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Renormalization group and scale transformations for renormalized field operators

J. Lukierski*
ZIF, Universität Bielefeld, Bielefeld, West Germany
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First we show that the renormalized regularized Lagrangian density satisfies the constraint equation which can be interpreted as the divergence equation for the local normalization current. The normalization current which we introduce in this paper generates a constant multiplication factor in front of the field operator and leads to the appearance of renormalization-group transformations. Using a nonlocal action principle we introduce the charge operator which can be identified with the quantum generator of renormalization-group transformations. Further, we consider the divergence of the canonical dilatation current, and we show how to define the renormalized dilatation charge operator which generates Callan-Symanzik corrections to canonical scaling. We obtain the result that both generators for the renormalization group and renormalized scale transformations exist in a cutoff-free limit in the sense of derivation and both are time-independent, i.e., they implement the symmetries of renormalized theory. Finally, we investigate in our operator framework the breaking of renormalized scale invariance owing to the presence of mass terms. Both off-shell (Gell-Mann–Low) and on-shell (conventional) renormalization schemes are discussed. The dependence of the renormalized scale transformations on a physical mass in different renormalization schemes is exhibited.

I. INTRODUCTION

Let us consider first the class of massless Lagrangian models described by an unrenormalized scale-invariant Lagrangian density.[1] The renormalization of such theories implies

(a) the modification of scaling properties leading to the presence of noncanonical anomalies in "naive" dilatational Ward identities and

(b) the necessary appearance of a masslike parameter κ describing the off-shell normalization momenta of the vertex functions.[2]

The modified scaling properties of renormalized Green's functions are governed by the Callan-Symanzik (CS) equation[3] and their κ dependence is described by the Gell-Mann–Low (GML) renormalization-group (RG) equation.[4] It appears that in the massless case the CS and RG equations are related simply by a dimensional transformation.

Usually the scaling properties as well as the renormalization-group transformations are discussed in the language of time-ordered Green's functions. The most compact and elegant derivation of CS and RG equations was given in the framework of the Zimmermann extension of the Bogoliubov-Parasiuk-Hepp (BPH) renormalization method.[5] However, in the BPHZ renormalization scheme the local renormalized operators are defined only through finite-part prescriptions,[6] determining the renormalized Gell-Mann–Low expansion for the Green's functions. Because these finite-part prescriptions are not represented by the modification of the Lagrangian, i.e., by adding counterterms, *it is not possible in a regularization-free renormalization scheme to watch explicitly what is happening to the field operator in the course of the renormalization procedure.*

The purpose of this paper is to study the renormalization-group equations and scaling anomalies in the operator framework of renormalized quantum field theory (QFT). In order to be able to use the machinery of the Lagrangian formulation of QFT we shall assume that the renormalized theory is obtained as a cut off-free limit of a finite regularized renormalized Lagrangian theory. The condition for the proper choice of regularized renormalized Lagrangian density $\mathcal{L}_R(x;g,\kappa;\Lambda)$ is the existence of the finite renormalized field operator

$$\lim_{\Lambda\to\infty}\Phi_\Lambda(x;g,\kappa)=\Phi_R(x;g,\kappa)\,, \tag{1.1}$$

where Φ_Λ is the solution of regularized renormalized field equations.[7] One of the advantages of such a formulation is the possibility of using for any finite Λ the operator form of the quantum action principle[8]

$$\frac{\partial\Phi_\Lambda(x;\{\alpha\})}{\partial\alpha_i}=i\int d^4y\,\theta(x_0-y_0)\times\left[\Phi_\Lambda(x;\{\alpha\}),\frac{\partial\mathcal{L}_R(y;\{\alpha\};\Lambda)}{\partial\alpha_i}\right], \tag{1.2}$$

where $\Phi_\Lambda(x;\{\alpha\})$ describes the solution of regularized QFT described by the Lagrangian density $\mathcal{L}_R(x;\{\alpha\};\Lambda)$, and $\{\alpha\}=(\alpha_1\cdots\alpha_n)$ denotes the set of parameters (masses, coupling constants, etc.).

The present paper is divided into two parts. In Secs. II–IV we shall consider the massless renormalized theory, generated by a scale-invariant unrenormalized Lagrangian density. In such a model one can define two one-parameter Abelian symmetry groups, leaving the renormalized Green's functions invariant:

(a) renormalization-group transformations

$$G_n(p_1, \ldots, p_n; g, \kappa) = Z^{n/2}(\lambda; g) G_n(p_1, \ldots, p_n; \bar{g}(\lambda; g), \kappa/\lambda), \tag{1.3}$$

(b) renormalized scale transformations[9]

$$G_n(p_1, \ldots, p_n; g, \kappa) = l^n Z^{n/2}(l; g)\, G_n(p_1/l, \ldots, p_n/l; \bar{g}(l; g), \kappa), \tag{1.4}$$

where Z and $\bar{g}$ are defined as[10]

$$\lambda \frac{\partial}{\partial \lambda} \ln Z^{1/2}(\lambda; g) = \gamma(\bar{g}), \quad Z(1; g) = 1 \tag{1.5a}$$

$$\lambda \frac{\partial}{\partial \lambda} \bar{g}(\lambda; g) = \beta(\bar{g}), \quad \bar{g}(1; g) = g \tag{1.5b}$$

and are called effective wave renormalization and effective coupling constants, respectively. Let us recall that in the operator formalism a one-to-one correspondence is expected between the symmetry groups and the presence of time-independent quantum generators (the quantum Noether theorem[11]). In Secs. III and IV we are able to introduce the quantum generators for both symmetry groups (1.3) and (1.4), i.e., *we are able to describe the action of the renormalization-group and renormalized scale transformations as the mapping of the algebra of renormalized field operators.*

The second part of our paper (Secs. V and VI) contains a discussion of the massive renormalized theory. It appears that both renormalization-group transformations and renormalized scale transformations are modified. The renormalization group remains to define a symmetry, but the renormalized scale transformations are broken.

Our procedure in this paper is first, to derive the formulas for the generators in regularized renormalized theory (finite Λ), and then to discuss the limit $\Lambda \to \infty$. It appears that the commutators generating infinitesimal renormalization-group transformations as well as infinitesimal renormalized scale transformations do exist in the limit $\Lambda \to \infty$. We obtain therefore the result that the generators of both symmetry groups do exist as a limit $\Lambda \to \infty$ of corresponding cutoff-dependent generators in the sense of derivation,[12] and by performing multiple commutation one can define finite transformations of renormalized field operators.

The plan of this paper is as follows: First, in Sec. II we show that the properly renormalized Lagrangian satisfies for any finite cutoff Λ the operator equation[13]

$$\gamma_\Lambda/\kappa(g)\partial^\mu N_\mu^\Lambda = \left[\kappa \frac{\partial}{\partial \kappa} + \beta_{\Lambda/\kappa}(g) \frac{\partial}{\partial g}\right] \mathcal{L}_R(g, \kappa; \Lambda), \tag{1.6}$$

where the local current N_μ^Λ generates the multiplication of the field by a constant[14]

$$[N^\Lambda(t), \Phi_\Lambda(\vec{x}, t)] = -i\Phi_\Lambda(\vec{x}, t), \tag{1.7}$$

and $\beta_{\Lambda/\kappa}, \gamma_{\Lambda/\kappa}$ are Gell-Mann–Low coefficient functions in the presence of the cutoff Λ.[15] Owing to relation (1.7) we shall call N_μ^Λ a normalization current, and relation (1.6), the renormalization-group constraint (RGC) equation. Indeed, in Sec. III by using the nonlocal Schwinger-Peierls action principle we demonstrate that Eq. (1.6) can be treated as the conservation law for the time-independent renormalization-group charge operator R, which generates the renormalization-group transformations. In Sec. IV we consider the renormalized scale transformations. First, we calculate explicitly the divergence of the canonical dilatation current S_μ^Λ, generating for fixed value Λ the canonical scale transformations

$$[S^\Lambda(t), \Phi_\Lambda(\vec{x}, t)] = -i(x_\mu \partial^\mu + 1)\Phi_\Lambda(\vec{x}, t). \tag{1.8}$$

Using again the Schwinger-Peierls action principle we introduce the generator of renormalized scale transformations.[16] In the limit $\Lambda \to \infty$ such a generator can be identified *in the sense of a derivation* with the time-independent generator of dimensional transformations in the renormalized theory.[17] Further, we consider the Lagrangian models with the mass term. In Sec. V we study the off-shell (GML) renormalization procedure. The operator formulation of renormalization-group transformations can be constructed in complete analogy with the considerations in Secs. II and III. In the discussion of renormalized scale invariance, it is shown that the presence of mass leads to modification of the group law (due to the so-called "hard terms") and to symmetry breaking (due to the remaining soft "mass vertex insertion"). In Sec. VI the on-shell normalization is discussed, which leads to still another form of renormalized scale transformations, broken by the presence of nonvanishing physical mass.

The considerations in Sec. II–VI are formal, at least in three respects:

(a) The discussion in Secs. II–IV is based on the assumption of the existence of the limit (1.1) for a massless theory, which should be defined in such a way that the IR divergences related to the so-called exceptional momenta do not occur.[18]

(b) It should not be forgotten that there are immense mathematical difficulties related to the description of a transformation in QFT which connects two theories with different renormalized parameters (masses, coupling constants, etc.).[19]

(c) We discuss only one choice of counterterms leading to finite field equations.[7] For the composite operators occurring in our considerations to be made finite in the limit $\Lambda\to\infty$, one would have to consider other multiplicative and additive renormalizations, corresponding to different normal-product prescriptions.[5,6]

In order to illustrate the derivation of the RGC equation on the most exploited example in the literature, we discuss in Appendix A the renormalized regularized formulation of massless $g_0\phi^4$ theory.[20] In Appendix B we derive and discuss briefly the nonlocal Schwinger-Peierls action principle, which we use extensively in Secs. III–VI. In Appendix C we discuss for completeness the operator form of the renormalization-group and renormalized scale transformations for the so-called soft renormalization schemes.[21-24]

II. RENORMALIZATION-GROUP CONSTRAINT EQUATION

We shall consider the conventional renormalization of the scale-invariant Lagrangian field theory defined formally by the unrenormalized Lagrangian density $\mathcal{L}_{\rm nr}[\phi;g_0]$. Such a Lagrangian theory is renormalizable in the sense of Dyson's power-counting rule. In order to obtain a renormalized regularized theory one must perform the following two steps:

(i) Introduce regularization and the mass counterterm

$$\mathcal{L}_{\rm nr}[\phi,g_0]\to\mathcal{L}_{\rm reg}[\phi_\Lambda;g_0;\Lambda]\,. \tag{2.1}$$

We assume further that one can apply to the regularized Lagrangian the canonical formalism.[25] We introduce the cutoff parameter Λ in such a way that it has a dimension of mass.

(ii) Perform multiplicative renormalization

$$\mathcal{L}_{\rm reg}[\phi_\Lambda;g_0;\Lambda]\equiv\mathcal{L}_{\rm reg}\left[\tilde Z_3^{1/2}\left(\frac{\Lambda^2}{\kappa^2};g\right)\phi_\Lambda;g_0\left(\frac{\Lambda^2}{\kappa^2};g\right);\Lambda\right]=\mathcal{L}_R[\Phi_\Lambda,g,\kappa;\Lambda] \tag{2.2}$$

in such a way that relation (1.1) and the Gell-Mann–Low normalization conditions[4] are valid.

