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Asymptotically free ϕ^4 theory*

Richard A. Brandt

Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742 and Department of Physics, New York University, New York, New York 10003 †

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Exact properties of asymptotically free $g\phi^4$ theory with negative (renormalized) g are deduced by renormalization-group and other methods. It has been argued that the effective potential $\mathcal{V}(\chi)$ for the model approaches $-\infty$ for $\chi\rightarrow\infty$, so that the model is inconsistent with positivity. It is shown here how this difficulty may be avoided because of deduced results which imply that actually $\mathcal{V}(\chi) = \text{const}\times\chi^2$. These results are exact zero-momentum theorems which state that the proper vertex functions (except for the inverse two-point function) vanish whenever one of their four-momentum arguments vanish. These theorems are deduced as a consequence of the fact that the *exact* field equation of the theory is invariant, apart from mass terms and mass counterterms, to the transformation $\phi(x)\rightarrow\phi(x)+\text{const}$, which only adds a constant (reflection-symmetry breaking) term to the field equation. This partial symmetry and the associated theorems arise as a consequence of renormalization—they are not true order by order in perturbation theory. The perturbation series in g for the vertex functions is therefore *not* an asymptotic expansion when a momentum vanishes. This is either a remarkable property of the model or an indication that the model really is unstable after all.

I. INTRODUCTION

The scalar self-interacting $g\phi^4$ theory is asymptotically free if the renormalized coupling constant g is negative and sufficiently small in magnitude.[1] However, for $g<0$ the model has been widely believed to be inconsistent with positivity, but Symanzik[1] remarked that, because of the vanishing of the bare coupling constant g_0, the usual[2] argument for inconsistency did not apply. Coleman[3] then showed that, as a consequence of the asymptotic freedom (AFD), the effective potential[4] $\mathcal{V}(\chi)$ could be calculated exactly for large values of the classical field χ and approached $-\infty$ for $|\chi|\rightarrow\infty$, thus reinstating in a very convincing way the inconsistency with positivity. In this paper we will show how the theory may get around this difficulty and may yet be consistent. The AFD enables us to establish certain zero-momentum theorems which imply the effective vanishing of $\mathcal{V}(\chi)$ in the exact theory.[5] Although this resolves the positivity problem, it may lead to other difficulties which may ultimately invalidate the model anyway.

In terms of the bare charge g_0, mass m_0, and field operator ϕ_0, the Lagrangian which describes the model is

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0-\tfrac{1}{2}m_0^{\,2}\phi_0^{\,2}-\tfrac{1}{24}g_0\phi_0^{\,4}, \tag{1.1}$$

and the nonvanishing canonical commutation relation is

$$[\phi_0(x),\dot{\phi}_0(y)]\delta(x_0-y_0)=i\delta(x-y). \tag{1.2}$$

So formally the energy density operator $H\sim g_0\phi_0^{\,4}$ becomes large and negative in states with large $|\phi_0|$ if $g_0<0$. If all the quantities appearing in (1.1) were well defined, this argument could be made precise by using (1.2).[2,6] However, the bare objects g_0, m_0, ϕ_0, are each divergent in perturbation theory. All of the divergences in the theory can be absorbed in the renormalization of these objects, as described by the renormalization constants

$$Z_1,\,Z_3,\,\delta m^2, \tag{1.3}$$

if Green's functions are expressed in terms of the renormalized objects

$$\phi=Z_3^{-1/2}\phi_0,\quad g=(Z_3^{\,2}/Z_1)g_0,\quad m^2=m_0^{\,2}+\delta m^2. \tag{1.4}$$

Then (1.1) becomes

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m^2\phi^2-\tfrac{1}{24}g\phi^4\\&+\tfrac{1}{2}(Z_3-1)(\partial_\mu\phi\partial^\mu\phi-m^2\phi^2)\\&+\tfrac{1}{2}Z_3\delta m^2\phi^2-\tfrac{1}{24}(Z_1-1)g\phi^4,\end{aligned} \tag{1.5}$$

and the renormalized field equation

$$-(\Box+m^2)\phi=\tfrac{1}{6}gZ\phi^3-\Delta\phi, \tag{1.6}$$

where

$$Z=Z_1/Z_3,\quad \Delta\equiv\delta m^2, \tag{1.7}$$

is finite in perturbation theory, although the canonical commutation relation

$$[\phi(x),\dot{\phi}(y)]\delta(x_0-y_0)=iZ_3^{-1}\delta(x-y) \tag{1.8}$$

becomes divergent. The renormalized energy density is now[7] $H\sim gN_4(\phi^4)$, which is not necessarily unbounded below for $g<0$ because the finite normal product $N_4(\phi^4)$ is not positive-definite. The renormalized perturbation expansion is in fact perfectly consistent in finite orders.

The above theory with $g<0$ was the first one shown to be asymptotically free (AF).[1] If $|g|$ is sufficiently small and the renormalized perturbation series is an asymptotic one, the behavior of the theory in the deep Euclidean limit is therefore exactly computable, and thus Symanzik[1] could show that Z_3 is finite and $g_0=0$. The Baym[2] argument therefore does not apply, even though ϕ_0 is not divergent. Using the AFD, Coleman presented a new improved argument that the energy spectrum cannot be bounded below. If $\bar{g}\equiv \mathcal{V}^{(4)}(0)$ is negative and sufficiently small in magnitude, the renormalization-group equation[4] satisfied by $\mathcal{V}(\chi)$ could be exactly solved for large $|\chi|$ and the result is inconsistent with Symanzik positivity.[8]

However, there are further implications of AFD which indicate that $\bar{g}$ actually vanishes so that the above difficulty can be avoided. We will deduce these implications using techniques we have previously employed,[9-11] with Ng, in connection with AF non-Abelian gauge theories (NAGT's). In AF theories, because the renormalization constants such as (1.3) can be exactly computed, the renormalized field equations such as (1.6) are exactly known. It turns out that these exact field equations often are invariant to larger symmetry groups than are the classical field equations or the field equations in finite orders of perturbation theory.[9-11] New symmetries can thus be said to arise as a *consequence* of renormalization. We will show in this paper that (1.6) is a case in point. In finite orders, where Z is divergent, (1.6) has no trace of invariance under the "R transformation"[9-14]

$$\phi(x)\to\phi_r(x)\equiv\phi(x)+r \tag{1.9}$$

for $r=$ constant. In the exact theory, on the contrary, $Z=0$ and $Z\phi^2=c$-number, and so (1.6) is symmetric under (1.9) apart from mass terms (i.e., terms of dimension 1). The Ward-Takahashi identities appropriate to this partial symmetry are zero-momentum theorems which imply, among other things, that $\mathcal{V}(\chi)\propto\chi^2$ and so $\bar{g}=0$. These theorems are *not* satisfied in finite orders of perturbation theory.

