i2e_a}{\pi} \int d^2k\, &\Gamma_n^{ab}(k,p) S_E^a(k) \Gamma^a(k,q) S_E^a(k-q) \\ &\times \Gamma_m^{ab}(k-q,p') S_E^b(k-p) . \end{aligned} \quad (65)$$

(We have written the contribution from the inter-

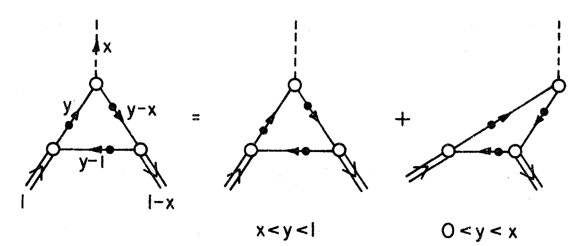

FIG. 13. "Time-ordering" in the infinite-momentum frame [Eq. (66)].

action with the quark a; to this we must add $F_{-}{}^{b}_{nm}$, the coupling to the antiquark $\bar{b}$.) Recall that

$$S_E(p) = \left[2p_+ - \frac{m^2-1}{p_-} - \pi A \epsilon(p_-)\right]^{-1} .$$

Let us now choose a Lorentz frame for which $q_- > 0$. Because the vertices depend only on k_-, we may perform the integration over k_+, giving the two contributions of Fig. 13, corresponding to the two "x_- time-orderings" of the process. Defining[35] $x = q_-/p_-$ ($0<x<1$ for spacelike q^2), we find

$$F_{-}{}^{a}_{nm} = 2e_a p_-{}^3 \left[\int_x^1 dy \frac{\Gamma_n^{ab}(y,p)\Gamma^a\left(\frac{y}{x},q\right)\Gamma_m^{ab}\left(\frac{y-x}{1-x},p'\right)}{\left(\mu_n{}^2 - \frac{\gamma_a-1}{y} - \frac{\gamma_b-1}{1-y}\right)\left(\frac{\mu_m{}^2}{1-x} - \frac{\gamma_a-1}{y-x} - \frac{\gamma_b-1}{1-y}\right)} - \int_0^x dy \frac{\Gamma_n^{ab}(y,p)\Gamma^a\left(\frac{y}{x},q\right)\Gamma_m^{ab}\left(\frac{y-x}{1-x},p'\right)}{\left(\mu_n{}^2 - \frac{\gamma_a-1}{y} - \frac{\gamma_b-1}{1-y}\right)\left(\frac{q^2}{x} - \frac{\gamma_a-1}{y} - \frac{\gamma_a-1}{x-y}\right)} \right] . \quad (66)$$

We recognize the energy denominators corresponding to the two "time" orderings. Indeed, calculations in the $A_-=0$ gauge are most efficiently performed by directly writing down the Feynman rules for old-fashioned perturbation theory in the infinite-momentum frame.[36] Using the bound-state equation and the formulas derived in Sec. II for the quark form factor [Eq. (47)], the equation above may be written as[37] (suppressing flavor indices)

$$F_{-}{}^{a}_{nm} = 2e_a p_- \left[\int_x^1 dy\, \phi_n(y)\Gamma\left(\frac{y}{x},q\right)\phi_m\left(\frac{y-x}{1-x}\right) - x\int_0^x dy\, \phi_n(y) G\left(\frac{y}{x};q^2\right)\left(\frac{\mu_m{}^2}{1-x} - \frac{\gamma_a-1}{y-x} - \frac{\gamma_b-1}{1-y}\right)\phi_m\left(\frac{y-x}{1-x}\right)\right] . \quad (67)$$

In manifestly covariant language, x satisfies the equation[38]

$$\mu_n{}^2 = \frac{q^2}{x} + \frac{\mu_m{}^2}{1-x} .$$

We are especially interested in the asymptotic behavior as $q^2 = -Q^2 \to -\infty$. Suppose we choose a frame where $x \approx 1 - \mu_m{}^2/Q^2$ as $Q^2 \to \infty$. In the first term, we let $z = (y-x)/(1-x)$ and insert the form of the quark form factor from Eq. (47a). After some manipulation, we may bring it into the form

$$F_{-}{}^{a}_{nm} = 2e_a p_-(1-x)\left\{\int_0^1 dz\, \phi_n[x+(1-x)z]\phi_m(z) + x^2\int_0^1 \int du\, dz \frac{\phi_n(xu) - \phi_n(x+(1-x)z)}{[x(1-u)+(1-x)z]^2} G(u;q^2)\phi_m(z)\right\} . \quad (68)$$

The equation may be described graphically as in Fig. 14: The first term is the bare coupling to the quarks; the second and third terms involve mesonic couplings to the photon. A gluon exchange has been extracted so that all quark constituents of mesons have momentum fractions in the interval (0, 1). Note that the extracted gluon cannot be infrared singular (at finite q^2). The expression above is exact; now let us consider its behavior as $x \to 1$. Using $\phi_n(x) \sim c_n(1-x)^{\beta_b}$ as $x \to 1$, we find for the first integral simply

$$c_n(1-x)^{\beta_b} \int_0^1 dz\,(1-z)^{\beta_b}\phi_m(z) ,$$

We cannot simply let $x \to 1$ in the second term because the integral becomes singular. Let $x(1-u) = v(1-x)$. Then we find for the second integral

$$-c_n \frac{(1-x)^{\beta_b}}{\mu_n{}^2} \int_0^1 dz\, \phi_m(z) \int_0^\infty dv \frac{(1+v)^{\beta_b} - (1-z)^{\beta_b}}{(v+z)^2} h_-^a(\mu_m{}^2 v) .$$

Putting it all together, we find for $q^2 \to -\infty$

$$F^a_{nm}(q^2) \approx 2e_a c_n \left(\frac{\mu_m{}^2}{-q^2}\right)^{1+\beta_b} \left[\int_0^1 dz\, \phi_m(z)(1-z)^{\beta_b} - \frac{1}{\mu_m{}^2}\int_0^1 dz\, \phi_m(z)\int_0^\infty dv \frac{(1+v)^{\beta_b} - (1-z)^{\beta_b}}{(v+z)^2} h_-^a(\mu_m{}^2 v)\right] . \quad (69)$$

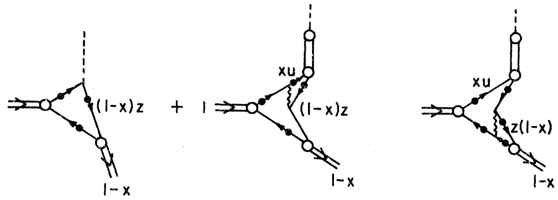

FIG. 14. Diagrammatic description of Eq. (68); all quark constituents of hadrons have momentum fractions in (0, 1).

The form factor is power behaved, but the power is dynamically determined.[39] In particular, the mechanism is not that described by Brodsky and Farrar,[9] a point which we discuss further below. We have not found a simple interpretation for the coefficient of the power; it seems to be complicated. Although the power is the same as one would obtain from a partonlike calculation (neglecting vertex corrections to the bare quark coupling), the coefficient is different. Thus, there does not seem to be any direct relation to the short-distance behavior of the theory. From the mathematics above, it is clear that the form factor probes the probability amplitude that the struck quark a carries all the initial-state momentum, while the antiquark b carries none. This is similar to Feynman's description,[8] although in the model we can go further to say that this amplitude goes as $(1-x)^{\beta_b}$.