In formula (2.2) there are introduced the renormalized charge g and the wave renormalization constant Z_3,

$$g=g\left(\frac{\Lambda^2}{\kappa^2};g_0\right),\qquad Z_3\left(\frac{\Lambda^2}{\kappa^2};g_0\right)=\tilde Z_3\left(\frac{\Lambda^2}{\kappa^2};g\left(\frac{\Lambda^2}{\kappa^2};g_0\right)\right), \tag{2.3}$$

related to the Gell-Mann–Low coefficient functions occurring in (1.6) (see Refs. 4 and 10)

$$\beta_{\Lambda/\kappa}(g)=\kappa\frac{\partial g}{\partial\kappa}\bigg|_{g_0=g_0(\Lambda^2/\kappa^2;g)},$$
$$\gamma_{\Lambda/\kappa}(g)=\tfrac12\kappa\frac{\partial}{\partial\kappa}\ln Z_3\bigg|_{g_0=g_0(\Lambda^2/\kappa^2;g)}=\tfrac12\kappa\frac{\partial}{\partial\kappa}\ln\tilde Z_3+\tfrac12\beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g}\ln\tilde Z_3\,. \tag{2.4}$$

From (2.2) one obtains immediately the following operator equation:

$$\kappa\frac{\partial\mathcal{L}_R}{\partial\kappa}\bigg|_{\substack{\phi_\Lambda=\tilde Z_3^{1/2}(\Lambda^2/\kappa^2;g)\Phi_\Lambda\\ g_0=g_0(\Lambda^2/\kappa^2;g)}}=0\,. \tag{2.5}$$

We shall show that relation (2.5) takes the form of the RGC equation (1.6). Let us observe first that

$$\kappa\frac{\partial\mathcal{L}_R}{\partial\kappa}\bigg|_g=\kappa\frac{\partial}{\partial\kappa}\tilde Z_3^{1/2}\left(\frac{\partial\mathcal{L}_{\rm reg}}{\partial\Phi_\Lambda}\Phi_\Lambda+\partial_\mu\frac{\partial\mathcal{L}_{\rm reg}}{\partial\Phi_{\Lambda,\mu}}\Phi_{\Lambda,\mu}\right)+\frac{\partial\mathcal{L}_{\rm reg}}{\partial g_0}\kappa\frac{\partial g_0}{\partial\kappa}\,, \tag{2.6a}$$

$$\beta_{\Lambda/\kappa}(g)\frac{\partial\mathcal{L}_R}{\partial g}=\beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g}\tilde Z_3^{1/2}\left|\frac{\partial\mathcal{L}_{\rm reg}}{\partial\Phi_\Lambda}\Phi_\Lambda+\partial_\mu\frac{\partial\mathcal{L}_{\rm reg}}{\partial\Phi_{\Lambda,\mu}}\Phi_{\Lambda,\mu}\right|+\beta_{\Lambda/\kappa}(g)\frac{\partial g_0}{\partial g}\frac{\partial\mathcal{L}_{\rm reg}}{\partial g_0}. \tag{2.6b}$$

If one differentiates the relation

$$g_0=g_0\left(\frac{\Lambda^2}{\kappa^2};g\left(\frac{\Lambda^2}{\kappa^2};g_0\right)\right), \tag{2.7}$$

one gets

$$0=\kappa\frac{\partial g_0}{\partial\kappa}+\frac{\partial g_0}{\partial g}\beta_{\Lambda/\kappa}(g)\,. \tag{2.8}$$

Substituting the field equation

$$\frac{\partial\mathcal{L}_{\rm reg}}{\partial\Phi_\Lambda}=\partial_\mu\frac{\partial\mathcal{L}_{\rm reg}}{\partial\Phi_{\Lambda,\mu}} \tag{2.9}$$

and the relations

$$\frac{\partial\mathcal{L}_{\rm reg}}{\partial\Phi_\Lambda}=\tilde Z_3^{1/2}\frac{\partial\mathcal{L}_R^\Lambda}{\partial\Phi_\Lambda}\,,\qquad \frac{\partial\mathcal{L}_{\rm reg}}{\partial\Phi_{\Lambda,\mu}}=\tilde Z_3^{1/2}\frac{\partial\mathcal{L}_R}{\partial\Phi_{\Lambda,\mu}} \tag{2.10}$$

and using relations (2.4), (2.6), (2.8), and (2.10), one obtains the result that the relation (1.6) is indeed valid if we choose

$$N_\mu^\Lambda(x)=\Pi_\mu^R(x;\Lambda)\Phi_\Lambda(x)\,, \tag{2.11}$$

where

$$\Pi_\mu^R = \frac{\partial \mathcal{L}_R}{\partial \Phi_{\Lambda,\mu}} . \tag{2.12a}$$

Because Π_0^R describes the canonical momentum, i.e., we have for any Λ

$$[\Pi_0^R(\vec{x},t;\Lambda), \Phi_\Lambda(\vec{x};t)] = -\, i\delta^3(\vec{x}-\vec{x}') , \tag{2.12b}$$

we conclude that N_μ^Λ describes the normalization current.

Remark 1. In formula (2.1) the mass counterterm is obtained as a value of the proper self-energy diagram at $p^2=0$ (the value of physical mass). Calculating the mass counterterm in perturbation theory one obtains

$$\delta m_0{}^2 = \Lambda^2 M(g_0) , \tag{2.13}$$

where

$$M(g_0) = \sum_{k=1}^{\infty} C_k (g_0{}^2)^k$$

and the factorization of Λ^2 follows from scale invariance. We see therefore that the mass counterterm is included in the replacement (2.1).

Remark 2. Relation (1.6) or (2.5) can be treated as an operator condition for the equivalence of renormalized theories with different off-shell GML parametrizations. In Appendix A we shall impose the RGC equation on the renormalized regularized Lagrangian of massless $\lambda\phi^4$ theory. It will be shown how in the presence of finite cutoff Λ the information contained in the RGC equation allows us to restore the conventional formulas relating renormalized and unrenormalized Lagrangians.

III. THE GENERATOR OF RENORMALIZATION-GROUP TRANSFORMATIONS

Let us integrate relation (1.6) over the four-dimensional volume between the time hyperplanes $t=t_1$ and $t=t_2$. One gets

$$\gamma_{\Lambda/\kappa}(g)[N^\Lambda(t_2) - N^\Lambda(t_1)] = \int_{t_1}^{t_2} d^4y \left[\kappa \frac{\partial}{\partial\kappa} + \beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g}\right]\mathcal{L}_R(y) . \tag{3.1}$$

Relation (3.1) can be written in an equivalent form as

$$R^\Lambda(t_2) = R^\Lambda(t_1) = R^\Lambda , \tag{3.2a}$$

where

$$R^\Lambda = \gamma_{\Lambda/\kappa}(g) N^\Lambda(t) - \int_{-\infty}^{t} d^4y \left[\kappa\frac{\partial}{\partial\kappa} + \beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g}\right]\mathcal{L}_R(y) . \tag{3.2b}$$

Let us observe that, owing to the Schwinger-Peierls nonlocal action principle[9] (see also Appendix B), one obtains

$$\frac{\partial}{\partial g}\Phi_\Lambda(x;g,\kappa) = i\int d^4y\, \theta(x_0-y_0) \times \left[\Phi_\Lambda(x;g,\kappa), \frac{\partial}{\partial g}\mathcal{L}_R(y;g,\kappa;\Lambda)\right] , \tag{3.3}$$

$$\frac{\partial}{\partial\kappa}\Phi_\Lambda(x;g,\kappa) = i\int d^4y\, \theta(x_0-y_0) \times \left[\Phi_\Lambda(x;g,\kappa), \frac{\partial}{\partial\kappa}\mathcal{L}_R(y;g,\kappa;\Lambda)\right] . \tag{3.4}$$

Using (3.3)–(3.5) and (1.7) one gets

$$[R^\Lambda, \Phi_\Lambda(x;g,\kappa)] = -i\left(\gamma_{\Lambda/\kappa}(g) + \beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g} + \kappa\frac{\partial}{\partial\kappa}\right) \times \Phi_\Lambda(x;g,\kappa) . \tag{3.5}$$

We see that the operator R^Λ generates an infinitesimal transformation of the renormalization group. One can introduce the finite renormalization-group transformations via multiple commutators

$$V_\Lambda(\lambda)\Phi_\Lambda(x;g,\kappa)V_\Lambda{}^{-1}(\lambda) = \sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{n!}[R^\Lambda, \ldots [R^\Lambda, \Phi_\Lambda(x;g,\kappa)]\ldots] = Z_{\Lambda/\kappa}{}^{1/2}(\lambda;g)\Phi_\Lambda\left(x;\bar{g}_{\Lambda/\kappa}(\lambda;g), \frac{\kappa}{\lambda}\right) , \tag{3.6}$$

where (we denote $u=\Lambda/\kappa$)

$$\lambda\frac{\partial}{\partial\lambda}\ln Z_u{}^{1/2}(\lambda;g) = \gamma_{u/\lambda}(\bar{g}_u) , \quad Z_u(\lambda;g)\big|_{\lambda=1} = 1 \tag{3.7}$$

and

$$\lambda\frac{\partial}{\partial\lambda}\bar{g}_u(\lambda;g) = \beta_{u/\lambda}(\bar{g}_u) , \quad g_u(\lambda;g)\big|_{\lambda=1} = g . \tag{3.8}$$

The parameters $Z_{\Lambda/\kappa}(\lambda;g)$ and $\bar{g}_{\Lambda/\kappa}(\lambda;g)$ denote the effective wave renormalization constant and effective coupling constant in the presence of finite cutoff Λ. Applying the transformation (4.6) twice we obtain

$$\lambda_1\lambda_2 Z_{\Lambda/\kappa}{}^{1/2}(\lambda_1\lambda_2;g)\Phi_\Lambda\left(x;\bar{g}_{\Lambda/\kappa}(\lambda_1\lambda_2;g), \frac{\kappa}{\lambda_1\lambda_2}\right) = \lambda Z_{\Lambda/\kappa}{}^{1/2}(\lambda_1;g)Z_{\Lambda/\kappa\lambda_1}{}^{1/2}(\lambda_2,\bar{g}_{\Lambda/\kappa}(\lambda_1;g)) \times \Phi_\Lambda(x;\bar{g}_{\Lambda/\kappa\lambda_1}(\lambda_2;\bar{g}_{\Lambda/\kappa}(\lambda_1;g))) . \tag{3.9}$$

We get the following multiplication law of finite transformations of the renormalization group in the presence of finite cutoff Λ:

$$\bar{g}_u(\lambda_1\lambda_2;g)=\bar{g}_{u/\lambda_1}(\lambda_2;\bar{g}_u(\lambda_1;g)), \tag{3.10a}$$

$$Z_u(\lambda_1\lambda_2;g)=Z_{u/\lambda_1}(\lambda_2;\bar{g}_u(\lambda_1;g))Z_u(\lambda_1;g). \tag{3.10b}$$

Let us consider now the cutoff-free limit $\Lambda\to\infty$ defining the renormalized theory. It is easy to check that the regularization which leads to finite field equations usually does not provide a finite cutoff-free limit for the renormalized Lagrangian or for the generator R^Λ, defined by (3.2b). However, observing that the GML functions for the regularization giving (1.1) do have finite limits, i.e.,

$$\gamma(g)=\lim_{\Lambda\to\infty}\gamma_{\Lambda/\kappa}(g),\quad \beta(g)=\lim_{\Lambda\to\infty}\beta_{\Lambda/\kappa}(g) \tag{3.11}$$

are well defined,[27] the limit

$$\lim_{\Lambda\to\infty}R^\Lambda=\bar{R} \tag{3.12}$$

exists in the sense of a derivation,[12] i.e., we have

$$[\bar{R},\Phi(x;g,\kappa)]=-i\left(\gamma(g)+\beta(g)\frac{\partial}{\partial g}+\kappa\frac{\partial}{\partial\kappa}\right)\Phi(x;g,\kappa), \tag{3.13}$$

and via multiple commutators one can define the renormalization group transformations for the renormalized field operator

$$V(\lambda)\Phi(x;g,\kappa)V^{-1}(\lambda)=Z^{1/2}(\lambda;g)\,\Phi\left(x;\bar{g}(\lambda;g),\frac{\kappa}{\lambda}\right), \tag{3.14}$$

where g and Z satisfy Eqs. (1.5) and the following multiplication law:

$$\begin{aligned}\bar{g}(\lambda_1\lambda_2;g)&=\bar{g}(\lambda_2;\bar{g}(\lambda_1;g)),\\ Z(\lambda_1\lambda_2;g)&=Z(\lambda_2;\bar{g}(\lambda_1;g))Z(\lambda_1;g).\end{aligned} \tag{3.15}$$

It is easy to see that under the assumption

$$V(\lambda)|0\rangle=|0\rangle \tag{3.16}$$

the transformation (3.14) implements the invariance (1.3) of the renormalized Green's functions.