Although our results may resolve the positivity problem, they imply that the theory has some rather peculiar properties. Because the zero-momentum theorems are not satisfied in finite orders, the perturbation series cannot be an asymptotic one at zero momentum. This is rather surprising because in all the previously known (two-dimensional) cases, perturbation theory is asymptotic.[15] This might indicate that four-dimensional models are intrinsically more complicated, although it might simply be a reflection of the fact that the model really is inconsistent after all. Our point is that there is at present no solid argument for this inconsistency.

It is very important to resolve this consistency problem one way or another. If the model is consistent, it is the only known AF theory with a known particle interpretation (unless the particle structure is destroyed because the perturbation series is also not asymptotic at the mass shell). It can then be added to the list, presently including only NAGT's,[16] of AF theories and may play a role in the ultimate theory which describes the strong interactions. It might even resolve the infrared difficulties of NAGT's and provide for quark and/or color confinement or spontaneous symmetry breaking. We will refrain from speculating about this in the present paper.

The main new results of this paper are the calculation of δm^2 by various renormalization-group (RG) techniques, the methods for deducing the consequences of the renormalization induced partial R invariance, and the methods for dealing with infinite symmetry-breaking terms. Various other properties of the renormalized ϕ^4 model are deduced in the course of the analysis.

The next three sections largely review previous work and are included in order to make the paper reasonably self-contained. Section II reviews the concepts of AFD, symmetry (ordinary and that due to renormalization), and partial symmetry. A simple partial diagrammatic model which exhibits a symmetry due to renormalization is recalled in Sec. III. Properties of the ϕ^4 model are reviewed in Sec. IV: the renormalization conditions, the renormalization group, and the effective potential. In Sec. V we calculate the renormalization constants (1.3) using various RG methods. The ϕ^2 renormalization constant ζ and the directional-dependent renormalization term $\sigma(\xi)(\xi\cdot\partial)^2\phi$ in the finite local field equation are also discussed. These results are used in Sec. VI to deduce that the theory is partially symmetric to the R transformation (1.9). The properties of the R-shifted theory are also discussed. The consequences of the symmetry are deduced in Sec. VIII. These are the zero-momentum theorems for the vertex functions and the lack of asymptotic convergence of the perturbative series. The possible existence of a zero-mass bound state and the possible use of normalization conditions at zero momentum are also discussed here. We conclude in Sec. VIII with a further discussion of our assumptions and their consequences and with a comparison with the related results in NAGT's.

II. PRELIMINARIES

A. Asymptotic freedom

For sufficiently small coupling-constant magnitude $|g|$, a quantum field theory exhibits essentially free field behavior in either the deep ultraviolet or deep infrared limit, depending on the slope of the function

$$\beta(g) \equiv m \frac{\partial}{\partial m} g \bigg|_{g_0 \text{ fixed}} \tag{2.1}$$

at $g=0$, where $\beta(0)=0$. The theory is AF (or ultraviolet free) if $\beta'(0)<0$ provided the coupling constant is in the domain of attraction of the origin, i.e., if it lies between the origin and the next zero, if it exists, of $\beta(g)$. For NAGT's,

$$\beta(g) = B_0 g^3 + \cdots , \tag{2.2}$$

and one has AFD if $B_0<0$ for any sign of g. This is the case for many NAGT's.[16] For ϕ^4 theory,

$$\beta(g) = b_0 g^2 + \cdots , \tag{2.3}$$

and so one has AFD for $g<0$ if $b_0>0$ or for $g>0$ if $b_0<0$. It turns out that $b_0>0$ and so a negative g is required for AFD.

AF theories are interesting because they have computable large-momentum behavior if the perturbation expansion is an asymptotic one and if, as one assumes throughout, the mass-insertion term in the Callan-Symanzik[17] equation is negligible. One obtains essentially free field behavior, with the at most logarithmic deviations governed by the "anomalous dimension" function

$$\gamma(g) = m \frac{\partial}{\partial m} \ln Z_3 \bigg|_{g_0 \text{ fixed}} \tag{2.4}$$

for $g \to 0$. In both NAGT's and ϕ^4 theory, one has

$$\gamma(g) = c_0 g^2 + \cdots . \tag{2.5}$$

The renormalized perturbation expansion is a formal power series for the Green's functions. We must assume that this series determines a unique theory which we will refer to as the *exact* theory determined by the given Lagrangian (and normalization conditions). The asymptotic nature of the series is *not* a sufficient condition for this,[15] but is sufficient to give the exact deep Euclidean behavior if the theory is AF.

B. Symmetry

The classical symmetries of a given Lagrangian are not always maintained in perturbation theory, although many are. The symmetries which are present order by order will remain valid in the exact theory. These in general include Lorentz invariance, the usual internal symmetries, the non-Abelian gauge invariance of NAGT's, and the symmetry under

$$\phi(x) \to -\phi(x) \tag{2.6}$$

in ϕ^4 theory. As another example, consider the (nonrenormalizable) derivative-coupling model described by

$$\mathcal{L}_I = G\bar{\psi}\gamma_\mu\psi\partial^\mu\phi . \tag{2.7}$$

This model is invariant to (1.9) in the absence of ϕ mass terms and counterterms. The presence of symmetries implies relations among Green's functions if the symmetry is unitarily implementable and implies Ward-Takahashi (WT) identities and zero-momentum theorems if the symmetry is spontaneously broken.