The form factor must be an analytic function of q^2 with only a right-hand cut. However, the wave functions are not analytic functions, so it is impossible to see this from our formulas. However, we can calculate the form factor directly for timelike q^2 and check, for example, that the asymptotic behavior agrees with the spacelike limit. So let us consider the "decay" of a virtual photon into two mesons n and m. Defining $\omega = p_-/q_-$, we have the relation $0<\omega<1$

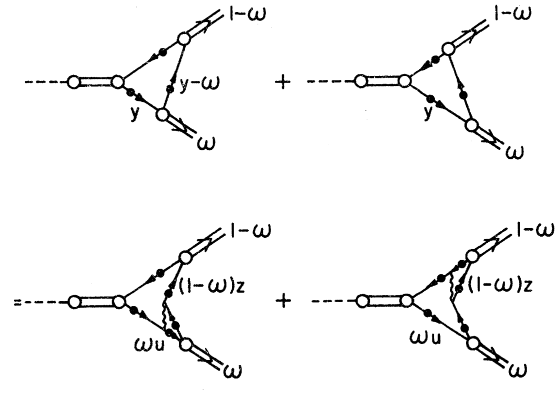

FIG. 15. Two forms for the timelike form factors.

$$q^2 = \frac{\mu_n^2}{\omega} + \frac{\mu_m^2}{1-\omega} . \qquad (70)$$

There are two solutions ω_R and ω_L, corresponding to whether the decay occurs with meson n moving to the right or to the left. Of course, by parity invariance, these two amplitudes must be equal, which implies that, when written in terms of ω, the form factor must be invariant under the substitution

$$\omega \to \frac{1-\omega}{1+\omega(\mu_m^2-\mu_n^2)/\mu_n^2} . \qquad (71)$$

We have not been able to demonstrate this invariance.

Proceeding in a manner similar to the spacelike case, we find (see Fig. 15 for notation)

$$F_{-nm}^{a} = 2e_a q_- \left[\int_\omega^1 dy\, G(y;q^2)\left(\frac{\mu_n^2}{\omega} - \frac{m_a^2-1}{y} - \frac{m_b^2-1}{\omega-y}\right)\phi_n\!\left(\frac{y}{\omega}\right)\phi_m\!\left(\frac{y-\omega}{1-\omega}\right) - \int_0^\omega dy\, G(y;q^2)\left(\frac{\mu_m^2}{1-\omega} - \frac{m_a^2-1}{1-y} - \frac{m_b^2-1}{y-\omega}\right)\phi_m\!\left(\frac{y-\omega}{1-\omega}\right)\phi_n\!\left(\frac{y}{\omega}\right)\right]. \qquad (72)$$

Notice that the coupling of the photon is entirely meson-dominated. Changing variables and using the bound-state equation leads to

$$F_{-nm}^{a} = 2e_a q_- \omega(1-\omega)\int_0^1\int_0^1 du\,dz\, \frac{G(\omega u;q^2) - G(\omega + z(1-\omega);q^2)}{[z(1-\omega)+\omega(1-u)]^2}\,\phi_n(u)\phi_m(z) . \qquad (73)$$

To obtain the asymptotic behavior as $\omega \to 1$, we let $\omega(1-u) = (1-\omega)v$:

$$F_{nm}^{a}(q^2) \approx 2e_a c_n \left(\frac{\mu_m^2}{q^2}\right)^{1+\beta_b} \left\{\frac{1}{\mu_m^2}\int_0^1 dz\,\phi_m(z)\int_0^\infty \frac{dv\, v^{\beta_b}}{(z+v)^2}\left[h_+^a(\mu_m^2(1+v)) - h_+^a(\mu_m^2(1-z))\right]\right\} . \qquad (74)$$

It is reassuring to find the same power as for the spacelike asymptotic behavior.[40]

Inserting the explicit sum over states for the Green's function, the form of F_{nm}^{a} is, from Eqs. (64) and (72),

$$F_{nm}^{a} = e_a \sum_k \frac{g_k^V \gamma_{knm}(q^2)}{\mu_k^2 - q^2 - i\epsilon} , \qquad (75)$$

with an obvious definition for $\gamma_{knm}(q^2)$. It is important to note that, although real, γ_{knm} is a function of ω, hence, of q^2. However, by analyticity, there can be no singularities in q^2 other than the poles displayed. Therefore, up to an additive polynomial, the function above must be unchanged if we replace $\gamma_{knm}(q^2)$ by $\gamma_{knm}(\mu_k^2)$, the three-meson coupling constant. In fact, because $F_{nm}\to 0$ as $q^2\to\infty$, no polynomial is present. On the contrary, since $q^2F_{nm}\to 0$ as well, we must have the superconvergence relation

$$\sum_k g_k^V\gamma_{knm}(\mu_k^2)=0\ . \tag{76}$$

In summary, the form factor will satisfy a meson-dominated dispersion relation. Because this dispersion relation is unsubtracted, the so-called "bare quark coupling" term which we found convenient to extract on the spacelike region is purely a matter of language and has no counterpart in unitarity.[41]

Having concluded our discussion of the behavior in this model, let us compare it with the picture of form factors put forth by Brodsky and Farrar[9] whose structure closely resembles the one here. Those authors would obtain the asymptotic behavior from Eq. (73) by letting $q^2\to\infty$ ($\omega\to 1$) for fixed constituent momentum fractions u and z:

$$F^a_{nm}(q^2)\sim\frac{2e_a}{(q^2)^{1+\delta}}\left\{\mu_m^2\int_0^1 dz\ \phi_m(z)[1-h_+^a(\mu_m^2(1-z))]\right\}\times\left[\int_0^1\frac{du\ \phi(u)}{(1-u)^{1+\delta}}\right]\ , \tag{77}$$

where we have inserted a parameter δ to facilitate comparison with Ref. 9. (Of course, $\delta=1$ in this superrenormalizable theory.) The first factor involving ϕ_m is the wave function at the origin in coordinate space plus a correction h_+^a because of vertex corrections to the photon coupling. The second factor involving ϕ_n is divergent, because $\phi_n(u)$ does not vanish sufficiently rapidly as $u\to 1$. In more physical terms, this divergence informs us that the large momentum does not flow through the gluon exchanged. Indeed, a glance at the correct calculation, Eq. (74), reveals that the momentum continues to flow with the quark and is taken up by exchanging the infinity of gluons which bind the quarks. On the other hand, for a renormalizable theory ($\delta=0$), the demands on $\phi_n(u)$ are less stringent and the last factor may be finite. In any case, the result depends on the rate at which $\phi_n(x)$ vanishes at $x=1$, which is *not* a property to be inferred from the short-distance structure of the theory. Although $\phi_n(u)$ is determined by the light-cone structure ($x_-=0$), the behavior as $u\to 1$ ($p_-\to r_-$) determines the probability amplitude for finding the antiquark at large distances ($x_+\to\infty$) along the light cone. This last property would seem to carry over to renormalizable theories as well.[42] We shall speculate further on this subject in Sec. VI.