Remark 1. The RGC equation (1.6) can also be written as

$$\partial^\mu R_\mu^\Lambda(x)=0, \tag{3.17}$$

where

$$R_\mu^\Lambda(x)=N_\mu^\Lambda(x)+\int d^4x\, a_\mu^{\rm ret}(x-x')\left[\kappa\frac{\partial}{\partial\kappa}+\beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g}\right]\mathfrak{L}_R(x') \tag{3.18}$$

and $a_\mu^{\rm ret}$ is a retarded solution of the equation

$$\partial^\mu a_\mu^{\rm ret}(x)=\delta^4(x). \tag{3.19}$$

It is easy to see that choosing the *noncovariant* solution of (3.19),

$$a_\mu^{\rm ret}(\vec{x},t)=(\vec{0},\,\theta(t)\delta^3(\vec{x})), \tag{3.20}$$

one obtains the generator (3.2) of the renormalization group as a charge operator related to the current (3.18)

$$R^\Lambda=\int d^3x\,R_0^\Lambda(\vec{x},t). \tag{3.21}$$

We see therefore that the current (3.18) can be called a *renormalization-group current.*

It can be shown that the value of R^Λ defined by means of the formula (3.21) does not depend on the choice of the nonlocal four-vector $a_\mu^{\rm ret}$, satisfying Eq. (3.19).

Remark 2. An interesting conjecture is the statement that R^Λ is a c number, which in general can be set equal to zero. In such a case instead of the *renormalization-group transformations* (3.13) and (3.14) one obtains the *renormalization-group identities:*

(i) differential form

$$\bar{R}=0\Rightarrow\left[\kappa\frac{\partial}{\partial\kappa}+\beta(g)\frac{\partial}{\partial g}+\gamma(g)\right]\Phi(x;g,\kappa)=0, \tag{3.22}$$

(ii) global form

$$V(\lambda)=1\Rightarrow\Phi(x;g,\kappa)=Z^{1/2}(\lambda;g)\Phi\left(x;\bar{g}(\lambda;g),\frac{\kappa}{\lambda}\right). \tag{3.23}$$

In such a case one can say that the parameter κ is a purely redundant variable.

IV. ANOMALIES IN THE TRACE OF ENERGY-MOMENTUM TENSOR AND SCALE TRANSFORMATIONS FOR RENORMALIZED FIELD OPERATOR

Let us first write the formula for the canonical scale transformations of the unrenormalized regularized field operator

$$\begin{aligned}\phi'(x;\Lambda)&=U(l)\phi(x;\Lambda)U^{-1}(l)\\ &=l\phi(lx;\Lambda).\end{aligned} \tag{4.1}$$

Because the cutoff parameter Λ occurring in the regularized Lagrangian has the dimension of mass, the regularized unrenormalized Lagrangian transforms under (4.1) as follows:

$$U(l)\mathfrak{L}_{\rm reg}[\phi,g_0;\Lambda]U^{-1}(l)=l^4\mathfrak{L}_{\rm reg}[\phi',g_0;\Lambda/l]. \tag{4.2}$$

The canonical dilatation current, generating the canonical transformation (4.1), is given via the

Noether theorem by the usual formula

$$S_\mu^\Lambda = x^\nu \tilde{T}_{\nu\mu}^{\rm reg}, \tag{4.3}$$

where $\tilde{T}_{\mu\nu}^{\rm reg}$ denotes the regularized improved energy-momentum tensor, defined as[28]

$$\tilde{T}_{\mu\nu}^{\rm reg} = T_{\mu\nu}^{\rm reg} + \tfrac{1}{3}(g_{\mu\nu}\partial^\lambda N_\lambda^\Lambda - \partial_\mu N_\nu^\Lambda), \tag{4.4a}$$

the four-vector N_μ is the normalization current given by the formula (2.11) (see Ref. 29), and

$$T_{\mu\nu}^{\rm reg} = \frac{\partial \mathcal{L}_{\rm reg}}{\partial \phi_{\Lambda,\mu}} \phi_{,\nu} - \delta_{\mu\nu}\mathcal{L}_{\rm reg}. \tag{4.4b}$$

Assuming that the field equations are satisfied, one gets the divergence condition

$$\partial^\mu S_\mu^\Lambda = \tilde{T}_\mu^{{\rm reg}\,\mu} = -\Lambda \frac{\partial \mathcal{L}_{\rm reg}}{\partial \Lambda}\bigg|_{g_0} \tag{4.5}$$

describing the trace of a regularized energy-momentum tensor. Formula (4.5) expresses canonical scale symmetry breaking by the regularization procedure.[30]

In order to study the scale invariance of the renormalized theory one should use the identity

$$\Lambda \frac{\partial \mathcal{L}_{\rm reg}}{\partial \Lambda}\bigg|_{g_0} = \left(\Lambda \frac{\partial}{\partial \Lambda} + \kappa \frac{\partial}{\partial \kappa}\right) \mathcal{L}_R \bigg|_g, \tag{4.6}$$

which follows directly from relation (2.2). Observing that formulas (4.4a) and (4.4b) remain the same under the change of the renormalization of the field operator, we get the following formula for the trace of the regularized renormalized improved energy-momentum tensor $\tilde{T}_{\mu\nu}^R$:

$$\partial^\mu S_\mu^\Lambda = \tilde{T}_\mu^{R\mu} = -\left(\kappa \frac{\partial}{\partial \kappa} + \Lambda \frac{\partial}{\partial \Lambda}\right) \mathcal{L}_R, \tag{4.7}$$

where

$$S_\mu^\Lambda = x^\nu \tilde{T}_{\nu\mu}^R. \tag{4.8}$$

The formal canonical dilatation charge operator is defined as

$$S^\Lambda(t) = \int d^3x\, S_0^\Lambda(\vec{x}, t) \tag{4.9}$$

and satisfies relation (1.8).

In order to define the renormalized scale transformations, let us recall that for fixed cutoff Λ the theory is determined by the value of the renormalized coupling constant g and the choice of the normalization point $p^2 = -\kappa^2$, and only if we perform *both* changes

$$p \to p' = p/l, \quad \kappa \to \kappa' = \kappa/l \tag{4.10}$$

can the normalization conditions be preserved. We shall assume that the renormalized scale transformations relate two, "the same," renormalized regularized theories, i.e., those described by the same values of g and Λ and the same normalization condition. In order to include the shift of κ into the definition of renormalized scale transformations, we introduce the following formula for the generator of renormalized scale transformations:

$$S_R^\Lambda(t) = S^\Lambda(t) + \int_{-\infty}^t d^4x\, \kappa \frac{\partial \mathcal{L}_R(x)}{\partial \kappa}. \tag{4.11}$$

Using (3.4) one gets

$$[S_R^\Lambda(t), \Phi_\Lambda(\vec{x}, t; g, \kappa)] = -i\left(x^\mu \partial_\mu + 1 - \kappa \frac{\partial}{\partial \kappa}\right) \Phi_\Lambda(\vec{x}, t; g, \kappa). \tag{4.12}$$

The generator $S_R^\Lambda(t)$ depends on time in accordance with the formula

$$\frac{dS_R^\Lambda(t)}{dt} = -\int d^3x\, \Lambda \frac{\partial \mathcal{L}_R(\vec{x}, t)}{\partial \Lambda}. \tag{4.13}$$

In order to study the limit $\Lambda \to \infty$, one can introduce the generator of renormalized dimensional transformations

$$D_R^\Lambda(t) = S_R^\Lambda(t) + \int_{-\infty}^t d^4x\, \Lambda \frac{\partial \mathcal{L}_R(x)}{\partial \Lambda}, \tag{4.14}$$

where it follows from (4.13) that

$$\frac{dD_R^\Lambda(t)}{dt} = 0, \qquad D_R^\Lambda(t_1) = D_R^\Lambda(t_2) = D_R^\Lambda, \tag{4.15}$$

and the dimensional transformation of the renormalized theory is realized as the following quantum symmetry mapping:

$$[D_R^\Lambda, \Phi_\Lambda(x; g, \kappa)] = -i\left(x^\mu \partial_\mu + 1 - \kappa \frac{\partial}{\partial \kappa} - \Lambda \frac{\partial}{\partial \Lambda}\right) \Phi_\Lambda(x; g, \kappa). \tag{4.16}$$

If we make a rather plausible assumption that the Λ dependence of the renormalized field operator Φ_Λ does not contain unusual oscillations for large Λ, we can supplement (1.3) with the condition

$$\lim_{\Lambda \to \infty} \Lambda \frac{\partial \Phi(x; g, \kappa)}{\partial \Lambda} = 0. \tag{4.17}$$

Substituting from (4.16) the relation (4.12) and using (4.17) one obtains

$$\lim_{\Lambda \to \infty} [D_R^\Lambda - S_R^\Lambda(t), \Phi_\Lambda(\vec{x}, t; g, \kappa)] = 0. \tag{4.18}$$

One can therefore say that the renormalized scale transformations in a cutoff-free limit coincide with the dimensional transformations, and that *the time-dependent generator $S_R^\Lambda(t)$ in the limit*

$\Lambda\to\infty$ *converges in the sense of derivation to the cutoff-free generator of dimensional mapping.* Because the dimensional tranformations always (owing to simple dimensionality counting) leave the theory invariant in such a way, one can conclude that the renormalized theory is also invariant under the renormalized scale transformations (4.12).

The global renormalized scale transformations can be written as follows:

$$U_R(l)\Phi(x;g,\kappa)U_R^{-1}(l)=l\Phi\left(lx;g,\frac{\kappa}{l}\right). \tag{4.19}$$

Using the renormalization-group invariance one obtains the second form of the renormalized scale transformations. Let us introduce the generator which adds to (4.12) the infinitesimal renormalization-group transformation:

$$\tilde{S}_R^\Lambda(t)=S_R^\Lambda(t)+R^\Lambda. \tag{4.20}$$

Using (4.20) as a generator one gets the following infinitesimal change of the field operator:

$$[\tilde{S}_R^\Lambda(t),\Phi_\Lambda(\vec{x},t)]=-i\left(x_\mu\partial^\mu+1+\gamma_{\Lambda/\kappa}(g)+\beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g}\right)\times\Phi_\Lambda(\vec{x},t). \tag{4.21}$$

In the limit $\Lambda\to\infty$ we obtain

$$[\tilde{S}_R,\Phi_R(x)]=-i\left(x_\mu\partial^\mu+1+\gamma(g)+\beta(g)\frac{\partial}{\partial g}\right)\Phi_R(x), \tag{4.22}$$

where the time independence of $\hat{S}_R$ follows from (4.18). The global form of the transformation (4.22) appears as follows:

$$\tilde{U}_R(l)\Phi(x;g,\kappa)\tilde{U}_R^{-1}(l)=Z^{1/2}(\lambda;g)\ \Phi(lx;\bar{g}(l;g),\kappa). \tag{4.23}$$

The Abelian group multiplication law is ensured by the relations (3.15).

We see from formulas (4.22) and (4.23) that the second form of the renormalized scale transformations exhibits more explicitly the Callan-Symanzik corrections to the canonical scaling laws. If the renormalized scale symmetry is not spontaneously broken, i.e.,

$$\tilde{U}_R(l)\,|0\rangle=|0\rangle, \tag{4.24}$$

formula (4.23) implies the invariance (1.4) of the renormalized Green's functions.

Remark 1. The renormalized scale transformations of the renormalized field operators in the Thirring model[31] have been introduced by Kupsch, Rühl, and Yunn[32] in the discussion of finite conformal transformations. They describe the renormalized scale transformation as a product of a canonical scale transformation and a transformation changing the normalization point.[33] This decomposition corresponds on the level of infinitesimal transformations to relation (4.12), where the generator of renormalized scale transformations is a sum of the canonical one and the generator describing the scale transformations of κ.

In the case of the Thirring model the function β is identically equal to zero, and in formula (4.23) we should set $\bar{g}(\lambda;g)\equiv g$. Because, from the discussion of conformal Ward identities, it follows[34] that the anomalies of the conformal current are described also by the same two coefficient functions γ and β, we expect that it will not be so difficult to generalize the formulas obtained in Ref. 32 for finite conformal transformations if $\beta=0$ to the case when $\beta\neq0$.