It is also useful to consider partial symmetries. These are symmetries which are only broken by terms in the Lagrangian of dimension less than 4.[18-20] Suitable Green's function relations and/or WT identities will then remain valid. As a simple example which is well understood and will be of use to us in the following, consider the Lagrangian

$$\mathcal{L}' = \mathcal{L} + C\phi \tag{2.8}$$

obtained from the symmetric [under (2.6)] ϕ^4 theory Lagrangian (1.5) by the addition of the term $C\phi$ of dimension 1, linear in the field ϕ. The renormalizations (1.4) which renormalize (1.1) also suffice to renormalize (2.8).[19,20]

Although it is true that symmetries of perturbation theory remain valid in the exact theory, the converse is not necessarily true. It is possible that the exact theory possesses *more* symmetry than does the perturbation theory.[9-11] To see how this can come about, consider massless ϕ^4 theory with no mass counterterm. The renormalized field equation is

$$-\Box\phi = \tfrac{1}{6} g Z \phi^3 , \tag{2.9}$$

where, order by order in perturbation theory, the renormalization constant Z is divergent:

$$Z = \lim_{\lambda\to\infty} Z(\lambda) , \qquad Z(\lambda) = \sum_{n=0}^{\infty} g^n Z_n(\lambda) , \quad Z_n(\lambda) \sim (\ln\lambda^2)^n . \tag{2.10}$$

This divergence cancels a similar divergence in ϕ^3 to make (2.9) finite. Equation (2.9) is thus not symmetric to (1.9) in perturbation theory. However, suppose that the sum in (2.10) is such that Z vanishes and sufficiently fast so that $Z\phi^2$ also vanishes. Then the exact Eq. (2.9) is invariant to (1.9), the R symmetry having arisen as a consequence of renormalization, and the exact vertex functions $\Gamma^{(n)}(p_1, \ldots, p_{n-1})$ will satisfy the consequent WT identities[11-14]

$$\Gamma^{(n)}(p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_{n-1}) = 0 . \tag{2.11}$$

This situation should be contrasted to that in the model (2.7), where invariance to (1.9) and the theorems (2.11) (for vanishing ϕ momenta) are valid order by order in perturbation theory.

In reality, a mass counterterm is necessary in (2.9) and so we might as well have a mass term also and consider (1.6). Then, with $Z=0$ and $Z\phi^2$ no worse than a c number, (1.6) will be invariant to (1.9) apart from mass terms. It will be seen in Sec. VII that (2.11) remains valid for $n>2$ in the exact theory, just as happens in the model (2.7) in perturbation theory when mass terms are included there.

III. SIMPLE MODEL

In this section we will illustrate how $Z=0$ can occur by summation of an infinite set of Feynman diagrams in ϕ^4 theory. The chosen diagrams will not contain self-energy parts and will not satisfy crossing symmetry, and so the model is not to be taken seriously. The model, and its generalization to NAGT's, was already discussed in Ref. 11, but we repeat it here in a slightly different way which is more relevant to what will follow. $Z=0$ will be seen to occur for either sign of g, but neglected diagrams invalidate this conclusion in general. We will see in Sec. V that $Z=0$ is valid when *all* diagrams are summed for $g<0$. Nothing is known about Z when all diagrams are summed for $g>0$.

The simplest nontrivial unrenormalized cutoff four-point vertex function is $g_0^2 H_\Lambda(p^2)$, where $H_\Lambda(p^2)$ is the $s=p^2$ channel bubble diagram (see Fig. 1)

$$H_\Lambda(p^2) \equiv \int d^4k[k^2(k+p)^2]^{-1}[\Lambda^2/(\Lambda^2-k^2)]\,. \quad (3.1)$$

The crossed diagrams which also contribute in order g_0^2 will be ignored. The function (3.1) is logarithmically divergent when the cutoff is removed,

$$H_\Lambda(p^2) \underset{\Lambda\to\infty}{\sim} \ln(\Lambda^2/p^2)\,, \quad (3.2)$$

and so can be renormalized by a single subtraction. This subtraction cannot be made at $p=0$, where (3.1) is infrared divergent, and so must be made at a different momentum l, with $l^2=\mu^2$ an arbitrary number. The renormalized amplitude is

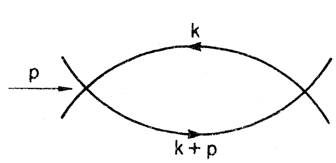

FIG. 1. Feynman diagram for the four-point scattering amplitude in order g^2. The solid lines between vertices are the free propagators $\sim 1/k^2$. The crossed diagrams are neglected.

$$H(p^2;\mu^2) = \int d^4k(k^2)^{-1}\{[(k+p)^2]^{-1} - [(k+l)^2]^{-1}\}\,, \quad (3.3)$$

which is finite except for $p^2=0$, where it is infrared divergent:

$$H(p^2;\mu^2) \underset{p^2\to 0}{\sim} \ln(p^2/\mu^2)\,. \quad (3.4)$$

The model we take for the complete unrenormalized four-point vertex function is that obtained by summing the iterations of (3.1):

$$T_\Lambda(p^2;g_0) = g_0 + g_0^2 H_\Lambda(p^2) + g_0^3[H_\Lambda(p^2)]^2 + \cdots$$
$$= \frac{g_0}{1-g_0 H_\Lambda(p^2)}\,. \quad (3.5)$$

This is illustrated in Fig. 2. The corresponding renormalized amplitude is

$$T(p^2;g,\mu^2) = g + g^2 H(p^2;\mu^2) + \cdots$$
$$= \frac{g}{1-gH(p^2;\mu^2)}\,. \quad (3.6)$$

In the model this is the only nontrivial amplitude, and so there is no mass or wave-function renormalization:

$$\delta m^2 = 0\,, \quad Z_3 = 1\,. \quad (3.7)$$

The vertex-function renormalization constant Z_1^{-1} is given by the value of the partially renormalized amplitude

$$1 + H_\Lambda(p^2)T(p^2;g,\mu^2) \quad (3.8)$$

at $p^2=\mu^2$,

$$Z_1^{-1} = 1 + gH_\Lambda(\mu^2) \underset{\Lambda\to\infty}{\sim} \ln(\Lambda^2/\mu^2)\,,$$

and we have

$$T(p^2;g,\mu^2) = \lim_{\Lambda\to\infty} T_\Lambda(p^2;g_0(\Lambda)) \quad (3.9)$$

with $g_0(\Lambda) = g\,Z_1(\Lambda)$. Thus

$$Z_1 = \lim_{\Lambda\to\infty}[1+gH_\Lambda(\mu^2)]^{-1} = 0\,. \quad (3.10)$$

The exact field equation (2.9) in the model is consequently R-invariant and the implied zero-momentum theorem[21]

$$T(0;g,\mu^2) = 0 \quad (3.11)$$

is correspondingly satisfied, because of (3.4), in the exact theory. This is in contrast to what hap-

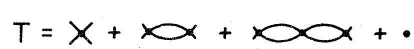

FIG. 2. Model for the exact four-point scattering amplitude. The vertices are the coupling constant g and the propagators are the free ones.

pens in finite orders of perturbation theory where Z_1 and $T(0;g,\mu^2)$ are both divergent.