V. DEEP-INELASTIC SCATTERING

We would now like to take up deep-inelastic scattering $e^-h\to e^-X$ by discussing the Bjorken limit of virtual Compton scattering. Because the theory is superrenormalizable, one expects to find exact scaling, and this will be borne out by explicit calculations. Our interests here are two-fold: (1) How does it work that the model scales and, without creating any quark-antiquark pairs by polarizing the vacuum, manages to have only hadronic final states? (2) The short-distance expansion and the light-cone behavior are simply related by the so-called moment sum rules for the structure functions. One might be concerned that infrared singularities might somehow destroy this relation. Since this worry will be settled in favor of the naive expectations, we relegate to Appendix B a brief discussion of these issues.[43]

Diagrams contributing to the imaginary part of the virtual Compton amplitude are shown in Fig. 16. Confident that quark singularities cancel as usual, the only final states which will contribute to leading order in $1/N$ are the single-meson states (Fig. 17). Thus to calculate the imaginary part of Compton scattering, we simply need to square the form factors. To be specific, define the structure function W as

$$W_{\mu\nu}=\left(p_\mu-\frac{q_\mu q\cdot p}{q^2}\right)\left(p_\nu-\frac{q_\nu q\cdot p}{q^2}\right)\frac{1}{\mu_n^2}W(q^2,\nu)\ , \tag{78}$$

where as usual, we define $p\cdot q=-\mu_n\nu$. Then, e.g.,

$$W_{--}=(2\pi)^2\sum_m|F_{-\,nm}^{\,a}(q^2)+F_{-\,nm}^{\,b}(q^2)|^2\times\delta((p-q)^2-\mu_m^2)\ . \tag{79}$$

We are interested in the asymptotic behavior $q^2\to-\infty$ for fixed $x_{\rm Bj}\equiv q^2/2p\cdot q$. Of course, we must somehow smooth the δ functions if we are to calculate directly from the formula above. We proceed as in the discussion of the hadronic total cross section by writing

$$W_{--}\approx 4\lim_{\rm Bj}|F_{-\,nm(\nu)}^{\,a}(q^2)+F_{-\,nm(\nu)}^{\,b}(q^2)|^2\ , \tag{80}$$

where $m(\nu)$ is defined by $(p-q)^2\approx 2\mu_n\nu(1-x_{\rm Bj})=\mu_m^2=m(\nu)\pi^2$. Defining $x=q_-/p_-$ as before, we have, in the Bjorken limit, $x\approx x_{\rm Bj}$. (A simple way to see this is to take $q_+\to-\infty$ for fixed $q_->0$ and fixed p.) Returning to Eq. (68) for the form factor, let us again take its asymptotic behavior but now for fixed x ($\mu_m^2\to\infty$). Since $\phi_m(z)\underset{m\to\infty}{\sim}\sqrt{2}\sin m\pi z$,

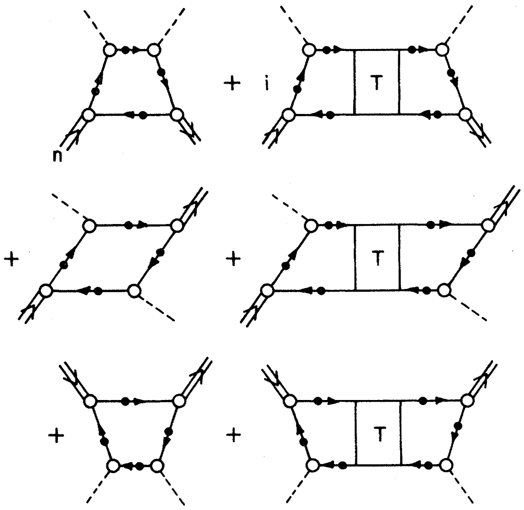

FIG. 16. Contributions to the virtual Compton scattering amplitude.

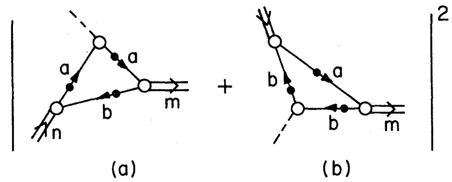

FIG. 17. Inelastic production as the square of form factors.

$$F_{-\,nm}^{\,a} \approx 2e_a p_- \frac{(1-x)}{\mu_n^{\,2}} \times \int_0^{\mu_m^{\,2}} d\xi\, \phi_n\!\left(x + \frac{(1-x)\xi}{\mu_m^{\,2}}\right)\varphi_m\!\left(\frac{\xi}{\mu_m^{\,2}}\right) \approx \frac{2e_a p_- x}{(-q^2)}\, \phi_n(x) \int_0^\infty \phi^a(\xi)\ . \tag{81}$$

Using Eq. (28) for the last integral, we conclude that this term gives

$$\lim_{\text{Bj}} F_{-\,nm}^{\,a}(q^2) \approx p_- \frac{\pi}{\sqrt{2}}\, \frac{e_a m_a x \phi_n(x)}{(-q^2)}\ . \tag{82}$$

This is in fact the exact result, since the other terms in Eq. (68) vanish more rapidly, as we will now argue. For the region $z \approx 0$, the dominant contribution comes from $u \approx 1$. Letting $z = \xi/\mu_m^{\,2}$, $u = 1 - \eta/(-q^2)$, we find that

the integrals are dominated by their behaviors for $z \approx 0, 1$. Consider the first term coming from the bare coupling: The region $z \approx 1$ is suppressed relative to $z \approx 0$ because ϕ_n vanishes there. Letting $z = \xi/\mu_m^{\,2}$, we have

$$\frac{x}{(1-x)q^2} \int_0^{(-q^2)} dy\, g_-(\eta) \int_0^\infty \frac{d\xi\, \varphi^a(\xi)}{(\eta+\xi)^2} \left[\phi_n\!\left(x - \frac{(1-x)\eta}{\mu_m^{\,2}}\right) - \varphi_n\!\left(x + \frac{(1-x)\xi}{\mu_m^{\,2}}\right)\right] \approx \frac{x}{(-q^2)\mu_m^{\,2}}\, \phi_n'(x) \int_0^{-q^2} d\eta\, g_-(\eta) \int_0^\infty \frac{d\xi\, \varphi^a(\xi)}{\eta+\xi}\ . \tag{83}$$

The integral over ξ behaves as η^{-1} for large. Since $g_-(\eta) \to 1$ as $\eta \to \infty$, the integral over η diverges logarithmically. Altogether then, we find this region gives a contribution of order $\ln(-q^2)/q^4$. Similarly, one can show that the region $z \approx 1$ is of order q^{-4}.