Remark 2. One can also obtain the renormalized scale transformations by using the notion of symmetry rearrangement, introduced by Umezawa and co-workers.[35] It appears that the canonical scale transformations of asymptotic fields due to the renormalization procedure are rearranged into the renormalized scale transformations (4.19) or (4.23) (see Ref. 36).

Remark 3. If the conjecture that $\bar{R}=0$ is valid (see Sec. III, remark 2) one gets from (4.20) that

$$\tilde{S}_R^\Lambda(t)=S_R^\Lambda(t)\ \text{or}\ U_R(l)=\tilde{U}_R(l). \tag{4.25}$$

In such a case formulas (4.19) and (4.23) represent two forms of a unique formula describing the renormalized scale transformations.

V. RENORMALIZATION GROUP AND RENORMALIZED SCALE TRANSFORMATIONS FOR MASSIVE THEORY: OFF-SHELL NORMALIZATION

Our method, based on a discussion of invariance properties of the renormalized regularized Lagrangian density, can also be applied to the massive theory. In this section we shall discuss the Gell-Mann–Low off-shell normalization scheme. For the massless theory the off-shell normalization is a necessity; however if, the physical mass $m\neq0$ one can choose also the mass m as the normalization point. In Sec. VI we shall discuss the conventional on-shell normalization; besides, we shall consider briefly in Appendix C the so-called soft renormalization schemes.[21-24]

In the presence of a mass term relations (2.1) and (2.2) are replaced by

$$\mathcal{L}_{nr}[\phi; g_0, m_0^2] \to \mathcal{L}_{reg}[\phi_\Lambda; g_0, m_0^2; \Lambda], \tag{5.1}$$

$$\mathcal{L}_{reg}[\phi_\Lambda; g_0, m_0^2; \Lambda] \equiv \mathcal{L}_{reg}\left[\tilde{Z}_3^{1/2}\left(\frac{\Lambda^2}{\kappa^2}, \frac{m^2}{\kappa^2}; g\right)\Phi_\Lambda; g_0\left(\frac{\Lambda^2}{\kappa^2}, \frac{m^2}{\kappa^2}; g\right), m^2 M\left(\frac{\Lambda^2}{m^2}; g_0\left(\frac{\Lambda^2}{\kappa^2}, \frac{m^2}{\kappa^2}; g\right)\right); \Lambda\right]$$
$$= \mathcal{L}_R[\Phi_\Lambda; g, \kappa, m^2; \Lambda], \tag{5.2}$$

where the mass counterterm has the form

$$\delta m^2 = m_0^2 - m^2 = m^2\left(M\left(\frac{\Lambda^2}{m^2}; g_0\right) - 1\right). \tag{5.3a}$$

We see that provided

$$\lim_{x\to\infty} \frac{M(x; g_0)}{x} = M(g_0), \tag{5.3b}$$

we obtain if $m \to 0$ the formula (2.13) for the massless theory.

(i) Renormalization-group transformations. Equation (2.5), in the presence of the unrenormalized mass m_0, takes the form

$$\kappa \frac{\partial \mathcal{L}_R}{\partial \kappa}\bigg|_{g_0, m_0} = 0 \tag{5.4}$$

and leads again to the RGC equation describing in the operator framework the renormalization-group invariance. Introducing the GML coefficient functions

$$\beta_{\Lambda/\kappa}\left(g; \frac{m^2}{\kappa^2}\right) = \kappa \frac{\partial g}{\partial \kappa}\bigg|_{g_0 = g_0(\Lambda^2/\kappa^2, m^2/\kappa^2; g)}, \tag{5.5a}$$

$$\gamma_{\Lambda/\kappa}\left(g; \frac{m^2}{\kappa^2}\right) = \frac{1}{2}\kappa\frac{\partial}{\partial\kappa}\ln\tilde{Z}_3 + \tfrac{1}{2}\gamma_{\Lambda/\kappa}\left(g; \frac{m^2}{\kappa^2}\right)\frac{\partial}{\partial g}\ln\tilde{Z}_3, \tag{5.5b}$$

we obtain after obvious generalization of the discussion in Sec. II the operator relation

$$\gamma_{\Lambda/\kappa}\left(g; \frac{m^2}{\kappa^2}\right)\partial^\mu N_\mu = \beta_{\Lambda/\kappa}\left(g; \frac{m^2}{\kappa^2}\right)\frac{\partial \mathcal{L}_R}{\partial g} + \kappa\frac{\partial \mathcal{L}_R}{\partial \kappa}. \tag{5.6}$$

Defining

$$R^\Lambda(m) = \gamma_{\Lambda/\kappa}\left(g; \frac{m^2}{\kappa^2}\right)N(t) - \int_{-\infty}^t d^4y\left[\beta_{\Lambda/\kappa}\left(g; \frac{m^2}{\kappa^2}\right)\frac{\partial}{\partial g} + \kappa\frac{\partial}{\partial\kappa}\right]\mathcal{L}_R(y), \tag{5.7}$$

one gets that the operator (5.7) in the limit $\Lambda\to\infty$ generates the renormalization-group transformations

$$[R(m), \Phi(x; g, \kappa, m)] = -i\left[\gamma\left(g; \frac{m^2}{\kappa^2}\right) + \beta\left(g; \frac{m^2}{\kappa^2}\right)\frac{\partial}{\partial g} + \kappa\frac{\partial}{\partial\kappa}\right]\Phi(x; g, \kappa; m). \tag{5.8a}$$

If the counterterms in (5.1) and the renormalization constants in (5.2) have been chosen properly, the limits

$$\gamma\left(g; \frac{m^2}{\kappa^2}\right) = \lim_{\Lambda\to\infty}\gamma_{\Lambda/\kappa}\left(g; \frac{m^2}{\kappa^2}\right), \tag{5.8b}$$

$$\beta\left(g; \frac{m^2}{\kappa^2}\right) = \lim_{\Lambda\to\infty}\beta_{\Lambda/\kappa}\left(g; \frac{m^2}{\kappa^2}\right) \tag{5.8c}$$

will exist.

In order to write the global transformation formula, one should introduce the solutions of the following two equations:

$$\lambda\frac{\partial}{\partial\lambda}\ln Z^{1/2}\left(\lambda; g, \frac{m^2}{\kappa^2}\right) = \gamma\left(\bar{g}; \frac{\lambda^2 m^2}{\kappa^2}\right), \quad Z\left(\lambda; g, \frac{m^2}{\kappa^2}\right)\bigg|_{\lambda=1} = 1 \tag{5.9}$$

$$\lambda\frac{\partial}{\partial\lambda}\bar{g}\left(\lambda; g, \frac{m^2}{\kappa^2}\right) = \beta\left(\bar{g}; \frac{\lambda^2 m^2}{\kappa^2}\right), \quad \bar{g}\left(\lambda; g, \frac{m^2}{\kappa^2}\right)\bigg|_{\lambda=1} = g. \tag{5.10}$$

We obtain

$$V_m(\lambda)\Phi(x; g, \kappa, m)V_m^{-1}(\lambda) = Z^{1/2}\left(\lambda; g, \frac{m^2}{\kappa^2}\right)\Phi\left(x; \bar{g}\left(\lambda; g, \frac{m^2}{\kappa^2}\right), \frac{\kappa}{\lambda}; m\right). \tag{5.11}$$

We see that in the presence of physical mass m the renormalization-group transformation remains a symmetry.

(ii) Anomalies of the trace of the energy-momentum tensor and the renormalized scale transformations. The "naive" canonical dilatation current $S_\mu^{(0)}$ satisfies the formal divergence relation

$$\partial^\mu S_\mu^{(0)} = \tilde{T}_\mu^{(0)\mu} = -m_0\frac{\partial \mathcal{L}_{nr}}{\partial m_0}, \tag{5.12}$$

where $\tilde{T}_{\mu\nu}^{(0)}$ describes the "naive" unrenormalized energy-momentum tensor in its modified form.[24] Introducing the regularization (5.1) one gets in place of (4.5)

$$\partial^\mu S_\mu^\Lambda = \tilde{T}_\mu^{\text{reg}\,\mu} = -\left(m_0\frac{\partial}{\partial m_0} + \Lambda\frac{\partial}{\partial\Lambda}\right)\mathcal{L}_{reg}\bigg|_{g_0}, \tag{5.13}$$

and, further, using the relation which follows from (5.2)

$$\left(m_0\frac{\partial}{\partial m_0}+\Lambda\frac{\partial}{\partial\Lambda}\right)\mathcal{L}_{\rm reg}\Big|_{g_0}=\left(m\frac{\partial}{\partial m}+\kappa\frac{\partial}{\partial\kappa}+\Lambda\frac{\partial}{\partial\Lambda}\right)\mathcal{L}_R\Big|_g, \tag{5.14}$$

we obtain

$$\partial^\mu S^\Lambda_\mu=\tilde{T}^{R\mu}_\mu=-\left(m\frac{\partial}{\partial m}+\kappa\frac{\partial}{\partial\kappa}+\Lambda\frac{\partial}{\partial\Lambda}\right)\mathcal{L}_R. \tag{5.15}$$

Introducing

$$\Delta_m\beta_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)=m\frac{\partial g}{\partial m}\Big|_{g_0=g_0(\Lambda^2/\kappa^2,\,m^2/\kappa^2;\,g)}, \tag{5.16a}$$

$$\Delta_m\gamma_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)=\tfrac{1}{2}m\frac{\partial}{\partial m}\ln\tilde{Z}_3+\tfrac{1}{2}\Delta\beta_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)\frac{\partial}{\partial g}\ln\tilde{Z}_3, \tag{5.16b}$$

$$m\frac{\partial m_0}{\partial m}\Big|_{g_0=g_0(\Lambda^2/\kappa^2,\,m^2/\kappa^2;\,g)}=\kappa_{\Lambda/\kappa}{}^2\left(g;\frac{m^2}{\kappa^2}\right)m^2, \tag{5.16c}$$

one can write, using relation (5.2), the following formula for the mass derivative:

$$m\frac{\partial}{\partial m}\mathcal{L}_R=+\Delta\gamma_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)\partial^\mu N_\mu-\Delta\beta_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)\frac{\partial\mathcal{L}_R}{\partial m}+m\frac{\partial m_0{}^2}{\partial m}\frac{\partial\mathcal{L}_{\rm reg}}{\partial m_0{}^2}. \tag{5.17}$$

If we define

$$\tilde{\beta}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)=-\Lambda\frac{\partial}{\partial\Lambda}g\Big|_{g_0=g_0(\Lambda^2/\kappa^2;\,m^2/\kappa^2;\,g)}=\beta_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)+\Delta\beta_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right), \tag{5.18}$$

$$\tilde{\gamma}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)=-\tfrac{1}{2}\Lambda\frac{\partial}{\partial\Lambda}\ln\tilde{Z}_3-\tfrac{1}{2}\beta_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right),\qquad \frac{\partial}{\partial g}\ln\tilde{Z}_3=\gamma_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)+\Delta\gamma_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right), \tag{5.19}$$

the divergence of the canonical dilatation current in renormalized regularized theory can be written as

$$\partial^\mu S^\Lambda_\mu=-\tilde{\gamma}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)\partial^\mu N^\Lambda_\mu+\tilde{\beta}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)\frac{\partial\mathcal{L}_R}{\partial g}-\tfrac{1}{2}\kappa_{\Lambda/\kappa}{}^2\left(g;\frac{m^2}{\kappa^2}\right)m^2\tilde{Z}_3\Phi_\Lambda{}^2-\Lambda\frac{\partial}{\partial\Lambda}\mathcal{L}_R, \tag{5.20}$$

where we used formulas (5.16)–(5.18) and the following obvious relation:

$$\frac{\partial\mathcal{L}_{\rm reg}}{\partial m_0{}^2}=\tfrac{1}{2}\phi_\Lambda{}^2(x). \tag{5.21}$$

The formal charge operator generating the renormalized scale transformations takes the form

$$\tilde{S}^\Lambda_R(t;m)=S^\Lambda(t)+\tilde{\gamma}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)N^\Lambda(t)-\tilde{\beta}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)\int_{-\infty}^{t}d^4x\frac{\partial\mathcal{L}_R(x)}{\partial g} \tag{5.22}$$

and satisfies the following ET commutation relations:

$$[\tilde{S}^\Lambda_R(t;m),\Phi_\Lambda(\vec{x},t;g,\kappa,m)]=-i\left[x_\mu\partial^\mu+1+\tilde{\gamma}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)+\tilde{\beta}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)\frac{\partial}{\partial g}\right]\Phi_\Lambda(\vec{x},t;g,\kappa,m). \tag{5.23}$$