IV. THE ϕ^4 MODEL

The ϕ^4 theory is described by the renormalized Lagrangian (1.5).[22] In the renormalized perturbation expansions for the model, one obtains in each order finite expressions for the Green's functions $G^{(n)}(p_1\cdots p_{n-1})\equiv G^{(n)}(P_n)$ and one-particle-irreducible (1PI) vertex functions $\Gamma^{(n)}(P_n)$ (these vanish for n odd) in terms of the physical charge g and mass m defined by the renormalization conditions

$$\Gamma(m^2)=0\,,\quad \Gamma'(m^2)=i\,,\quad \Gamma^{(4)}(P_m)=g\,, \tag{4.1}$$

where we have written the simplest vertex function as

$$\Gamma^{(2)}(p)\equiv\Gamma(p^2)\,,$$

and P_m represents the triplet P_3 of momenta satisfying $p_i p_j=\frac{1}{3}m^2(4\delta_{ij}-1)$. Equation (4.1) implies that the propagator

$$G(p^2)\equiv G^{(2)}(p)=-[\Gamma(p^2)]^{-1} \tag{4.2}$$

has a pole at $p^2=m^2$ with residue i.

With the conditions (4.1), the vertex functions are unique functions of the momenta, g, and m in each order. The renormalization constants (1.3) are then given by

$$Z_1^{-1}=1+\tilde{\Xi}(P_m)\,, \tag{4.3}$$

$$Z_3=1+\tfrac{1}{6}gZ_1\tilde{\Pi}'(m^2)\,, \tag{4.4}$$

$$\delta m^2=-\tfrac{1}{6}g\frac{Z_1}{Z_3}\tilde{\Pi}(m^2)\,, \tag{4.5}$$

where the partially renormalized proper vertex part $\tilde{\Xi}(P_3)$ is the amputated Fourier transform of

$$\langle 0|T[\phi^3(w)\phi(x)\phi(y)\phi(z)]|0\rangle, \tag{4.6}$$

and the partially renormalized proper self-energy part $\tilde{\Pi}(p^2)$ is the amputated Fourier transform of

$$\langle 0|T[\phi^3(x)\phi(y)]|0\rangle\,. \tag{4.7}$$

This function is illustrated in Fig. 3. The unrenormalized final loop integrations in (4.3)–(4.5) are cut off as in (3.1) so that Z_1 and Z_3 are logarithmically divergent and δm^2 is quadradically divergent for $\Lambda\to\infty$.

In each order, the Γ's satisfy Callan-Symanzik[17] equations and, in the deep Euclidean limit, approach the vertex functions Γ_{as} of an appropriately normalized zero-mass theory.[23] The Γ_{as} are obtained by summing in each order the leading power behaviors of the Γ's with all the nonleading *logarithmic* corrections. The Γ_{as} satisfy the renormalization-group (RG) equations[24]

$$\left[m\frac{\partial}{\partial m}+\beta(g)\frac{\partial}{\partial g}-n\gamma(g)\right]\Gamma^{(n)}_{as}(P;m^2,g)=0 \tag{4.8}$$

for $P\notin\mathcal{E}$ = set of exceptional momenta[23] = set of (Euclidean) momenta an (even) partial sum of which vanish. Here

$$\beta(g)=b_0g^2+b_1g^3+\cdots\,,\quad b_0=3/32\pi^2 \tag{4.9}$$

and

$$\gamma(g)=c_0g^2+c_1g^3+\cdots\,,\quad c_0=1/2^{11}3\pi^2 \tag{4.10}$$

is the anomalous dimension of the renormalized field ϕ. If the Γ's also involve l insertions of the finite composite operator $N(\phi^2)$, the term $+l(2\gamma-\eta)\Gamma_{as}$ is added to (4.8), where[23]

$$\eta(g)=h_0g+h_1g^2+\cdots\,,\quad h_0=\tfrac{1}{3}b_0 \tag{4.11}$$

is the anomalous dimension of ϕ^2. The solutions to (4.8) satisfy

$$\Gamma^{(n)}_{as}(\lambda P;m^2,g)=\lambda^{4-n}\Gamma^{(n)}_{as}(P;m^2,\bar{g}(\lambda))[A(g,\lambda)]^n\,, \tag{4.12}$$

where $\bar{g}(\lambda)$ is the effective charge defined by

$$\lambda\frac{\partial}{\partial\lambda}\bar{g}(\lambda,g)=\beta(\bar{g}(\lambda,g))\,,\quad \bar{g}(1,g)=g \tag{4.13}$$

and

$$A(g,\lambda)=\exp\left[-\int_1^\lambda\frac{d\lambda'}{\lambda'}\gamma(\bar{g}(\lambda'))\right]\,. \tag{4.14}$$

We have written $\lambda P_n=(\lambda p_1,\ldots,\lambda p_{n-1})$.

If $g<0$ and the power series (4.9) is an asymptotic one, (4.13) can be solved to give[1]

$$\bar{g}(\lambda)\underset{\lambda\to\infty}{\sim}-(b_0\ln\lambda^2)^{-1}-(b_1\ln\ln\lambda^2/b_0{}^3\ln^2\lambda^2)+O(1/\ln^2\lambda^2) \tag{4.15}$$

as an exact asymptotic statement. Equation (4.12) thus gives the exact behavior of $\Gamma(\lambda P)$ for $\lambda\to\infty$ for $P\notin\mathcal{E}$ if the perturbation expansion is an

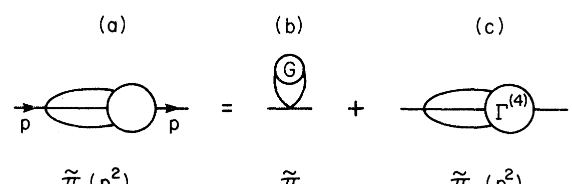

FIG. 3. The partially renormalized proper self-energy part (a) and its decomposition into the partially disconnected piece (b) and connected piece (c). The blobs in (a), (b), and (c) are fully renormalized and the lines between vertices in (a) and (c) are the full propagators G. The explicit loop integrations are regularized but not renormalized. The value of this function at $p^2=m^2$ is proportional to the mass renormalization additive counter term $\delta m^2=\Delta=\Delta_1+\Delta_2$, where Δ_1 comes from the diagrams (b) and Δ_2 comes from the diagrams (c).

asymptotic one and, as we assume throughout, the mass-insertion term in the CS equation remains negligible in the exact theory. The model thus has a computable large-momentum behavior.