The contribution of the antiquark may similarly be shown to come from the bare coupling term

$$F_{-\,nm}^{\,b}(q^2) = 2e_b p_-(1-x) \int_0^1 dz\, \phi_n(z(1-x))\phi_m(z)\ . \tag{84}$$

Letting $z = 1 - \xi/\mu_m^{\,2}$, we find[44]

$$F_{-\,nm}^{\,b}(q^2) \approx \frac{2e_b p_-(1-x)}{\mu_m^{\,2}}\, \phi_n(1-x) \int_0^\infty d\xi\, \phi_m\!\left(1 - \frac{\xi}{\mu_m^{\,2}}\right) \approx \frac{(-)^m p_-}{\sqrt{2}}\, \pi\, \frac{e_b m_b}{-q^2}\, x\phi_n(1-x)\ . \tag{85}$$

Inserting these results into Eq. (79), we find for the structure function

$$\lim_{\text{Bj}} \frac{1}{4\mu_n^{\,2}} W(q^2,\nu) = \frac{2\pi^2 x^2}{(-q^2)^2}\, | e_a m_a \varphi_n(x) + (-1)^m e_b m_b \varphi_n(1-x)|^2\ , \tag{86}$$

or

$$\lim_{\text{Bj}} \nu^2 W(q^2,\nu) = 2\pi^2[e_a^{\,2} m_a^{\,2} \varphi_n(x)^2 + e_b^{\,2} m_b^{\,2} \varphi_n^{\,2}(1-x)]\ , \tag{87}$$

which is exactly what one would obtain from the "handbag" diagrams[45] [Fig. 18(a)]. Actually, the specification of the "handbags" requires care in order that they lead to a gauge-invariant result. The correct correspondence, as one might have guessed from the short-distance expansion, is to choose $A_- = 0$ in a frame where $x \to x_{\rm Bj}$ and to take bare (pointlike) couplings of the photon to the quarks. In the Bjorken limit, the high momentum flows through the quark propagator between the photons, so this quark may be chosen to be bare. However, the other three quark propagators should be fully dressed in order to obtain the correct, gauge-invariant asymptotic limit. In other gauges, Eq. (87) would *not* be given by the "handbags" alone. This is discussed further in Appendix B.

In going from Eq. (86) to Eq. (87), the interference term proportional to

$$(-)^m 2e_a m_a e_b m_b \phi_n(x)\phi_n(1-x) \tag{88}$$

has been dropped on the grounds that the phase factor $(-1)^m = e^{-i(p-q)^2/\pi}$ oscillates infinitely rapidly as $(p-q)^2 \to \infty$. One might naively expect this interference term to survive, because final states of only one parity will contribute (m is *either* even *or* odd). However, we must remember that this formula was arrived by smoothing the δ-function discontinuity. The proper interpretation of all these terms (including this phase) should be determined by taking the limit $(p-q)^2 \to \infty$ in some direction other than along the positive real axis.[46] We present an argument along these lines in Appendix C, showing that, to this order, the interference term contributes only to the real part of the Compton amplitude. One expects this interference term to correspond to the "crossed handbag" Fig. 18(b). One can check by direct calculation that the leading contribution from Fig. 18(b) does not scale.

The physical picture which goes with the mathematics here is quite similar to the discussion of the total cross section (Sec. III), but we feel that it bears repeating. For $(p-q)^2$ off the real axis, the perturbation expansion converges and, because the theory is asymptotically free, the high-energy behavior may be simply calculated from the lowest-order graphs ("handbag" diagrams). For $(p-q)^2 > 0$, the perturbation expansion diverges and the discontinuity comes entirely from mesonic final states. Nevertheless, the asymptotic behavior is exactly the same *as if* we calculated the discontinuity of the handbag diagrams (Fig. 18) with pointlike couplings to bare quarks.[47] This resolves very clearly the paradox of how the parton model works without producing quark final states. This should also clarify the meaning of the "as if" nature of all calculations performed in the language of bare quanta (parton model, charmonium, etc.).

Notice that, as $x \to 1$, $\nu^2 W \sim (1-x)^{2\beta}$, which is the analog of the Drell-Yan-West relation[10] between the behavior of the structure function and the falloff of the form factor. Secondly, because the resonances in the final state not only contribute to, but in fact saturate, the scaling function, the model illustrates very nicely the duality between resonances and the scaling function.[11]

As discussed in Sec. IV the form factor may be thought of as meson-dominated. From the point of view of the color-singlet, asymptotic states of the theory, the virtual Compton amplitude may be depicted as in Fig. 19. This is, of course, a general *correspondence principle* of any field theory of confinement. Since normal analyticity and unitarity hold true, to any description in terms of bare quanta (parton model), there must correspond a description entirely in terms of physical (gauge-invariant, color-singlet) states.

To describe the situation again in a language similar to Feynman's,[8] consider the process in the brick-wall frame defined by

$$-q = \frac{1}{\sqrt{2}}(0, -2xP) \,,$$

$$p = \frac{1}{\sqrt{2}}\left(P + \frac{\mu_n^2}{2P}, P - \frac{\mu_n^2}{2P}\right) \,.$$

We shall analyze the process in old-fashioned perturbation theory in the infinite-momentum frame. The amplitude to find quark a in the initial state with momentum

$$k = \frac{1}{\sqrt{2}}\left(yP + \frac{m_a^2}{2yP}, yP - \frac{m_a^2}{2P}\right)$$

is $\varphi_n(y)$. The antiquark $\bar{b}$ has momentum

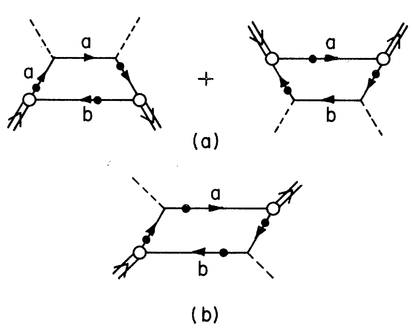

FIG. 18. The three "handbag" diagrams of the parton model: (a) scattering from the quark and antiquark, and (b) interference term corresponding to absorption by the quark and emission by the antiquark.

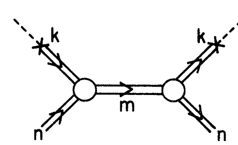

FIG. 19. Deep-inelastic scattering in terms of hadronic, asymptotic states.

$$p-k=\frac{1}{\sqrt{2}}\left((1-y)P+\frac{m_b^{\,2}}{2(1-y)P}\,,\,(1-y)P-\frac{m_b^{\,2}}{2(1-y)P}\right).$$

The amplitude that the struck quark combines with this antiquark to form the final-state hadron of mass $\mu_m^{\,2}\approx 4x(1-x)P^2$ is $\phi_m((y-x)/(1-x))$. As $\mu_m^{\,2}\to\infty$, the wave function oscillates rapidly so that we obtain contributions only for $y\approx x+\xi/4xP^2$. The "invariant mass" of the struck quark is $2(k_+-q_+)(k_--q_-)\approx(m_a^{\,2}/y+4xP^2)(y-x)\approx\xi$. Thus the rapid oscillation of the final-state wave function forces the quark's invariant mass to be finite.[48] In Feynman's language, this corresponds to the assumption of "finite interaction energy" in the infinite-momentum frame, so that the struck quark differs from being on-mass-shell by a *finite* amount $\Delta\approx\xi-m_a^{\,2}$. Asymptotic freedom then guarantees that we must get *exactly* the same result as for the "handbag".[49]

VI. SUMMARY AND SPECULATIONS

In this paper, we have clarified several aspects of the two-dimensional theory, in particular, questions concerning gauge invariance and confinement. We have shown how all previous attempts to make mathematically precise parton models fail here. By inserting a transverse momentum and assuming it is damped, it is clear how one could proceed *ad hoc* to construct a parton model patterned after what we have learned here. This could be considerably useful for phenomenology.