The conservation law for the generator S^Λ_R (for finite Λ) has the following form:

$$\frac{dS^\Lambda_R(t;m)}{dt}=-\tfrac{1}{2}\kappa_{\Lambda/\kappa}{}^2\left(g;\frac{m^2}{\kappa^2}\right)m^2\tilde{Z}_3\left(\frac{\Lambda^2}{\kappa^2},\frac{m^2}{\kappa^2};g\right)\int d^3x\,\Phi_\Lambda{}^2(\vec{x},t)-\int d^3x\,\Lambda\frac{\partial\mathcal{L}_R(\vec{x},t)}{\partial\Lambda}. \tag{5.24}$$

If we subtract properly the divergences occurring in the composite operators, in the limit $\Lambda\to\infty$ we obtain the following:

(a) If condition (4.17) is valid the term $\Lambda\partial\mathcal{L}_R/\partial\Lambda$ in (5.24) can be dropped,[37] and the product $\tilde{Z}_3\Phi_\Lambda{}^2$ should be replaced by the multiplicatively renormalized square of the field operator.[38]

(b) Introducing the limits

$$\tilde{\gamma}\left(g;\frac{m^2}{\kappa^2}\right)=\lim_{\Lambda\to\infty}\tilde{\gamma}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right),\qquad \tilde{\beta}\left(g;\frac{m^2}{\kappa^2}\right)=\lim_{\Lambda\to\infty}\tilde{\gamma}_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right) \tag{5.25}$$

and the equations

$$\lambda\frac{\partial}{\partial\gamma}\ln\tilde{Z}^{1/2}\left(\lambda;g,\frac{m^2}{\kappa^2}\right)=\tilde{\gamma}\left(g;\frac{\lambda^2m^2}{\kappa^2}\right), \tag{5.26a}$$

$$\tilde{Z}\left(\lambda;g,\frac{m^2}{\kappa^2}\right)\Big|_{\lambda=1}=1$$

$$\lambda\frac{\partial}{\partial\lambda}\hat{g}\left(\lambda;g,\frac{m^2}{\kappa^2}\right)=\tilde{\beta}\left(\tilde{g};\frac{\lambda^2m^2}{\kappa^2}\right), \tag{5.26b}$$

$$\tilde{g}\left(\lambda;g,\frac{m^2}{\kappa^2}\right)\Big|_{\lambda=1}=g$$

one obtains the global scale transformations in the presence of the mass parameter:

$$U_m^R(l)\Phi(x;g,\kappa,m)U_m^{R-1}(l)$$
$$=l\tilde{Z}^{1/2}\left(l;g,\frac{m^2}{\kappa^2}\right)\Phi\left(x;\bar{g}\left(l;g,\frac{m^2}{\kappa^2}\right),\frac{\kappa}{l};m\right). \tag{5.27}$$

This formula describes the generalization of the transformation law (4.23) in the presence of physical mass $m\neq 0$. Introducing [compare with (4.20)]

$$S_R^\Lambda(t;m)=\tilde{S}_R^\Lambda(t;m)-R^\Lambda(m), \tag{5.28}$$

one obtains the relation

$$[S_R^\Lambda(t;m),\Phi_\Lambda(\vec{x},t;g,\kappa,m)]$$
$$=-i\left[x^\mu\partial_\mu+1-\kappa\frac{\partial}{\partial\kappa}-\Delta_m\gamma_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)\right.$$
$$\left.-\Delta_m\beta_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)\frac{\partial}{\partial g}\right]\Phi_\Lambda(\vec{x},t;g,\kappa,m), \tag{5.29}$$

which is the generalization of relation (4.12) if $m\neq 0$. Indeed, at least in perturbation theory, we obtain that

$$\lim_{m\to 0}\Delta_m\beta_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)=\lim_{m\to 0}\Delta_m\gamma_{\Lambda/\kappa}\left(g;\frac{m^2}{\kappa^2}\right)=0, \tag{5.30}$$

and formula (5.29) in the limit $m\to 0$ coincides with (4.22).

We see therefore that *the "hard" part of the mass derivative $m(\partial/\partial m)\mathcal{L}_R$ has contributed to the modification of the renormalized scale transformations in the presence of the mass parameter, and the "soft" part is treated as the term* which breaks the renormalized scale invariance.[39]

VI. RENORMALIZED SCALE TRANSFORMATIONS FOR THE MASSIVE THEORY: ON-SHELL NORMALIZATION

One obtains the on-shell normalization by replacing (5.2) in the following way:

$$\mathcal{L}_{\rm reg}[\phi_\Lambda;g_0,m_0^2;\Lambda]\equiv\mathcal{L}_{\rm reg}\left[Z_3^{1/2}\left(\frac{\Lambda^2}{m^2};g\right)\Phi_\Lambda;g_0\left(\frac{\Lambda^2}{m^2};g\right),m^2M\left(\frac{\Lambda^2}{m^2};g\right);\Lambda\right]$$
$$=\mathcal{L}_R[\Phi_\Lambda;g,m^2;\Lambda]. \tag{6.1}$$

In such a formalism one cannot perform the renormalization-group transformations. In order to discuss the renormalized scale transformations let us observe that from (6.1) we get the relation

$$\left(m_0\frac{\partial}{\partial m_0}+\Lambda\frac{\partial}{\partial\Lambda}\right)\mathcal{L}_{\rm reg}\Big|_{g_0}=\left(m\frac{\partial}{\partial m}+\Lambda\frac{\partial}{\partial\Lambda}\right)\mathcal{L}_R\Big|_g, \tag{6.2}$$

and we get for the divergence of the canonical dilatation current in the renormalized regularized theory the following formula:

$$\partial^\mu S_\mu^\Lambda=\tilde{T}_\mu^{R\mu}=-\left(m\frac{\partial}{\partial m}+\Lambda\frac{\partial}{\partial\Lambda}\right)\mathcal{L}_R. \tag{6.3}$$

Introducing

$$\tilde{\beta}_{\Lambda/m}(g)=-m\frac{\partial}{\partial m}g\left(\frac{\Lambda^2}{m^2};g_0\right), \tag{6.4}$$

$$\tilde{\gamma}_{\Lambda/m}(g)=-\tfrac{1}{2}m\frac{\partial}{\partial m}\ln Z_3+\tfrac{1}{2}\tilde{\beta}_{\Lambda/m}(g)\frac{\partial}{\partial g}\ln Z_3, \tag{6.5}$$

one gets from (6.1)

$$m\frac{\partial}{\partial m}\mathcal{L}_R=\tilde{\gamma}_{\Lambda/m}(g)\partial^\mu N_\mu^\Lambda-\tilde{\beta}_{\Lambda/m}(g)\frac{\partial\mathcal{L}_R}{\partial g}$$
$$+\tfrac{1}{2}\tilde{\kappa}_{\Lambda/m}{}^2(g)m^2\tilde{Z}_3\Phi_\Lambda{}^2, \tag{6.6}$$

where

$$\tilde{\kappa}_{\Lambda/m}{}^2(g)=\left(2+m\frac{\partial}{\partial m}\right)M\left(\frac{\Lambda^2}{m^2};g\right). \tag{6.7}$$

If we define the generator

$$\tilde{S}_R^\Lambda(t;m)=S^\Lambda(t)+\tilde{\gamma}_{\Lambda/m}(g)N^\Lambda(t)$$
$$-\tilde{\beta}_{\Lambda/m}(g)\int_{-\infty}^t d^4x\frac{\partial\mathcal{L}_R(x)}{\partial g}, \tag{6.8}$$

then the following "nonconservation law" is satisfied:

$$\frac{d\tilde{S}_R(t;m)}{dt}=-\tfrac{1}{2}\tilde{\kappa}_{\Lambda/m}{}^2(g)m^2\tilde{Z}_3\int d^3x\,\Phi_\Lambda{}^2(\vec{x},t)$$
$$-\int d^3x\,\Lambda\frac{\partial}{\partial\Lambda}\mathcal{L}_R(\vec{x},t). \tag{6.9}$$

In the limit $\Lambda \to \infty$ we obtain the following infinitesimal renormalized scale transformations:

$$[\tilde{S}_R(t;m), \Phi(\vec{x}, t; g, m)] = -i\left(x_\mu \partial^\mu + 1 + \tilde{\gamma}(g) + \tilde{\beta}(g)\frac{\partial}{\partial g}\right)\Phi(\vec{x}, t; g, m), \qquad (6.10)$$

where

$$\tilde{\beta}(g) = \lim_{\Lambda\to\infty} \tilde{\beta}_{\Lambda/m}(g), \quad \tilde{\gamma}(g) = \lim_{\Lambda\to\infty} \gamma_{\Lambda/m}(g). \qquad (6.11)$$

From the formula for $d\tilde{S}_R^\Lambda/dt$ it follows that in the cutoff-free limit the renormalized scale transformations are broken by the renormalized mass insertion term [described in the limit $\Lambda \to \infty$ by the renormalized field square operator]. The global form of the renormalized scale transformations obtained from (6.10) appears as

$$\tilde{U}(l)\Phi(x;g,m)\tilde{U}^{-1}(l) = l\tilde{Z}^{1/2}(\lambda;g)\,\Phi(lx;\tilde{g}(\lambda;g),m), \qquad (6.12)$$

where

$$\lambda\frac{\partial}{\partial\lambda}\tilde{g}(\lambda;g) = \tilde{\beta}(\tilde{g}), \quad \tilde{g}(1;g) = g$$
$$\lambda\frac{\partial}{\partial\lambda}\ln\tilde{Z}^{1/2}(\lambda;g) = \tilde{\gamma}(\tilde{g}), \quad \tilde{Z}(1;g) = 1. \qquad (6.13)$$

In the case of on-mass-shell normalization we have only one form (6.12) of the renormalized scale transformations.

Remark 1. It should be stressed that the coefficient functions (6.11) are not equal to the ones introduced in Sec. III for the massless theory. The coefficients (3.11) can be obtained as the limits

$$\lim_{m\to 0}\tilde{\gamma}\left(g;\frac{m^2}{\kappa^2}\right) = \gamma(g),$$
$$\lim_{m\to 0}\tilde{\beta}\left(g;\frac{m^2}{\kappa^2}\right) = \beta(g), \qquad (6.14)$$

and one gets formally the functions (6.10) from the functions $\tilde{\beta}(g, m^2/\kappa^2)$ and $\tilde{\gamma}(g; m^2/\kappa^2)$ by setting $\kappa^2 = -m^2$. Because in the general case $\tilde{\beta}(g;0) \neq \tilde{\beta}(g;-1)$ and $\tilde{\gamma}(g;0) \neq \tilde{\gamma}(g;-1)$, the coefficient functions (6.11) and (3.11) do not coincide.

Remark 2. In addition to the two previously discussed renormalization schemes, other ways of normalizing the massive renormalized theory may also be used:

(*a*) *Intermediate normalization, at* $p = 0$ (*used in the BPHZ renormalization scheme*). The discussion in our framework of such a parametrization would follow the considerations of this section.

(*b*) *Soft renormalization schemes*. There are several versions,[21-24] but all are characterized by the property that the so-called soft mass parameter m_s enters only in the mass δ term as a multiplicative factor, i.e., we have

$$m_0^2 = m_s^2 \kappa_{\Lambda/\kappa}^2(g), \qquad (6.15)$$

and the RGC equation has the form

$$\gamma_{\Lambda/\kappa}(g)\partial^\mu N_\mu = \beta_{\Lambda/\kappa}(g)\frac{\partial\mathcal{L}_R}{\partial g} + \alpha_{\Lambda/\kappa}(g) m_s \frac{\partial\mathcal{L}_R}{\partial m_s} + \kappa\frac{\partial\mathcal{L}_R}{\partial\kappa}, \qquad (6.16)$$

where

$$\alpha_{\Lambda/\kappa}(g) = \left(\kappa\frac{\partial}{\partial\kappa} + \beta_{\Lambda/\kappa}\frac{\partial}{\partial g}\right)\ln\kappa_{\Lambda/\kappa}(g), \qquad (6.17)$$

and $\gamma_{\Lambda/\kappa}(g), \beta_{\Lambda/\kappa}(g)$ are defined by formula (2.4) for the massless theory.