Coleman's[3] argument that the energy in the above theory cannot be bounded below is based on properties of the effective potential

$$\mathcal{V}(\chi; m,g) \equiv \sum_n \frac{\chi^n}{n!}\Gamma^{(n)}(0\cdots 0; m,g)\,. \tag{4.16}$$

The asymptotic part of the dimensionless function $\tilde{\mathcal{V}}(\chi/m,g)\equiv(\partial/\partial\chi)^4\mathcal{V}(\chi;m,g)$ satisfies the RG equation[4]

$$\left(\chi\frac{\partial}{\partial\chi}+\tilde{\beta}\frac{\partial}{\partial g}-4\tilde{\gamma}\right)\tilde{\mathcal{V}}_{\rm as}(\chi/m,g)=0\,, \tag{4.17}$$

where $\tilde{\beta}=\beta/(1+\gamma)$ and $\tilde{\gamma}=\gamma/(1+\gamma)$. To solve (4.17), one uses the normalization condition

$$\tilde{\mathcal{V}}(1,\bar{g})=\bar{g}\,, \tag{4.18}$$

where $\bar{g}$ is not $g=\Gamma^{(4)}(P_m)$ or even $\Gamma^{(4)}(000)$ but involves an infinite sum of $\Gamma^{(n)}$'s. Then, if $\bar{g}<0$ and is sufficiently small in magnitude, one has exactly

$$\mathcal{V}(\chi;m,g)\underset{\chi\to\infty}{\sim}\left(\frac{-1}{b_0\ln\chi^2}\right)\left(\frac{\chi^4}{4!}\right)\exp\left[-4\int_0^\infty\frac{d\lambda'}{\lambda'}\tilde{\gamma}(\bar{g}(\lambda'))\right]\,, \tag{4.19}$$

where $\bar{g}(\lambda;\bar{g})$ is the effective charge (4.13) defined by $\tilde{\beta}(g)$ and $\bar{g}$. This result contradicts Symanzik positivity,[8] and thus strongly suggests that the model cannot be stable.

Our purpose in the remainder of this paper is to show that this conclusion may not be valid because of other properties of the model which follow from the AFD. To this end, we turn our attention next to the calculation of the renormalization constants which appear in the renormalized field equation.

V. RENORMALIZATION CONSTANTS

A. Multiplicative

In AF field theories such as the one under consideration, the computability of the large-momentum behavior enables one to exactly compute the behavior of the multiplicative renormalization constants $Z_i(\lambda)$ for large cutoff $\lambda=\Lambda/m$.[25] This large-cutoff behavior is all that is relevant, e.g., in the field equation (1.6). The simplest way to proceed is to use the formal asymptotic definitions

$$p^2G(p^2)\underset{p^2\to\infty}{\longrightarrow} iZ_3^{-1} \tag{5.1}$$

and

$$\Gamma^{(4)}(\lambda P)\underset{\lambda\to\infty}{\longrightarrow} gZ_1\,. \tag{5.2}$$

For $n=2$, (4.1), (4.2), (4.12), and (4.15) give[1]

$$G(p^2)\underset{p^2\to\infty}{\sim}\frac{i}{p^2A(g)}\left[1-\frac{2c_0}{b_0^2\ln p^2/m^2}+O(\ln\ln p^2/\ln^2 p^2)\right]\,, \tag{5.3}$$

where

$$A(g)\equiv A^2(g,\infty) \tag{5.4}$$

in terms of (4.14). Thus Z_3 is finite and positive for sufficiently small $|g|$:

$$Z_3=A(g)<1\,. \tag{5.5}$$

The Källén-Lehmann spectral function $\rho(a)$ is thus positive for large a and is integrable:

$$\rho(a)=2c_0(Ab_0^2a)^{-1}[\ln(a/m^2)]^{-2}+O((\ln a)^{-3}\ln\ln a)\,. \tag{5.6}$$

This means that the bare field operator $\phi_0=Z_3^{1/2}\phi$ has finite (cutoff-independent) matrix elements, and that the renormalized field equal-time commutator (1.8) is finite.

Next (4.1), (4.12) with $n=4$, and (4.15) give

$$g\,Z_1(\lambda;g)\equiv\lim_{\lambda\to\infty}\Gamma^{(4)}(\lambda P;m^2,g)=\bar{g}(\lambda)A^2(g)\,, \tag{5.7}$$

and so the bare coupling constant is

$$g_0(\lambda)=g\,Z_1/Z_3^2=\bar{g}(\lambda)\underset{\lambda\to\infty}{\longrightarrow}0\,. \tag{5.8}$$

We have previously mentioned the significance of the vanishing of g_0.

The same results (5.5) and (5.7) can be deduced more rigorously from the Ng-Young[25] RG equations for the Z_i. For the ϕ^4 model, these equations imply

$$Z_i(\lambda;g)\underset{\lambda\to\infty}{\sim}\exp\left[\int_0^\lambda\frac{d\lambda'}{\lambda'}\gamma_i(\bar{g}(\lambda'))\right]\,, \tag{5.9}$$

with

$$\gamma_i(g)=\lambda\frac{\partial}{\partial\lambda}\ln Z_i(\lambda;g)\,, \tag{5.10}$$

so that

$$\gamma_3(g)=\gamma(g)=c_0g^2+\cdots \tag{5.11}$$

and

$$\gamma_1(g)=2\gamma(g)+\beta(g)/g=b_0g+\cdots\,. \tag{5.12}$$

Substitution of (5.11) and (5.12) into (5.9) gives (5.5) and (5.7), respectively.

Yet a third way to deduce (5.5) and (5.7) is directly from the expressions (4.3) and (4.4) in terms of cutoff (with $\Lambda=\lambda m$) vertex functions. Equations (4.3) and (4.12) for $n=2$ and $n=6$ give (5.7), and (4.4) and (4.12) for $n=2$ and $n=4$ give (5.5).