We have considered hadronic form factors and shown them to be power-behaved for large q^2. The power is dynamically determined by the coupling constant, and the physical picture is not that of Brodsky and Farrar.[9] To what extent can we expect our conclusions to apply to the four-dimensional theory? In four-dimensional QCD, the coupling constant g is dimensionless. However, if we believe confinement occurs, then there will arise dynamically some transverse-momentum cutoff or Regge slope α' which sets the scale of hadron mass splittings. To say this in another way, the confinement phase is a nonperturbative solution in which there are neither quarks nor massless gluons. This will be expressed by the replacement of the long-range force typical of massless exchanges by a damping factor or coherence length which determines the scale of hadron sizes or masses. Thus, even in the four-dimensional theory, quite apart from quark masses, there exists a dimensional parameter in the theory which we may take to be the Regge slope $\alpha'\sim 1$ GeV^{-2}. The BF arguments, which are essentially perturbative, might well fail for the confinement phase.[50] Instead, we can imagine that after integration over $p_\perp$, our equations look much as in this paper. If we wish to make contact with a parameter of our two-dimensional model, we would set $\pi g^2N=\alpha'^{-1}$. But if we argue that the form factor is controlled by the details of the confinement mechanism which determines the hadron wave function and not by the short-distance structure, how can we account for the phenomenological success of the dimensional counting rules[51]? Our meson form factors go as $(q^2)^{-1-\beta}$, whereas BF would predict $(q^2)^{-1}$.[52] As 't Hooft has speculated, in a theory such as this one in which the light mesons obey quadratic mass formulas,[53] the nonstrange quark masses are very small, of order $m\approx 15$ MeV. But then β turns out to be about 0.025. Thus, in effect, *light quarks reproduce the dimensional counting rules*. For strange quarks, which are substantially heavier, $m_s\approx 200$ MeV, we find $\beta_s\approx 0.33$, which may begin to be a testable difference. As we have seen, the form factor will be controlled by the lightest quark, so to discriminate, one would have to formulate tests sensitive to the strange quark component. Perhaps with sufficiently accurate measurements, one could see that the pion form factor is a single power but that the kaon form factor is the sum of two different powers.[54]

Since we are suggesting that the picture here applies to four dimensions, we may attempt to anticipate how spin will alter our results. We recognize the factor $m_a^{\,2}/Q^2$ as coming from the helicity-flip coupling to the quarks. As Feynman has emphasized,[55] the "transverse" transition form factors, such as $\pi\to\rho_\perp$, would fall less rapidly by one power of Q. Thus, we anticipate $F_{\pi\rho_\perp}(Q^2)\sim Q^{-1-2\beta}$, an extremely slow decrease. Presumably, this could be tested in ρ leptoproduction, whose contribution to νW_2 in the Bjorken limit would fall as $(Q^2)^{-2\beta}$. Thus, if $\beta=0$, there would be exclusive channels, such as "γ"$p\to\rho^0p$, which would scale (Fig. 20). This prediction is dramatically different from the BF result,[56] which leads to $F_{\pi\rho_\perp}\sim(Q^2)^{-2}$. The available data[57] show that $\mu^-p\to\mu^-\rho^0p$ falls dramatically at very small Q^2 (from $Q^2=0$ out to $Q^2\approx 0.3$ GeV^2). Thereafter, this exclusive reaction remains a nearly constant fraction of the total cross section, from $Q^2\approx 0.3$ out to $Q^2\approx 3$ GeV^2.[58] It should not be prohibitively

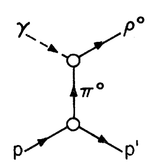

FIG. 20. An exclusive reaction which almost scales in the Bjorken limit (ρ in current fragmentation region, p' in target fragmentation region).

difficult to perform accurate measurements in the timelike region of the pion form factor and of $e^-e^+ \to \pi\rho$ to test these predictions directly.

Turning to deep-inelastic phenomena, we have shown how the model achieves consistency with its short-distance expansion in $e^-h \to e^-X$. We obtained the canonical scaling result, and it would be interesting and worthwhile to discuss inclusive electroproduction $e^-h \to e^-h'X$ in various limits (current, target, and hole fragmentation). A related, more interesting question is whether we obtain parton-model predictions for reactions which are not controlled by short-distance arguments, such as inclusive annihilation, $e^-e^+ \to hX$, and the Drell-Yan process, $hh' \to e^-e^+X$. These will be discussed elsewhere.

One could envisage other applications to purely hadronic phenomena, but since the two-dimensional model is not dual, it is not clear how useful this will be. It might be interesting to see whether there is a Pomeron in this model.

Perhaps the most interesting question to be asked of the two-dimensional theory is whether there are "baryon" bound states in the color-singlet channel formed from N quarks in a totally antisymmetric state. Unfortunately, the $1/N$ expansion seems ill-suited for this purpose, so a different "non-perturbative" approach must be invented.

To conclude, let us attempt to abstract a few lessons from the two-dimensional case which may be helpful to the solution of four-dimensional QCD. Although it has not been discussed here, it seems exceedingly difficult to solve even the two-dimensional model in any gauge other than the light-cone gauge, even to leading order in $1/N$. We might speculate that the compatible choice of gauge, the use of the proper infinite-momentum frame, and the $1/N$ expansion will also lead to simplifications in four dimensions. The essential new complication comes from showing that the transverse degrees of freedom, especially the massless gluons, are actually damped and that only massive hadrons arise. Having chosen $A_-=0$ and eliminated A_+ as a dynamical variable leaves only $\vec{A}_\perp$ and the quark fields, so that spurious degrees of freedom are conveniently absent. To leading order in $1/N$, we can continue to neglect quark loops but can no longer suppress gluon dynamics. Hopefully, this self-coupled glue will lead to a dual model of mesons.

ACKNOWLEDGMENTS

A portion of this work was performed during a visit to the Weizmann Institute of Science, whose warm hospitality is happily acknowledged. Discussions there with A. Schwimmer and G. Veneziano were quite stimulating, and I also acknowledge the advice of Y. Kannai of the Department of Pure Mathematics on the mathematics of unbounded operators in Hilbert space. I have benefitted from conversations and seminars with most of my colleagues in the Theory Department at Fermilab, but discussions with W. A. Bardeen have been especially valuable. His penetrating insights and demanding questions sharpened my understanding considerably. Finally, I pay grateful tribute to the encouragement received from H. Jackins.

Note added. A method of solution of four-dimensional QCD has been proposed which, for a transverse lattice spacing on the order of $\sqrt{\alpha'}$, leads to equations as in the two-dimensional model. See W. A. Bardeen and R. B. Pearson, Phys. Rev. D 14, 547 (1976).

Note added in proof. It has been shown that there exists a natural self-adjoint extension of 't Hooft's Hamiltonian and that its spectrum is discrete. See P. Federbush and A. Tromba, Univ. of Michigan Mathematics Department report (unpublished).