More detailed discussion of the renormalization-group transformations and renormalized scale transformations for the field operators in the soft renormalization framework is presented in Appendix C.

VII. SYMMETRY MAPPINGS: REMARKS ABOUT RIGOROUS FORMULATION AND PROBLEM OF DIVERGENCES

We shall refer to *symmetry mappings* as *transformations which leave the Green's functions invariant but relate two theories with different values of the numerical parameters*[40] (masses, coupling constants, etc.). Examples of such symmetry mappings were considered in this paper, i.e.,

(a) renormalization-group transformations (for finite Λ and $\Lambda \to \infty$, arbitrary masses),

(b) renormalized scale transformations [for $\Lambda \to \infty$ and $m = 0$ (see Ref. 41)],

(c) dimensional transformations (for finite Λ and $\Lambda \to \infty$, arbitrary mass).

As follows from our discussion, the symmetry mappings are generated in QFT by formal charge operators which are nonlocal in time. One can say that in the case of symmetry mapping we replace the broken local symmetry, generated by a local nonconserved current, by a symmetry mapping, generated by a conserved current which is nonlocal in time. Indeed, as the simplest soluble models show,[42] the derivative of the field operator with respect to the parameters cannot be expressed by the canonical variables at a given time. In this paper we use the nonlocal Schwinger-Peierls principle which defines such a derivative as an explicitly given operation which is nonlocal in time.

The description of the change of the parameters in QFT leads directly to the problem of inequivalent representations.[19] Even if we consider the theory with cutoff (ultraviolet divergences removed, finite Λ), the numerical changes of the parameters do not represent sufficiently smooth perturbations, and in order to stay in the same Fock space, one has to introduce local changes of the parameters. Already in the simplest case of a mass shift in free-field theory one can observe that the global change $m \to m + \delta m$ cannot be represented by unitary mappings in the Fock space generated by the free field with mass m. In order to obtain unitary implementability one has to consider local changes of the mass parameter

$$m \to m + \delta m \, \xi(x) , \tag{7.1}$$

where, for example, $\xi(x) \subset D(R^4)$.[43] We see that the change $m \to m + \delta m$ can be obtained only if we perform an infinite number of steps, each one of which can be implemented unitarily.[44] In such a simple case we have the following two possibilities[45]:

(a) One can introduce a large space of states $\mathcal{H}$, containing reducible representations of canonical commutation relations (CCR) and all irreducible sectors which are needed in order to pass continuously from the global value of the mass parameter m to the value $m + \delta m$.

(b) One can consider only the mappings of the field operators. In such an algebraic approach *the change of a mass parameter for free fields is an automorphism of the field algebra which is not unitarily implementable.* More explicitly, one can express the creation and annihilation operators of the field with mass $m + \delta m$ as a linear form in terms of creation and annihilation operators of the field with mass m.[46] However, even when the unitary transformation performing the global mass shift does not exist, the relation between the operators is finite and well defined.

In principle both possibilities (a) and (b) are correct, but the second possibility seems to be more promising in the complicated case of nonlinear interactions. In particular, the transformation changing the normalization of the field [see (1.7)] is perfectly well defined in an algebraic framework, but if we wish to represent such a change as a transformation on the space of states, we will be forced to introduce mutually orthogonal Fock spaces for every value of the normalization constant. Similarly, the existence of the quantum generators in the sense of a derivation[12] suggests that one should not ask how the space of states transforms under symmetry mappings (it will always be a wild change), but how the transformation of the field operators appears, expressed in terms of multiple commutators.

For the regularized theory one should introduce the local changes of the parameters, i.e., replace the numerical parameters α_i by external space-time-dependent fields $\alpha_i(x)$. Formula (1.2) takes the form

$$\frac{\delta \Phi_\Lambda[x;\{\alpha\}]}{\delta \alpha_i(y)} = i \int d^4z \, \theta(x_0 - y_0) \times \left[\Phi_\Lambda[x;\{\alpha\}], \frac{\delta \mathcal{L}_R[z;\{\alpha\};\Lambda]}{\delta \alpha_i(y)} \right] , \tag{7.2}$$

where $\mathcal{L}_R[z;\{\alpha\};\Lambda]$ is the functional of external fields $\alpha_i(x)$, converging to the usual Lagrangian density $\mathcal{L}_R(z;\alpha_i;\Lambda)$ if $\alpha_i(y) \to \alpha_i$.[47] Having formula (7.2) one can describe the shift $\alpha_i \to \alpha_i + \xi(y)\delta\alpha_i$ as generated by

$$\delta \mathcal{L}_R[x_0;\{\alpha\};\Lambda] = \int_0^{x_0} d^4z \int d^4y \, \xi(y) \frac{\delta \mathcal{L}_R[z;\{\alpha\};\Lambda]}{\delta \alpha_i(y)} \delta \alpha_i . \tag{7.3}$$

If we assume that $\delta \mathcal{L}_R[z]/\delta\alpha_i(y)$ is local, i.e., proportional to $\delta^4(z-y)$, the function $\xi(y)$ introduces a space-time cutoff as well as regularizes the product of the θ function and the commutator on the right-hand side of (6.5).[48]

However, formula (7.2) is not easily tractable for explicit calculations with nonlinear interaction terms. In particular, it is not known how the counterterms in the regularized Lagrangian density $\mathcal{L}_R[z;\{\alpha\};\Lambda]$ [see (7.2)] depend on the derivatives of the external fields $\alpha_i(x)$.

Finally, a few words about the limit $\Lambda \to \infty$. For finite Λ, the theory is canonical and difficulties with symmetry mappings correspond to the problem of the dependence of the irreducible representations of CCR on the dynamics. However, in the limit $\Lambda \to \infty$, owing to the presence of infinite renormalization constants, the field algebra ceases to be a canonical one. In such a case the role of basic algebraic objects can be attached to the asymptotic fields[49] and to the S matrix, mapping unitarily "in" onto "out" fields. The Borchers classes of interpolating fields giving the same value of the S matrix now play the role of different representations of CCR. It should be added that the Schwinger-Peierls nonlocal action principle in its standard form (1.2) or (7.2) should be applied to the dynamical changes which do not modify the ingoing asymptotic fields determining the four-dimensional algebra of asymptotic fields.[50] Because fixing the normalization conditions for asymptotic fields means giving up the normalization conditions for the interpolating field, we see that the discussion of symmetry mappings via the Schwinger-Peierls action principle leads to the consideration of a noncanonical algebra of renormalized field operators. Only in such an algebra without a normalization condition does the limit $\Lambda\to\infty$ have a very good chance to exist, and also it does make sense to consider the renormalized field operators parametrized by continuous values of coupling constants.

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APPENDIX A: RGC EQUATIONS FOR MASSLESS Φ^4 THEORY

Let us write the required renormalized Lagrangian for the massless $g_0\phi^4$ theory,

$$\begin{aligned}\mathcal{L}_R(\Phi_\Lambda; g, \kappa; \Lambda) &\equiv \mathcal{L}_R[\Phi_\Lambda; g, \tilde{Z}_3, \tilde{Z}_1; \Lambda]\\ &= \tfrac{1}{2}\tilde{Z}_3\left(\frac{\Lambda^2}{\kappa^2}; g\right)\partial^\mu\Phi_\Lambda\partial_\mu\Phi_\Lambda\\ &\quad - g\tilde{Z}_1\left(\frac{\Lambda^2}{\kappa^2}; g\right)(\Phi^4)_\Lambda - \tfrac{1}{2}\Lambda^2\tilde{m}^2\left(\frac{\Lambda^2}{\kappa^2}; g\right)\Phi_\Lambda^2 ,\end{aligned} \tag{A1}$$

where $(\Phi^4)_\Lambda$ denotes the regularization of the local power Φ^4 by means of a kernel which is symmetric, i.e.,

$$\Phi\,\frac{\delta(\Phi^4)_\Lambda}{\delta\Phi} = 4(\Phi^4)_\Lambda , \tag{A2}$$

and is local in time. The paraemter κ describes the normalization point for the propagator

$$p^2 G_\Lambda^{(2)}(p^2)\,|_{p^2=-\kappa^2} = 1 \tag{A3}$$

and provides the definition of the renormalized coupling constant

$$\Gamma_\Lambda^{(4)}(p_1, \ldots, p_4)\,|_{p_i p_j = (\kappa^2/3)(4\delta_{ij}-1)} = g \tag{A4}$$

The wave renormalization constants $\tilde{Z}_3$ and $\tilde{Z}_1$ can be determined by the asymptotic behavior of $G_\Lambda^{(2)}(p^2)$ and $\Gamma_\Lambda(p_1, \ldots, p_4)$. Introducting the three functions

$$\begin{aligned}\tilde{\gamma}_{\Lambda/\kappa}(g) &= \tfrac{1}{2}\kappa\frac{\partial}{\partial\kappa}\ln\tilde{Z}_3,\\ \tilde{\delta}_{\Lambda/\kappa}(g) &= \kappa\frac{\partial}{\partial\kappa}\ln\tilde{Z}_1 ,\\ \tilde{\kappa}_{\Lambda/\kappa}(g) &= \kappa\frac{\partial}{\partial\kappa}\tilde{m}^2 ,\end{aligned} \tag{A5}$$

we obtain

$$\kappa\frac{\partial}{\partial\kappa}\mathcal{L}_R = \tilde{\gamma}_{\Lambda/\kappa}(g)\,\tilde{Z}_3\,\partial^\mu\Phi\partial_\mu\Phi - \tilde{\delta}_{\Lambda/\kappa}(g)\,g(\Phi^4)_\Lambda - \tfrac{1}{2}\Lambda^2\tilde{\kappa}_{\Lambda/\kappa}{}^2(g)\Phi^2 \tag{A6}$$

and

$$\frac{\partial}{\partial g}\mathcal{L}_R = \frac{1}{2}\frac{\partial\ln\tilde{Z}_3}{\partial g}\tilde{Z}_3\partial^\mu\Phi\partial_\mu\Phi - \frac{\partial\ln\tilde{Z}_1}{\partial g}g\tilde{Z}_1(\Phi^4)_\Lambda - \tilde{Z}_1(\Phi^4)_\Lambda - \tfrac{1}{2}\Lambda^2\frac{\partial\tilde{m}^2}{\partial g}\Phi^2. \tag{A7}$$

The conventional Callan-Symanzik coefficient functions are obtained if we observe that the renormalized coupling constant g also depends on κ. These functions are defined by the following full derivatives with respect to the κ parameter:

$$\begin{aligned}\beta_{\Lambda/\kappa}(g) &= \kappa\frac{\partial}{\partial\kappa}g,\\ \gamma_{\Lambda/\kappa}(g) &= \tfrac{1}{2}\kappa\frac{d}{d\kappa}\ln\tilde{Z}_3\\ &= \tilde{\gamma}_{\Lambda/\kappa}(g) + \tfrac{1}{2}\beta_{\Lambda/\kappa}(g)\frac{\partial\ln\tilde{Z}_3}{\partial g},\\ \delta_{\Lambda/\kappa}(g) &= \kappa\frac{d}{d\kappa}\ln\tilde{Z}_1\\ &= \tilde{\delta}_{\Lambda/\kappa}(g) + \beta_{\Lambda/\kappa}(g)\frac{\partial\ln\tilde{Z}_1}{\partial g},\\ \kappa_{\Lambda/\kappa}{}^2(g) &= \kappa\frac{d}{d\kappa}\tilde{m}^2\\ &= \tilde{\kappa}_{\Lambda/\kappa}{}^2(g) + \beta_{\Lambda/\kappa}(g)\frac{\partial\tilde{m}^2}{\partial g}.\end{aligned} \tag{A8}$$