We can similarly exactly calculate the behavior of the ϕ^2 renormalization constant ζ defined by

$$\phi^2=\zeta N(\phi^2)+D(0)\,, \tag{5.13}$$

where

$$D(x) \equiv \langle 0 | T[\phi(x)\phi(0)] | 0 \rangle = \left(\frac{1}{2\pi}\right)^4 \int d^4p\, e^{ip\cdot x} G(p^2) \underset{x\to 0}{\sim} \frac{-1}{4\pi^2 Z_3} \frac{1}{x^2}. \tag{5.14}$$

Equation (5.13) can be written more precisely by introducing a cutoff λ:

$$\phi_\lambda{}^2 = \zeta(\lambda) N(\phi^2) + D_\lambda(0), \tag{5.15}$$

where

$$D_\lambda(0) = \left(\frac{1}{2\pi}\right)^4 \int d^4p\, G(p)[\lambda^2 m^2/(\lambda^2 m^2 - p^2)]^2 \sim \lambda^2 m^2. \tag{5.16}$$

The normal product $N(\phi^2)$ is of course cutoff-independent. Equation (4.11) now gives

$$\zeta(\lambda) \underset{\lambda\to\infty}{\sim} (\ln\lambda^2)^{h_0/b_0} = (\ln\lambda^2)^{1/3}, \tag{5.17}$$

so that the divergences in (5.15) are explicit. Another way to exhibit this behavior is via the local limit of the point-separated field product:

$$\phi(x)\phi(0) \underset{x\to 0}{\sim} (\ln x^2 m^2)^{1/3} N(\phi^2(0)) + D(x). \tag{5.18}$$

B. Additive

The final renormalization constant we would like to calculate is the additive one, $\Delta = \delta m^2 = m^2 - m_0{}^2$. Equation (4.5) immediately gives

$$\Delta = \Delta_1 + \Delta_2, \tag{5.19}$$

with

$$\Delta_1 = \tfrac{1}{2} g \frac{Z_1}{Z_3} \frac{1}{(2\pi)^4} \int d^4p\, G(p^2) = \tfrac{1}{2} g \frac{Z_1}{Z_3} D(0) \tag{5.20}$$

and

$$\Delta_2 = g \frac{Z_1}{Z_3} \frac{1}{(2\pi)^8} \int d^4p\, d^4q\, G^{(2)}(p) G^{(2)}(q) G^{(2)}(p+q-k) \Gamma^{(4)}(p,q,k) \bigg|_{k^2=m^2}. \tag{5.21}$$

This is illustrated in Fig. 3. The Δ_1 piece is present because we have not normal-ordered the Lagrangian. Its behavior for large cutoff $\Lambda^2 = \lambda^2 m^2$ is already known from (5.5), (5.7), (4.15), and (5.16):

$$\Delta_1(\Lambda) \sim \Lambda^2/\ln\Lambda^2. \tag{5.22}$$

To exactly compute $\Delta_2(\Lambda)$ for large cutoff, a further assumption must be abstracted from perturbation theory. We know of three procedures.

(i) Assume, as is true in each order of perturbation theory, that the leading divergence in (5.21) is the same as for $k=0$. The Λ dependence in the cutoff

$$\Lambda^4(\Lambda^2 - p^2)^{-1}(\Lambda^2 - q^2)^{-1} \tag{5.23}$$

introduced in (5.21) can then be shifted to the vertex functions, and (4.12), (4.15), and (5.7) give

$$\Delta_2(\Lambda) \underset{\Lambda\to\infty}{\sim} \Lambda^2/\ln^2\Lambda^2. \tag{5.24}$$

(ii) Use

$$\zeta^{-1}(\lambda) = \frac{dm_0{}^2}{dm^2}\bigg|_{g_0,\Lambda \text{ fixed}} \tag{5.25}$$

to deduce an RG equation for m_0 and assume that the effective mass-insertion term is negligible in the exact theory. We write

$$\frac{m_0{}^2}{m^2} = 1 + f(\lambda; g) \tag{5.26}$$

and consider the quadratically divergent part

$$f_{as}(\lambda; g) = \lambda^2(f_1 g + f_2 g^2 + \cdots) \tag{5.27}$$

of f. We easily deduce the RG equation

$$\left[1 - \lambda\frac{\partial}{\partial\lambda} + \beta(g)\frac{\partial}{\partial g}\right] f_{as}(\lambda; g) = 0, \tag{5.28}$$

whose solution is

$$f_{as}(\lambda; g) = \lambda^2 f_{as}(1; \bar{g}(\lambda)). \tag{5.29}$$

Thus, if (5.27) is an asymptotic expansion,

$$f(\lambda; g) \underset{\lambda\to\infty}{\sim} \lambda^2[f_1\bar{g}(\lambda) + f_2\bar{g}^2(\lambda) + \cdots], \tag{5.30}$$

and so

$$\Delta_1 \sim m^2\lambda^2\bar{g}(\lambda) \sim m^2\lambda^2/\ln\lambda^2, \tag{5.31}$$

$$\Delta_2 \sim m^2\lambda^2\bar{g}^2(\lambda) \sim m^2\lambda^2/\ln^2\lambda^2, \tag{5.32}$$

exactly as (5.22) and (5.24).

(iii) Use Lehmann's[26] procedure and assume that $\dot{\phi}$ can be commuted at equal times on both sides of the field equation (1.6). We first use (1.8) to deduce the familiar constraint

$$Z_3{}^{-1} = \int da\, \rho(a) \tag{5.33}$$

on the spectral representation

$$\langle 0|[\phi(x),\phi(y)]|0\rangle=\int_0^\infty da\,\rho(a)\Delta(x-y;a), \tag{5.34}$$

where

$$(\Box+a)\Delta(x;a)=0 \tag{5.35}$$

and

$$\dot{\Delta}(x;a)\delta(x_0)=i\delta(x). \tag{5.36}$$

We next commute $\dot{\phi}(y)$ with (1.6) and then take $y_0\to x_0$ to obtain

$$\Delta=m^2-Z_3\int da\,a\rho(a)+\tfrac{1}{2}gZD(0). \tag{5.37}$$

The third term in (5.37) arises from the ϕ^3 term in (1.6) and is not present in the pseudoscalar-meson theory considered by Lehmann.[26] It exactly cancels the Δ_1 piece of Δ, and we are left with the sum rule

$$\Delta_2(\Lambda)=m^2-Z_3\int_0^{\Lambda^2} da\,a\rho(a). \tag{5.38}$$

Using (5.6), we obtain

$$\Delta_2(\Lambda)\sim-\frac{2c_0}{b_0^{\,2}}\frac{\Lambda^2}{\ln^2\Lambda^2}, \tag{5.39}$$

consistent with the previous two results (5.24) and (5.32).