APPENDIX A

Following the suggestion of W. A. Bardeen, we will demonstrate the equivalence of the bound-state equation to a potential theory problem. Define a function $F_n(z)$, analytic except for a cut on $(0,1)$, by

$$F_n(z)=\frac{1}{\pi}\int_0^1 \frac{dx\,\phi_n(x)}{x-z}\ . \tag{A1}$$

Writing $z=x+iy$ and $F_n(z)=U_n(x,y)+iV_n(x,y)$, we have that $\mathrm{Im}F_n(x,0)=V_n(x,0)$ and our bound-state equation may be written as

$$\left(\mu_n^2-\frac{\gamma_a-1}{x}-\frac{\gamma_b-1}{1-x}\right)V_n(x,0)=-\pi\frac{\partial}{\partial x}U_n(x,0)\ . \tag{A2}$$

By the Cauchy-Riemann equations,

$$\frac{\partial U_n}{\partial x}=\frac{\partial V_n}{\partial y}\ . \tag{A3}$$

Therefore, we may state our eigenvalue problem in the following terms: Find functions $V_n(x,y)$ and eigenvalues μ_n^2 satisfying Laplace's equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V_n(x,y)=0 \tag{A4}$$

in the upper half plane subject to the boundary conditions on the real axis:

(1) $V_n(x,0)=0\ ,\quad x\notin(0,1)$ (A5a)

(2) $\left(\mu_n^2-\dfrac{\gamma_a-1}{x}-\dfrac{\gamma_b-1}{1-x}\right)V_n(x,0)=-\pi\dfrac{\partial}{\partial y}V_n(x,0)\ ,$ (A5b)

$x\in(0,1)\ .$

In addition, we require that $V_n(x,y)$ vanish at infinity (in fact, as z^{-1}). This is actually a useful form for obtaining approximate solutions to the equations. For example, consider the conformal transformation[59]

$$\xi = \sin^{-1}(2z-1) \tag{A6}$$

mapping the upper half plane onto a rectangle whose boundary has the real axis as its inverse image (Fig. 21). Letting $\xi = \rho + i\sigma$, we may restate the problem as the following: Find $V_n(\rho,\sigma)$ satisfying Laplace's equation inside the rectangle with boundary conditions

$$(1)\ V_n\left(\pm\frac{\pi}{2},\sigma\right)=0,\quad \sigma>0 \tag{A7a}$$

$$(2)\ \left(\frac{\mu_n^2}{2}-\frac{\gamma_a-1}{1+\sin\rho}-\frac{\gamma_b-1}{1-\sin\rho}\right)V_n(\rho,0) = \frac{-\pi}{\cos\rho}\frac{\partial}{\partial\sigma}V_n(\rho,0)\ , \tag{A7b}$$

$$(3)\ V_n(\rho,\infty)=0,\quad \rho\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\ . \tag{A7c}$$

Because of Laplace's equation, we may write the general solution satisfying boundary conditions (1) and (3) as

$$V_n(\rho,\sigma)=\sum_{m=0}^{\infty} e^{-2m\sigma}[a_m^n \sin 2m\rho + b_m^n e^{-\sigma}\cos(2m+1)\rho]\ . \tag{A8}$$

The remaining problem of satisfying boundary condition (2) may be formulated as a recursion relation for the coefficients a_m^n and b_m^n. This is now in a form amenable to numerical solution.

The case when $\gamma_a=\gamma_b=1$ is especially simple and can be easily solved numerically to yield accurate eigenvalues and eigenfunctions.[60] We shall not elaborate this here.

APPENDIX B

We would like to expand on some of the details of the short-distance and light-cone structure of the model. The Fourier transform of matrix elements of the product of two current will take the familiar form

$$\sum_n f_n(q)\,\langle\alpha|O_n|\beta\rangle$$

where O_n are a complete set of local operators. The light-cone limit generally requires that we keep our operators of a given "twist", so that derivatives which are suppressed in the short-distance limit contribute equally in the light-cone limit. By gauge invariance, the covariant derivative must appear, i.e.,

$$q^{\mu_1}q^{\mu_2}\cdots q^{\mu_n}\langle\alpha|\bar{q}(x)D_{\mu_1}D_{\mu_2}\cdots D_{\mu_2}q(x)|\beta\rangle\ , \tag{B1}$$

where $D_\mu=\partial_\mu+gA_\mu$. Consequently, it may seem somewhat surprising that we recover the "handbag" diagrams in deep-inelastic scattering, since these do not require gluonic operators. We can realize the light-cone limit by taking $q_+\to-\infty$ for fixed $q_->0$. But then, in the $A_-=0$ gauge,

$$q^\mu D_\mu \approx q_+\partial_-\ . \tag{B2}$$

Thus it is precisely because our gauge was chosen compatibly with our infinite-momentum boost that no gluons occur. Had we taken instead $q_-\to\infty$ for fixed q_+ in this gauge, then the physics would *not* simply correspond to the "handbag" diagrams and *would* involve an infinity of gluon field operators as well. What appears trivial in one gauge will appear horrendously complex in another. (These remarks will apply equally to the four-dimensional problem.)

We would now like to illustrate how the infrared behavior might lead us to worry whether the light-cone structure would not be more complicated than it turned out to be. Consider for this purpose the lowest-order vertex correction to the quark coupling (Fig. 22). With a vector current, this diagram is given by

$$\Gamma^{(1)}\gamma_-=ig^2N\int\frac{d^2l}{(2\pi)^2}\frac{\gamma_-S(l)\gamma_\mu S(l-q)\gamma_-}{(k_--l_-)^2}\ . \tag{B3}$$

In particular, for the γ_- component, we find simply

$$\Gamma^{(1)}=-\left(\frac{g^2N}{\pi}\right)\int_0^1\frac{dz}{(y/x-z)^2[q^2-m^2/z(1-z)]}\ , \tag{B4}$$

where $y/x\equiv k_-/q_-$, in the notation used earlier in the text. If we consider the short-distance limit, $q^2\to\infty$ for fixed y/x, we easily find the naive power-counting result

$$\Gamma^{(1)}\sim\frac{g^2N}{\pi q^2}\left(\frac{x}{y}-\frac{x}{y-x}\right) = O\left(\frac{g^2N}{q^2}\right)\ . \tag{B5}$$

However, for the form factor *and for deep-inelastic scattering*, we found that the dominant behavior came from $y\approx x+O(m^2/q^2)$ so that $\Gamma^{(1)}\sim O(g^2N/m^2)$ instead. To see this more explicitly,

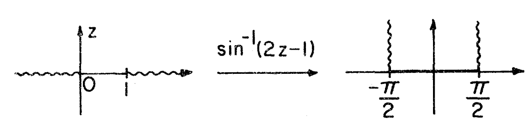

FIG. 21. A conformal transformation of the potential problem (Appendix A).

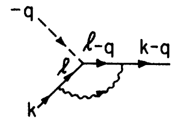

FIG. 22. Lowest-order vertex correction to quark coupling of photon.

suppose we let $y/x \approx 1+\eta m^2/(-q^2)$ and $z\approx 1-\zeta m^2/(-q^2)$. Then we find

$$\Gamma^{(1)} \approx -\frac{g^2N}{\pi m^2}\int_0^\infty \frac{d\zeta}{(\eta+\zeta)^2}\left(\frac{\zeta}{1+\zeta}\right) . \tag{B6}$$

This is precisely why we obtained corrections to the parton-model description of the form factor and, naively, we might expect a similar result for deep-inelastic scattering where a similar limit had to be taken. [Recall Eq. (83).] In the latter case, however, all such scaling contributions cancel each other leaving us with the famous parton-model result. This cancellation, seen for example in Eq. (83), is presumably related to the gauge-invariance arguments presented above but the precise connection has not been made. I believe this example serves as a warning that, *in processes for which the result is not guaranteed by short-distance arguments, these cancellations may not occur, and we may not recover the parton model.* In any case, it makes clear how the infrared structure may substantially complicate the discussion of light-cone singularities.