If we use the relation

$$\partial^\mu\Phi\partial_\mu\Phi = \partial^\mu(\Phi\partial_\mu\Phi) - \Phi\Box\Phi \tag{A9}$$

and substitute the field equation

$$\tilde{Z}_3\Box\Phi = -g\tilde{Z}_1\frac{\delta(\Phi^4)_\Lambda}{\delta\Phi} - \Lambda^2\tilde{m}^2\Phi \tag{A10}$$

we obtain for the regularizations satisfying Eq. (A2) the following equation:

$$\begin{aligned}\left[\kappa\frac{\partial}{\partial\kappa} + \beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g}\right]\mathcal{L}_R - \gamma_{\Lambda/\kappa}(g)\tilde{Z}_3\partial^\mu(\Phi\partial_\mu\Phi) &= [4g\gamma_{\Lambda/\kappa}(g) - g\delta_{\Lambda/\kappa}(g) - \beta_{\Lambda/\kappa}(g)]\tilde{Z}_1(\Phi^4)_\Lambda\\ &\quad - \tfrac{1}{2}\Lambda^2[\kappa_{\Lambda/\kappa}{}^2(g) - 2\gamma_{\Lambda/\kappa}(g)\tilde{m}^2]\Phi^2.\end{aligned} \tag{A11}$$

Because in the $\lambda\Phi^4$ model formula (2.11) gives

$$N_\mu^\Lambda(x) = \tilde{Z}_3\Phi(x)\partial_\mu\Phi(x), \tag{A12}$$

we obtain

$$\partial^\mu N_\mu^\Lambda(x) = \left[\kappa \frac{\partial}{\partial \kappa} + \beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g}\right]\mathcal{L}_R \quad (A13)$$

provided that

$$\beta_{\Lambda/\kappa}(g) = 4g\gamma_\Lambda(g) - g\delta_{\Lambda/\kappa}(g), \quad (A14a)$$

$$\kappa_{\Lambda/\kappa}{}^2(g) = 2\gamma_{\Lambda/\kappa}(g)\tilde{m}^2. \quad (A14b)$$

Using formula (A8) one can write relation (A14a) as

$$\kappa \frac{\partial}{\partial \kappa}(\tilde{Z}_1 \tilde{Z}_3{}^{-2} g) = 0, \quad (A15)$$

i.e., one can define the unrenormalized coupling constant g_0 independent of κ as follows:

$$g_0 = \tilde{Z}_1 \tilde{Z}_3{}^{-2} g. \quad (A16)$$

Similarly relation (A14b) leads to the relation

$$\kappa \frac{\partial}{\partial \kappa}(\tilde{Z}_3{}^{-1}\tilde{m}^2) = 0 \quad (A17a)$$

or

$$Z_3{}^{-1}\tilde{m}^2 = M(g_0), \quad (A17b)$$

and one gets the definition of the unrenormalized mass counterterm (2.13)

$$\delta m_0{}^2 = \tfrac{1}{2}\Lambda^2 \tilde{Z}_3{}^{-1}\tilde{m}^2 = \tfrac{1}{2}\Lambda^2 M(g_0). \quad (A18)$$

It should be stressed that relations (A16) and (A18) came out from purely algebraic considerations with the renormalized regularized Lagrangian (A1) expressed only in terms of renormalized parameters.

APPENDIX B: NONLOCAL QUANTUM ACTION PRINCIPLE AND NONCONSERVED CURRENTS

Schwinger's quantum action principle can be written as (see e.g. Ref. 8)

$$\delta\langle \alpha_2, t_2 | \alpha_1, t_1\rangle = i\langle \alpha_2, t_2 | \delta W_{21} | \alpha_1, t_1\rangle \ , \quad (B1)$$

where

$$W_{21} = \int_{t_1}^{t_2} d^4x (x) \ . \quad (B2)$$

Writing

$$\delta\langle \alpha_2, t_2|\alpha_1, t_1\rangle = \delta(\langle \alpha_2, t_2|)|\alpha_1, t_1\rangle + \langle \alpha_2, t_2|(\delta|\alpha_1, t_1\rangle), \quad (B3)$$

we obtain (B1) for *any* variation of W_{21} if

$$\delta|\alpha_1, t_1\rangle = -i\,\delta F^{\text{nonl}}(t_1)|\alpha_1, t_1\rangle \ , \quad (B4)$$

where

$$\delta F^{\text{nonl}}(t) = \delta\int_{-\infty}^{t} d^4x \mathcal{L}(x) \ . \quad (B5)$$

We see that the change of state vectors at time t induced by the change of the action operator (B2) is given by an infinitesimal nonlocal unitary transformation

$$U = I - i\,\delta F^{\text{nonl}}(t_1) \ , \quad (B6)$$

and the changes of operators $\hat{O}(t)$ are given by the formula

$$\delta\hat{O}(t) = i[\hat{O}(t), \delta F^{\text{nonl}}(t)] \ . \quad (B7)$$

If the change of W_{21} is due to the variation $\delta\alpha$ of the parameter α in $\mathcal{L}$ one can write

$$\delta F^{\text{nonl}}(t) = \int_{-\infty}^{t} d^4x' \frac{\partial\mathcal{L}(x')}{\partial\alpha}\delta\alpha \ , \quad (B8)$$

and the dependence of $\hat{O}(t)$ on α satisfies the generalized Heisenberg equation

$$\frac{\partial\hat{O}(t)}{\partial\alpha} = i[\hat{O}(t), D^{\text{nonl}}(t)], \quad (B9)$$

where the nonlocal operator playing the role of the generator has the form

$$D^{\text{nonl}}(t) = \int_{-\infty}^{t} d^4x' \frac{\partial\mathcal{L}(x')}{\partial\alpha} = \int_{-\infty}^{+\infty} dt'\theta(t-t')\int d^3x' \frac{\partial\mathcal{L}(\vec{x}', t')}{\partial\alpha} \ . \quad (B10)$$

The operator $\theta(t)$ can be treated as an inverse of time differentiation, i.e., we have

$$\frac{d}{dt}D^{\text{nonl}}(t) = \int d^3x' \frac{\partial\mathcal{L}(\vec{x}', t)}{\partial\alpha} \quad (B11)$$

or

$$\partial^\mu D_\mu^{\text{nonl}}(x) = \frac{\partial\mathcal{L}(x)}{\partial\alpha} \ , \quad (B12)$$

where

$$D^{\text{nonl}}(t) = \int d^3x D_0^{\text{nonl}}(\vec{x}, t) \quad (B13)$$

and D_μ^{nonl} has three vanishing space components.

Relation (B12) can be also solved by a covariant inverse of the four-dimensional gradient operator,

$$\partial^\mu a_\mu^{\text{ret}}(x) = \delta^4(x) \ , \quad (B14)$$

where $a_\mu^{\text{ret}}(x, x_0) = 0$ for $x_0 > 0$. A good choice for $a_\mu^{\text{ret}}(x)$ is

$$a_\mu^{\text{ret}}(x) = \partial_\mu \Delta^{\text{ret}}(x) = \frac{1}{2\pi}x_\mu \delta'(x^2) \ , \quad (B15)$$

where Δ^{ret} is the retarded Green's function for the free massless field. Using (B14) we obtain

$$D_\mu^{\text{nonl}}(x) = \int d^4x' \Delta^{\text{ret}}_{,\mu}(x - x')\frac{\partial\mathcal{L}(x')}{\partial\alpha} \quad (B16)$$

and

$$D^{\rm nonl}(t) = \int d^3x \int d^4x' \Delta^{\rm ret}_{,0}(\vec{x}-\vec{x}', t-t') \, \frac{\partial \mathcal{L}(x')}{\partial \alpha} \,. \tag{B17}$$

Formula (B9) with $D^{\rm nonl}(t)$ given by (B17) describes a covariant version of the nonlocal Schwinger-Peierls action principle.

APPENDIX C: RENORMALIZATION-GROUP AND RENORMALIZED SCALE TRANSFORMATIONS IN SOFT RENORMALIZATION SCHEMES

If follows from formulas (5.17) and (6.6) that in the two renormalization schemes discussed in Secs. V and VI the change of the physical mass implies the appearance in the trace of the energy-momentum tensor of "soft" as well as "hard" terms.[39] However, one can treat the nonvanishing mass term as a perturbation performed on the massless theory. In such a way one introduces the soft mass parameter m_s, and the modification of the regularized renormalized Lagrangian (2.2) appears as

$$\begin{aligned}\mathcal{L}_R(\Phi_\Lambda; g, \kappa; \Lambda) \to \mathcal{L}_R^{m_s}(\Phi_\Lambda; g, \kappa; \Lambda) &= \mathcal{L}_R(\Phi_\Lambda; g, \kappa; \Lambda) \\ &\quad - \tfrac{1}{2} m_s^2 \kappa_{\Lambda/\kappa}^2(g) \tilde{Z}_3 \Phi_\Lambda^2 \,.\end{aligned} \tag{C1}$$

The function $\kappa_{\Lambda/\kappa}^2(g)$ can be determined from the additional normalization condition, describing the dependence of the inverse propagator on the mass parameter.[51]

(a) Renormalization-group transformations. In order to obtain the RGC equation (6.16) one should use formulas (5.4) and (C1). The time-independent generator of the renormalization-group transformations

$$\begin{aligned}R^\Lambda &= \gamma_{\Lambda/\kappa}(g) N^\Lambda(t) \\ &\quad - \int_{-\infty}^{t} d^4y \left[\beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g} + \alpha_{\Lambda/\kappa}(g)\, m_s \frac{\partial}{\partial m_s} + \kappa \frac{\partial}{\partial \kappa}\right] \\ &\quad \times \mathcal{L}_R(y)\end{aligned} \tag{C2}$$

leads to the formula

$$\begin{aligned}[R^\Lambda, \Phi_\Lambda(x; g, \kappa, m_s)] = -i\Big(&\gamma_{\Lambda/\kappa}(g) + \beta_{\Lambda/\kappa}(g)\frac{\partial}{\partial g} \\ &+ \alpha_{\Lambda/\kappa}(g) m_s \frac{\partial}{\partial m_s} + \kappa\frac{\partial}{\partial \kappa}\Big) \\ &\times \Phi_\Lambda(x; g, \kappa, m_s)\,,\end{aligned} \tag{C3}$$

which has a limit as $\Lambda \to \infty$ and can be written in integrated form as

$$\begin{aligned}V(\lambda)\Phi(x; g, \kappa, m_s)V^{-1}(\lambda) \\ = Z^{1/2}(\lambda; g)\Phi\left(x; \bar{g}(\lambda; g), \frac{\kappa}{\Lambda}, \bar{m}_s(\lambda; g)\right),\end{aligned} \tag{C4}$$

where

$$\lambda \frac{\partial}{\partial \lambda} \ln \bar{m}_s(\lambda; g) = \alpha(\bar{g}) \tag{C5}$$

and

$$\alpha(g) = \lim_{\Lambda\to\infty} \alpha_{\Lambda/\kappa}(g)\,.$$

We see that the soft mass parameter plays the role of a κ-dependent coupling constant, and it is changed under the transformations of the renormalization group. In the limit $m_s \to 0$ we obtain from (C2) and (C4) the formulas for the massless theory discussed in Sec. III.

(b) Renormalized scale transformations. The divergence of the renormalized regularized dilatation current is given by the formula

$$\begin{aligned}\partial^\mu S_\mu^\Lambda &= \tilde{T}^{R\mu}_{\ \mu} \\ &= -\left(\kappa\frac{\partial}{\partial\kappa} + m_s\frac{\partial}{\partial m_s} + \Lambda\frac{\partial}{\partial\Lambda}\right)\mathcal{L}_R^{m_s}\,.\end{aligned} \tag{C6}$$

One can introduce the following formula for the global renormalized scale transformations:

$$U_R(l)\Phi_\Lambda(x; g, \kappa, m_s)U_R^{-1}(l) = l\,\Phi_\Lambda\left(lx; g, \frac{\kappa}{l}, \frac{m_s}{l}\right). \tag{C7}$$

In the limit $\Lambda \to \infty$ the transformations (C7) coincide with the dimensional transformations and describe the symmetry of the theory. Using (C2) one can also write in the limit $\Lambda \to \infty$ another form of the renormalized scale transformations

$$\begin{aligned}\tilde{U}_R(l)\Phi(x; g, \kappa, m_s)\tilde{U}_R^{-1}(l) \\ = lZ^{1/2}(\lambda; g)\Phi(lx; \bar{g}(\lambda; g), \kappa, e^{-1/l^2}\bar{m}_s(l; g))\,.\end{aligned} \tag{C8}$$

Formulas (C7) and (C8) generalize in the presence of the nonvanishing soft mass parameter m_s two forms (4.19) and (4.23) of the renormalized scale transformations for the massless theory.