Let us consider the above calculation in more detail. The precise form of (1.6) is the finite local field equation[22]

$$-(\Box+m^2)\phi(x)=\lim_{\xi\to 0}[\tfrac{1}{6}gZ(\xi)\phi(x+\xi)\phi(x)\phi(x-\xi)-\Delta(\xi^2)\phi(x)+\sigma(\xi^2)(\xi\cdot\partial)^2\phi(x)], \tag{5.40}$$

which rigorously describes the theory. The limit $\xi\to 0$ must be taken in a spacelike direction, and we shall take the simplest limit with $\xi_0=0$ and average over the spatial directions of $\vec{\xi}$ to maintain manifest rotational invariance.[27] To proceed, we must *assume* that the equal-time limit $y_0\to x_0$ and the $\vec{\xi}\to 0$ limit commute.[28] Because of the presence of the directional-dependent final term in (5.40), which in the kth order of perturbation theory behaves like

$$\sigma(\xi^2)\sim(\ln\xi^2)^k/\xi^2, \tag{5.41}$$

when we commute the right side of (5.40) with $\dot{\phi}(y)$ and take $y_0\to x_0$ inside the $\xi\to 0$ limit, we obtain a contribution of the form

$$(\ln\lambda^2)^k\vec{\nabla}^2\delta(\vec{x}-\vec{y}). \tag{5.42}$$

This is not inconsistent because a similar term actually arises from the left side when it is properly evaluated. We gave the method for doing this long ago.[28] The general result is

$$F(x)\equiv\int_0^\infty da\,a\rho(a)\dot{\Delta}(x;a)\underset{x_0\to 0}{\sim}\sum_{r=0}^\infty K_r\vec{\nabla}^{2r}\delta(\vec{x}), \tag{5.43}$$

where

$$K_r=\lim_{n\to\infty}K_{rn}, \tag{5.44}$$

with

$$K_{rn}=\int da\,a\rho(a)\left(\frac{\partial}{\partial a}\right)^r e^{-a/n}. \tag{5.45}$$

In perturbation theory,

$$a\rho(a)\sim(\ln a)^k, \tag{5.46}$$

and so only K_0 and K_1 are nonvanishing:

$$F(x)\to K_0\delta(\vec{x})+K_1\vec{\nabla}^2\delta(\vec{x}),\quad K_0\sim\Lambda^2(\ln\Lambda^2)^k,\quad K_1\sim(\ln\Lambda^2)^k, \tag{5.47}$$

and this exactly matches (5.42).

In the exact theory, the behaviors (5.46) add up to (5.6) and so only K_0 in (5.44) is nonvanishing:

$$F(x)\to K_0\delta(\vec{x}),\quad K_0\sim\Lambda^2/\ln^2\Lambda^2. \tag{5.48}$$

K_1, for example, vanishes like $(\ln^2\Lambda^2)^{-1}$. Correspondingly, the exact expression for $\xi_i\xi_j\sigma(\xi^2)$ vanishes for $\xi\to 0$, and so the directional-dependent term is not present in the exact field equation. However, it will not be necessary for us to use this result.

VI. PARTIAL R INVARIANCE

The results of Sec. V explicitly specify the finite local field equation (5.40) which describes the model. For notational simplicity, we shall write (5.40) formally as (1.6) with

$$Z=Z(\xi=0)=Z_1/Z_3=\lim_{\lambda\to\infty}\bar{g}(\lambda)A^2/g=0, \tag{6.1}$$

$$\Delta=\Delta(\xi=0)=\delta m^2=\lim_{\lambda\to\infty}\delta m^2(\lambda), \tag{6.2}$$

and where the direction-dependent term $\sim\partial\partial\phi$ has been suppressed. Note that (6.1) does not imply that (5.40) is trivial—the vanishing of Z is compensated by the divergence of the local field product $\phi^3(x)$. Equation (5.40) is finite in each order of perturbation theory, and we assume that it remains valid in the exact theory.

We introduce now into (1.6) the field

$$\phi_r(x)=\phi(x)+r, \tag{6.3}$$

where r is a constant.[29] The $Z\phi^3$ term becomes the sum of $Z\phi_r^{\,3}$, a term proportional to

$$r^3 Z, \tag{6.4}$$

a term proportional to

$$r^2 Z\phi, \tag{6.5}$$

and a term proportional to

$$rZ\phi^2. \tag{6.6}$$

In view of (5.7) and (4.15), (6.4) and (6.5) vanish because the renormalized field is finite:

$$Z=Z\,\phi=0. \tag{6.7}$$

To evaluate (6.6), we use also (5.15) and (5.17), which give

$$\begin{aligned} Z\phi^2 &= \lim_{\lambda\to\infty} Z(\lambda)[\zeta(\lambda)N(\phi^2)+D_\lambda(0)] \\ &= 0+2\Delta_1/g, \end{aligned} \tag{6.8}$$

because the normal product $N(\phi^2)$ is a finite operator. The result (5.20) has also been used. We thus see that the new field (6.3) satisfies the field equation

$$-(\Box+m^2)\phi_r=\tfrac{1}{6}gZ\phi_r^{\,3}-\Delta\phi_r+C_r, \tag{6.9}$$

which differs from the field equation (1.6) satisfied by ϕ only by the constant term

$$C_r=r(\Delta_2-m^2). \tag{6.10}$$

The presence of the direction-dependent term in (5.40) of course does not change this conclusion.[30]

The field equation (5.40) is thus seen to be partially invariant to the R transformation (1.9). The form

$$-(\Box+m^2)\phi=\tfrac{1}{6}gZ[\phi^3-3\phi D(0)]-\Delta_2\phi \tag{6.11}$$

of (1.6) is invariant apart from the mass terms $m^2\phi$ and $\Delta_2\phi$. The consequences of this partial symmetry will be deduced in the following section, after the structure of (6.9) is discussed below. The procedure followed up to now is similar to that we have previously used to argue that gauge theories with AFD are invariant under the analog of (1.9),[9-11] but is more rigorous because here we know the exact field equation (5.40) whereas only a gauge invariantly regularized version is known in the gauge field case. Because we here obtain only partial R symmetry, we must now deviate from the methods used in Refs. 9–11.