APPENDIX C

In this appendix, we present an alternate derivation of the scaling result Eq. (87). Its main purpose is to justify further the neglect of interference terms. Our discussion here parallels the discussion of the total annihilation cross section given in Ref. 6. The contribution to W comes from the imaginary part of

$$\begin{aligned} T = -2\pi(1-x)^2\int_0^1\int_0^1 dz\,dz' G(z,z';(p-q)^2) \\ \times [e_a^2\phi_n(x+(1-x)z)\phi_n(x+(1-x)z') + e_b^2\phi_n(z(1-x))\phi_n(z'(1-x)) \\ +2e_ae_b\phi_n(x+(1-x)z)\phi_n(z'(1-x))] . \end{aligned} \tag{C1}$$

Let us obtain the asymptotic behavior by taking $(p-q)^2\to\infty$ (fixed x) in some direction off the positive real axis. Then we may use the asymptotic expansion

$$G(z,z';(p-q)^2) \to \frac{\delta(z-z')}{(p-q)^2} + \frac{1}{(p-q)^4}\left[\left(\frac{m_a^2-1}{z}+\frac{m_b^2-1}{1-z}\right)\delta(z-z') - \frac{1}{(z-z')^2}\right] . \tag{C2}$$

Just as in $e^-e^+\to X$, the term in $(p-q)^{-2}$ contributes only to the real part. The first piece of the second term gives for the contribution to the e_a^2 term

$$\frac{-2\pi(1-x)^2}{(p-q)^4}e_a^2\int_0^1 dz\,\phi_n(x+(1-x)z)^2\left(\frac{m_a^2-1}{z}+\frac{m_b^2-1}{1-z}\right) . \tag{C3}$$

The integral diverges logarithmically at $z\approx 0$, which tells us that the leading behavior will be

$$\frac{-2\pi(1-x)^2}{(p-q)^4}e_a^2\phi_n(x)^2(m_a^2-1)\ln[-(p-q)^2] . \tag{C4}$$

Continuing back to $(p-q)^2>0$ and taking the imaginary part, we obtain

$$\frac{2\pi^2x^2}{q^4}e_a^2(m_a^2-1)\phi_n(x)^2 . \tag{C5}$$

The term in Eq. (87) involving $(z-z')^{-2}$ can easily be shown to cancel the 1 in this expression, leaving precisely the first term obtained in Eq. (87), obtained by direct calculation. A similar discussion of the term in e_b^2 leads to the second term in Eq. (87). The interference term is proportional to

$$\frac{2e_ae_b}{(p-q)^4}\int_0^1\int_0^1 dz\,dz'\,\phi_n(x+(1-x)z)\phi_n(z'(1-x))\left[\left(\frac{m_a^2-1}{z}+\frac{m_b^2-1}{1-z}\right)\delta(z-z')-\frac{1}{(z-z')^2}\right] . \tag{C6}$$

This is easily seen to be perfectly convergent; consequently it does not contribute to the imaginary part of T. Thus the interpretation given the oscillating phase found in Sec. V by smoothing δ functions is consistent with a more careful analysis of the asymptotic behavior.

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[1]Hereafter referred to as QCD.

[2]For a review with references, see H. D. Politzer, Phys. Rep. 14C, 129 (1974).

[3]G. 't Hooft, Nucl. Phys. B72, 461 (1974).

[4]G. 't Hooft, Nucl. Phys. B75, 461 (1974).

[5]G.'t Hooft, lectures given at Erice and Copenhagen Summer Schools, 1975 (unpublished).

[6]C. G. Callan, Jr., N. Coote, and D. J. Gross, Phys. Rev. D 13, 1649 (1976).

[7]T. Appelquist and H. D. Politzer, Phys. Rev. Lett. 34, 43 (1974); Phys. Rev. D 5, 1404 (1975). We disagree with the arguments of Ref. 6 regarding the dynamical suppression mechanism in this model.

[8]R. P. Feynman, *Photon-Hadron Interactions* (Benjamin, Reading, Mass., 1972).

[9]S. J. Brodsky and G. R. Farrar, Phys. Rev. Lett. 31, 1153 (1973); Phys. Rev. D 11, 1309 (1975). Our appellation is not meant to slight V. Matveev, R. Muradyan, and A. Tavkhelidze, Nuovo Cimento Lett. 7, 719 (1973), who also proposed dimensional scaling laws. It simply reflects the fact that our discussion most closely parallels the Bethe-Salpeter approach of Brodsky and Farrar (hereafter, BF).

[10]S. D. Drell and T. M. Yan, Phys. Rev. Lett. 24, 181 (1970); G. B. West, *ibid.* 24, 1206 (1970).

[11]E. D. Bloom and F. J. Gilman, Phys. Rev. Lett. 25, 1140 (1970); Phys. Rev. D 4, 2901 (1971). The first model having this property was the Veneziano-type amplitude of P. V. Landshoff and J. C. Polkinghorne, Nucl. Phys. B19, 432 (1970).

[12]They are related to the usual Hermitian fields A_μ^k of the adjoint representation by $A^j_{\mu,i} = (-i/2)A_\mu^k(\lambda_k)_{ij}$, where $\frac{1}{2}\lambda_k$ are the traceless, Hermitian matrix generators of SU(N). (Of course, we believe $N=3$ for the real world.)

[13]J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970).

[14]In the context of the massive Schwinger model S. Coleman [Ann. Phys. (N.Y.) 101, 239 (1976)] has argued that, unlike in four dimensions values of $|B|<1$ correspond to stable, well-defined "vacuua" which polarize the bound states. He has observed, however, that in the non-Abelian case, color-singlet bound states cannot have a nonzero dipole moment, so the background field leaves the physics of color-singlet states unchanged. I thank S. Coleman for a discussion about this.

[15]It is often useful to keep $A \neq 0$ as a check in gauge-invariant calculations.

[16]See Y. Frishman, CERN Report No. Ref. TH 2039-CERN, 1975 (unpublished), Appendix.

[17]Of course, strictly speaking, for finite λ, it is not a gauge parameter. However, calculations are always actually performed by using the $\lambda \to 0$ forms.

[18]This exercise was suggested to me by W. A. Bardeen.

[19]Notice that this definition is quite generally gauge invariant, independent of the $1/N$ expansion and the number of space-time dimensions.

[20]In four dimensions, we would define $\phi_n(x)$ to be the integral over p_+ and the transverse momentum $\vec{p}_\perp$. This should be the quantity which obeys a simple bound-state equation.

[21]See, e.g., M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, Vol. 1, 1972; Vol. 2, 1975).

[22]Strictly speaking, in order for this limit to exist, we must choose the phase of $\phi_n(x)$ carefully to avoid oscillations. We choose $\phi_n(x)$ to be everywhere real and, as $x \to 0$, we require $\phi_n(x) \geq 0$. Note that, in the equal-mass case, $\phi_n(1-x) = (-1)^n \phi_n(x)$ (ground state corresponds to $n=0$). Quite generally, having fixed the phase at $x=0$, we cannot readjust it at $x=1$.

[23]It has to have an inverse energy denominator to cancel the extra energy denominator arising from the fact that the T matrix always involves one more loop integration than the disconnected term.