*On leave from the University of Wroclaw, Wroclaw, Poland.

[1]A typical example: massless $g_0\phi^4$ in four dimensions.

[2]In principle even in QFT of one field with one coupling constant one can introduce *two* independent normalization momenta—one for the 2-point propagator and the second providing the interpretation of the renormalized coupling constant. For simplicity we shall consider further the theory of one real field with one coupling constant, and we assume the presence of one normalization parameter κ (see Ref. 1).

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[4]M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954).

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[7]We would like to stress that we do not assume the existence of the finite limit $\Lambda\to\infty$ for the Lagrangian $\mathcal{L}_R$ but for the field operator and the field equations.

[8]R. F. Peierls, Proc. R. Soc. London A214, 143 (1952); J. Schwinger, Proc. Nat. Acad. Sci. (USA) 37, 452 (1951); 37, 455 (1951). The problem of divergences occurring in (1.2) for a theory with finite cutoff Λ (necessity of space-time cutoff) is briefly discussed in Sec. VII.

[9]We refer to the renormalized scale transformations as the modified canonical scale transformations leading for $\beta=0$ to anomalous dimensions, and for $\beta\neq 0$ leading to the operator form of dimensionality $\hat{d}=1+\gamma+\beta\partial/\partial g$. The renormalized scale transformations take into consideration the scale dependence of effective renormalization constants, e.g., see J. Lukierski and A. T. Ogielski, Phys. Lett. 58B, 57 (1975), and leave the renormalized theory invariant [see (1.4)].

[10]See e.g. K. Symanzik, in Proceedings of the 1973 Summer Institute in Theoretical Physics, Mexico [Springer Lecture Notes in Physics No. 32 (Springer, New York 1975), p. 20].

[11]For a discussion of some mathematical problems related to the existence of the quantum generator defined as a space integral of the fourth component of a current see e.g. C. A. Orzalesi, Rev. Mod. Phys. 42, 381 (1970).

[12]The limit in the sense of a derivation on the elements of the algebra $\mathfrak{A}$ is defined as follows: $\lim_{n\to\infty}A_n \to \bar{A}$; if for all $a\in\mathfrak{A}$, $\lim_{n\to\infty}[A_n, a]=[A,a]$. The convergence of a charge operator (generator) in the sense of a derivation does not imply either strong or weak operator convergence or even convergence in the sense of a sesquilinear form (see Ref. 11). Concerning the notion of derivation see e.g. O. Bratteli and D. W. Robinson, Commun. Math. Phys. 42, 253 (1975).

[13]We shall denote by a lower subscript the *residual* dependence on Λ, which vanishes in the limit $\Lambda\to\infty$ (for example, $\Phi_\Lambda, \gamma_{\Lambda/\kappa}$, etc.).

[14]The transformation generated by $N^\Lambda(t)$ describes constant dilatation of one real scalar field, and it should not be confused with the gauge transformations which can be defined only if two real fields are present. The definition of gauge transformations for real multiplets is described by rotations; see e.g. J. Lopuszanski, Commun. Math. Phys. 14, 158 (1969).

[15]See Ref. 4. In the massless case the coefficients $\beta_{\Lambda/\mu}$ and $\gamma_{\Lambda/k}$ are identical to the Callan-Symanzik coefficient functions.

[16]Such a generator has already been calculated explicitly for the scale-invariant version of the soluble Zachariasen-Thirring model by J. Lukierski and A. T. Ogielski, Trieste Report No. ICTP/75/153; Ann. Phys. (N.Y.) 100, 192 (1976).

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[18]The existence of the renormalized Green's function for massless theories as tempered distributions has been shown by P. Blanchard and R. Seneor, Ann. Inst. Henri Poincaré 23, 147 (1975); but this proof does not prevent the existence of on-mass-shell IR singularities. For a description of IR divergences in massless theories see e.g. K. Symanzik, Commun. Math. Phys. 34, 7 (1973).

[19]The orthogonality of the Fock spaces corresponding to two different values of coupling constant $g\neq g'$ had been first observed by Van Hove; see L. Van Hove, Physica (Utrecht) 18, 145 (1952). It is often called the Van Hove phenomenon. Similar results have been shown for the mass shift transformation; for free-field theory see e.g. M. Guenin and G. Velo, Helv. Phys. Acta 41, 362 (1968); L. Rosen, J. Math. Phys. 13, 918 (1972); W. J. Eachus and L. Streit, Rep. Math. Phys. 4, 161 (1973). If we compare the free theory ($g=0$) and interacting theory ($g'\neq 0$), the Van Hove conclusion is a special case of a result due to Haag, known as the Haag theorem; see R. Haag, Kgl. Danske Vid. Selsk. Mat.—Fys. Medd. 29, No. 12 (1955); D. Hall and A. S. Wightman, *ibid.* 31, No. 5 (1957).

[20]It should be mentioned here that the regularized renormalized Lagrangian theories have their counterparts in statistical mechanics. In particular, massless $\lambda\Phi^4$ theory discussed in Appendix A after passing to Euclidean momenta describes the thermodynamical limit of the four-dimensional Ising model at critical temperature. In order to identify the Ising Hamiltonian with the Euclidean Lagrangian in $\lambda\Phi^4$ theory one should do the following: (a) Write the spin-wave propagator in the Ising model in a linear approximation,

i.e., make a replacement in the "free propagator" of spin excitations

$$4a^{-2}\sum_i \sin^2(\tfrac{1}{2}aq_i) \simeq q_i^{\,2} \quad (a \text{ is a lattice constant}).$$

(b) Choose the cutoff Λ in the Euclidean Lagrangian as fixed by the lattice constant a ($\Lambda = \Pi/a$) and introduce the regularization by restricting the Fourier transforms of the field operator to the values $p_i \leq \Lambda = (\Pi/a)$ ($i = 1, 2, 3, 4$). For more details see e.g. F. Jegerlehner, Lecture Notes, ZIF, Universität Bielefeld, 1976 (unpublished).

[21]G. 't Hooft, Nucl. Phys. B61, 455 (1973).

[22]F. Jegerlehner, Fortschr. Phys. 23, 71 (1975).

[23]S. Weinberg, Phys. Rev. D 8, 3497 (1973).

[24]M. Gomes and B. Schroer, Phys. Rev. D 10, 3525 (1974).

[25]It is sufficient to assume that the regularization is obtained via increasing the order of the differential operator in the free Lagrangian and/or by introducing the special nonlocalities in the interaction term. It has been shown for example by Zimmermann (see Ref. 26) that spacelike regularization is sufficient for deriving finite field equations. For simplicity we shall assume further that the regularized Lagrangian contains only first derivatives of the field. It should be stressed that in the case with higher derivatives in the free Lagrangian (e.g. the Pauli-Villars regularization) the arguments leading to the validity of the relation (1.6) are further valid provided that we use the canonical formalism adapted to the presence of higher derivatives.

[26]W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (M.I.T. Press, Cambridge, Mass., 1970), Vol. I.

[27]This conclusion follows from the formula expressing CS functions in terms of the renormalized Green's functions (see e.g. Ref. 10).

[28]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970); Y. Takahashi, Phys. Rev. D 3, 622 (1971).

[29]The definition of N_μ [see (2.11)] does not depend on the normalization of the field operator, i.e., it has the same form for unrenormalized as well as renormalized field operators.

[30]Any regularization procedure breaks the scale invariance by introducing a cutofflike parameter with a dimension of mass. In particular, in dimensional regularization the parameter plays the role of the coupling constant $g_0(\epsilon)$ in $D = 4-\epsilon$ dimensions. Setting $g_0(\epsilon) = M^\epsilon g_0$, where g_0 is dimensionless, we introduce the mass unit. See e.g. Ref. 21.

[31]See e.g. B. Klaiber, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. XA, p. 141.

[32]J. Kupsch, W. Rühl, and B. C. Yunn, Ann. Phys. (N.Y.) 89, 115 (1975).

[33]See Ref. 24, Sec. IIID [especially formula (3.95)].

[34]See e.g. B. Schroer, Lett. Nuovo Cimento 2, 627 (1971); N. K. Nielsen, Nucl. Phys. B65, 413 (1973).

[35]See e.g. H. Umezawa, in *Renormalization and Invariance in Quantum Field Theory*, Proceedings of the Capri Symposium, 1973 (Plenum, New York, 1974), p.275, and references quoted therein.

[36]J. Lukierski and A. Ogielski, Phys. Lett. 64B, 336 (1976); see also A. Ogielski, Ann. Inst. Henri Poincaré 25, 59 (1976).

[37]In order to justify this statement on a heuristic level one can repeat the argument following Eq. (4.13).

[38]The finite "renormalized square" (or normal product $N_2[\Phi](x)$ in its operator form) can be rigorously obtained also in the framework of counterterm technique; see e.g., S. A. Anikin and A. I. Zavialov, Teor. Mat. Fiz. 26, 162 (1976).

[39]"Hard" terms have "naive" canonical dimensionality $d = 4$ and "soft" terms are superrenormalizable ($d < 4$). The soft terms can be neglected in the divergence of the dilatation current if we consider its Green's functions with all external momenta approaching infinity. It is rather plausible that the soft terms can be also neglected if we discuss the asymptotic values of the form factors of the dilatation current.

[40]We assume that the Lagrangian density depends on coordinates and parameters. The coordinates are always integrated in the action principle. For example, if we write the Lagrangian for a family of free fields with continuous mass parameters (the so-called Licht fields), the mass variable becomes a coordinate [see e.g. J. Lukierski and W. Sienkiewicz, J. Math. Phys. 15, 344 (1974)].

[41]One can only define the renormalized scale transformations as a symmetry for nonvanishing masses if one uses the so-called soft parametrization (see Appendix C).

[42]For example, differentiating the Klein-Gordon (KG) equation with respect to the mass parameter one obtains the equation

$$(\Box - m^2)\,\frac{d\phi_0(x;m^2)}{dm^2} = \phi_0(x;\,m^2),$$

and $d\phi_0(x;m^2)/dm^2$ is related to $\phi_0(x;m^2)$ by a nonlocal operation (the inverse of the KG operator).

[43]It can be mentioned that if the number of space dimensions is ≥ 4, the space-time cutoff is not sufficient and one has to introduce nonlocal smearing in the bilinear mass term.

[44]One can use for example the decomposition of unity into the countable sum of the test functions belonging to Schwartz space D.

[45]We do not consider here the third possibility which is related to the mathematically somewhat obscure notion of the so-called pseudounitary operators. See e.g. W. Weidlich, Nuovo Cimento 30, 803 (1963).

[46]See e.g. the paper of Guenin and Velo in Ref. 19 and also J. Glimm and A. Jaffe, Ann. Inst. H. Poincaré 22, 109 (1975).

[47]Compare with the formalism used by Bogoliubov and Shirkov in their renormalization program of S-matrix elements, where $g \to g(x)$: See N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959); see also D. A. Kirzhnitz, Zh. Eksp. Teor. Fiz. 49, 1544 (1965) [Sov. Phys.—JETP 22, 1059 (1966)].

[48]The difficulty with the point $x_0 = y_0$ occurs also in the definition of Dyson's half-matrix $U(t, -\infty)$ in regularized theory, and it is due to the so-called "surface divergencies." See e.g. E.C. G. Stueckelberg, Phys. Rev. 81, 130 (1951); A. D. Sukhanov, Zh. Eksp. Teor.

Fiz. 43, 1400 (1962) [Sov. Phys.—JETP 16, 993 (1963)].

[49]It should be mentioned that just recently Buchholz proved the existence of the asymptotic condition in massless theory [DESY report, 1976 (unpublished)].

[50]One can introduce also an inhomogeneous form of Schwinger-Peierls nonlocal action principle, in the case when the asymptotic ingoing fields are modified.

[50]In order to be able to discuss scalar theories one should normalize the mass derivative of the inverse propagator at $m_s \neq 0$, for example at $p=0$, $m_s=\kappa$ (see Ref. 22). If one excludes scalar theories, one can normalize the mass derivative at $p^2=-\kappa^2$, $m_s=0$ (see Ref. 23).