[24]So far as we can tell, the gauge-invariant sector is always the same, regardless of whether the renormalized quark mass is positive, zero, or even imaginary. As emphasized in Ref. 16, there does, however, seem to be a problem developing the theory starting with a bare quark mass $m_a = 0$.

[25]For simplicity, we consider only flavor-conserving sources here.

[26]The dependence of $g_\Delta{}^n(q)$ on the momentum q is entirely kinematic, determined by Lorentz covariance.

[27]We find the assignment made in Ref. 6 of an anomaly to the vector bubble rather confusing or, at least, unconventional.

[28]E. Poggio, H. Quinn, and S. Weinberg, Phys. Rev. D 13, 1958 (1976).

[29]A. Casher, J. Kogut, and L. Susskind, Phys. Rev. D 10, 732 (1974).

[30]To higher order in $1/N$, when the meson resonances develop a width, smoothing of the thresholds will still be required for agreement. Notice the correspondence between the leading behavior in the $1/N$ expansion and generalized meson dominance and "new duality in electromagnetic interactions" developed by Sakurai and collaborators. See J. J. Sakurai, in *Laws of Hadronic Matter*, 1973 International School of Subnuclear Physics, edited by A. Zichichi (Periodici Scientifici, Milano, 1975).

[31]P. Landshoff and J. Polkinghorne, Phys. Rep. 5C, 1 (1972), and references therein.

[32]G. Preparata, in *Lepton and Hadron Structure*, proceedings of the International School of Subnuclear Physics "Ettore Majorana," 1974, edited by A. Zichichi (Academic Press, New York, 1975), p. 54.

[33]M. B. Einhorn and G. C. Fox, Nucl. Phys. B89, 45 (1975).

[34]We have been cavalier about the normalization of the proper vertex. As shown in Ref. 6, the correct definition is $(\pi/N)^{1/2}$ times $\Gamma_n(x,p)$.

[35]Although $x \neq x_{\rm Bj} \equiv q^2/2p\cdot q$, they become equal as $q^2 \to \infty$.

[36]J. D. Bjorken, J. B. Kogut, and D. E. Soper, Phys. Rev. D 3, 1382 (1971).

[37]Had we been calculating in a gauge where $A \neq 0$, we would have found at this point that all dependence on A cancelled. It appears that the two terms are separately gauge invariant, at least for this limited class of gauges.

[38]The two solutions for x correspond to whether meson n is right-moving or left-moving. We return to this point below.

[39]Recall that β_b is the root between 0 and 1 of the equation $\pi\beta_b \cot\pi\beta_b = 1 - \pi m_b{}^2/g^2N$. Of course, we must add

a similar contribution, F^{b}_{-nm}, attaching the photon to the antiquark $\bar{b}$, which will behave as $(q^2)^{-1-\beta_a}$. Since β is a monotonically decreasing function of m, the falloff is least rapid for the lightest quark, i.e., the photon prefers to strike the heavier quark since it is easiest to capture the lighter.

[40]We have not however, shown that the phase is $e^{-i\pi\beta_b}$ nor that the coefficient of the power is the same in the timelike and spacelike regions. Although much is simplified in the light-cone gauge, analyticity and parity invariance are often not manifest.

[41]Had the dispersion relation required a subtraction whose value was determined by the underlying field theory, we might have been justified in regarding the subtraction constant as a reminder of the pointlike coupling to constituent quarks.

[42]Although this requires further investigation, it would appear that in the present mode, all quarks retain finite invariant mass whereas for BF, the struck quark is driven to invariant masses of order q^2.

[43]All of these results differ from the conclusions of J. L. Cardy, Phys. Lett. 61B, 293 (1976), and UCSB Report No. TH-1, 1976 (unpublished). Our disagreements stem from (1) his having calculated certain contributions to the *real* part of the Compton amplitude, and (2) his interchange of the $q^2\to\infty$, $\lambda\to 0$ limits. As a historical aside, we received his first paper after having completed the work through Sec. IV.

[44]Recall from footnote 22 the phase factor $(-)^m$ comes from our convention on the overall phase of the wave functions.

[45]The fact that $\nu^2 W$ scales is due to the vector current; for a scalar current, νW would scale.

[46]This discussion is quite analogous to the analysis of the (su) diagram in dual models.

[47]As a technical matter, the leading contribution for the vector current is purely real and the leading contribution to the imaginary part is of order $m^2/(p-q)^4$. In principle, the first-order corrections involving gluon exchanges would be of the same order $[g^2N/(p-q)^4]$; however, they cancel out. (See Appendix B for details.) In this respect, the discussion of the scalar current might be simpler, since one could neglect all contributions depending on the dimensional parameters m^2 and g^2N.

[48]It is amusing how the "on-mass-shell" condition for the quarks is reproduced by the rapid oscillations of the high-mass wave functions $\phi_m(z)$.

[49]Wilson's operator-product expansion has been established only in perturbation theory, so one might have worried that it could fail somehow for nonperturbative solutions such as the one discussed in this paper.

[50]It has recently been emphatically emphasized that the properties of the confinement mechanism cannot be seen in perturbative calculations. See T. Appelquist, J. Carazzone, H. Kluberg-Stern, and M. Roth, Phys. Rev. Lett. 36, 768 (1976); 36, 1161(E) (1976); E. C. Poggio and H. R. Quinn, Harvard report, 1976 (unpublished); Y.-P. Yao, Phys. Rev. Lett. 36, 653 (1976). Perturbation-theory calculations are trustworthy only when justified by a short-distance argument.

[51]See the review by R. Blankenbecler, S. J. Brodsky, and D. Sivers, Phys. Rep. 23C, 1 (1976).

[52]Although we simply do not know how large-$p_\perp$ processes are controlled, we might well expect that if the BF arguments fail for the form factor, other applications of dimensional counting will also be in doubt.

[53]$M_0{}^2 \approx (m_a+m_b)/(3\alpha')^{1/2}$. See Ref. 5.

[54]Of course, for even heavier quarks, the distinction between our conclusions and those of BF would become even sharper, albeit more difficult to test. One nice reaction would be $e^-e^+\to F^-F^+$, where $F=(c\bar{s})$, the strange, charmed pseudoscalar meson. Others of interest not involving charm are $e^-e^+\to\phi\eta, \Omega^-\bar{\Omega}^-$.

[55]R. P. Feynman (unpublished). This is referred to in several places, e.g., in R. D. Field and D. J. Mellema, Caltech Report No. CALT-68-522, 1975 (unpublished). I thank R. Field for several discussions on this subject.

[56]G. F. Farrar and D. R. Jackson, Phys. Rev. Lett. 35, 1416 (1975).

[57]C. A. Heusch, invited talk at the International Conference on High Energy Physics, Palermo, 1975; and private communication.

[58]To the extent that ϕ is composed of only strange quarks, we would predict $F_{\eta\phi}(Q^2)\sim Q^{-1-2\beta_s}$. However, in the same approximation, ϕ leptoproduction by η exchange violates Zweig's rule and so is expected to be much smaller than ρ leptoproduction, in agreement with data (Ref. 57). Consequently, our prediction on the $F_{\eta\phi}$ from factor cannot be tested in leptoproduction. (See, however, Ref. 54 above.)

[59]This was suggested by R. Savit (private communication).

[60]This has been done by R. B. Pearson (private communication).