The structure of the theory described by (6.9) has been thoroughly studied in the literature.[19,20,31,32] It is the simplest one of symmetry breaking ($\phi\to-\phi$) by a linear term ($C\phi$) in the Lagrangian. The $C=0$ theory corresponding to (1.6) is obtained from the unrenormalized theory by renormalizing the bare quantities g_0, ϕ_0, m_0 by absorbing the divergences in the renormalization constants Z_1, Z_3, and δm^2. Because each connected diagram of the $C\neq 0$ theory is given, apart from an overall factor, by a diagram of the $C=0$ theory evaluated with a certain number of vanishing external momenta, all divergences in the $C\neq 0$ theory can also be removed by renormalization of g_0, ϕ_0, and m_0 and the implied renormalization of $C_0=-v\Gamma_0(0)=Z_3^{-1/2}C$ determined by $v_0\equiv\langle 0\,|\,\phi_0\,|0\rangle=Z_3^{1/2}\langle 0\,|\,\phi\,|0\rangle\equiv Z_3^{1/2}v$. The renormalized theory is then expressed in terms of the two parameters (g,m) plus the third parameter C, or more conveniently $v=\langle\phi\rangle$, and one has

$$F^{(n)}(g,m,v)=\sum_{j=0}^{\infty}\frac{v^j}{j!}F^{(n,j)}(g,m), \tag{6.12}$$

where $F^{(n)}(g,m,v)$ is a connected Feynman amplitude in the $C\neq 0$ theory with n external lines and $F^{(n,j)}(g,m)$ is the connected Feynman amplitude in the $C=0$ theory with $n+j$ external lines, n as in $F^{(n)}$ and the remaining j amputated and evaluated at zero momenta. This is illustrated in Fig. 4.

If Δ_2, and hence (6.10), is finite, the perturbation expansion obtained by iteration of (6.9) is finite, as described above. If, on the other hand, (6.10) is infinite, as is suggested by the calculations of Sec. V B, then the Green's function obtained by iteration of (6.9) will not be finite in finite orders, although (6.3), of course, is still a finite exact solution of (6.9). For example, the vacuum expectation value of (6.9) gives

$$\begin{aligned} -m^2 r &= \tfrac{1}{6}gZ[3rD(0)+r^3]-[\tfrac{1}{2}gZD(0)+\Delta_2]r \\ &\quad +r(\Delta_2-m^2), \end{aligned} \tag{6.13}$$

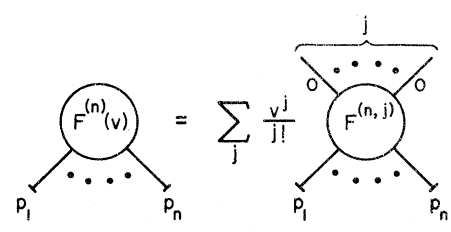

FIG. 4. Representation of a general Feynman amplitude $F^{(n)}(v)$ in the $g\phi^4+C\phi$ theory in terms of Feynman amplitudes $F^{(n,j)}$ in the symmetric $g\phi^4$ theory with j extra zero-momentum amputated external lines for $j=0, 1, 2, \ldots$. Here $v=\langle 0|\phi|0\rangle$ in the unsymmetric theory.

in which the Δ_1 piece of Δ is seen to cancel the divergence in $\langle \phi_r{}^3 \rangle$ and the Δ_2 piece is seen to cancel the divergence in C_r. [Note that $Z = 0$ makes (6.13) linear in r, unlike in finite orders of perturbation theory where (6.13) is a nonlinear equation for r.[19]] It is actually sufficient for our purposes to consider the theory defined by (6.9) with C_r unspecified but chosen so that $\langle \phi \rangle = r$ in each order. This theory is finite order by order and sums to the exact theory given by (6.9) and (6.10) since both theories are given by (6.12) in terms of the $C = 0$ theory.

From either point of view, the field equation (6.9) and (6.10) is finite and local. The fact that it has arisen from the symmetric field equation (1.6) by means of the R transformation (6.3) will be shown in the following section to have strong implications for the structure of the symmetric theory.

VII. CONSEQUENCES

We assume that (6.9), with Z, Δ, and

$$Z_3 = i \int d^4x\, \delta(x_0)[\phi(x), \dot{\phi}(0)],$$

the same as in the $C = 0$ theory (1.6), has a unique solution which approaches the $C = 0$ solution for $C_r \to 0$. We have two expressions for the exact Green's functions $G_r^{(n)}$ for this solution in terms of the Green's function $G^{(n)}$ of the $C = 0$ theory: that given by (6.3) $(p_{n+1} \equiv -p_1 - \cdots - p_n)$,

$$G_r^{(n)}(p_1 \cdots p_n) = G^{(n)}(p_1 \cdots p_n) + (2\pi)^4 r \sum_{i=1}^{n+1} \delta^4(p_i) G^{(n-1)}(p_1 \cdots p_{i-1} p_{i+1} \cdots p_n) + \cdots \tag{7.1}$$

and that given by (6.12).[33] The requirement that these $G_r^{(n)}$ be the same gives as consistency conditions zero-momentum theorems satisfied by the $C = 0$ theory which we now deduce.

The one-point function is just the vacuum expectation value of the field, and (6.3) gives

$$v \equiv \langle 0 | \phi_r | 0 \rangle = r. \tag{7.2}$$

For the two-point Green's function, (6.3) gives

$$G_r^{(2)}(p) = G^{(2)}(p) + (2\pi)^4 r^2 \delta^4(p) \tag{7.3}$$

and (6.12) gives

$$\begin{aligned} G_r^{(2)}(p) &= G_{r\,\mathrm{conn}}^{(2)}(p) + (2\pi)^4 v^2 \delta^4(p) \\ &= \sum_{j=0}^{\infty} \frac{v^j}{j!} G^{(2,j)}(p0 \cdots 0) + (2\pi)^4 v^2 \delta^4(p), \end{aligned} \tag{7.4}$$

where there are j zeros in $G^{(2,j)}(p0 \cdots 0)$. This is illustrated in Fig. 5. Equality of (7.3) and (7.4) implies $v = r$ [which also follows from the vacuum expectation value (7.2) of (6.3)] and

$$G^{(2,j)}(p0 \cdots 0) = 0 \text{ for } j > 0. \tag{7.5}$$

Decomposing $G^{(2,j)}$ into a sum of products of $G^{(2)}(p)$'s and $\Gamma^{(2+k)}$'s (see Fig. 6) then gives

$$\Gamma^{(2+2k)}(p0 \cdots 0) = 0, \quad k > 0 \tag{7.6}$$

where there are $2k$ zeros in the argument of $\Gamma^{(2+2k)}$. These are our first set of zero-momentum theorems and actually the only ones we will need for the effective potential. However, it is simple to deduce the complete set of theorems.

We proceed to consider the three-point Green's function. Equation (6.3) gives (symbolically)

$$G_r^{(3)} = 3rG^{(2)}\delta + r^3\delta\delta.$$

The second form for $G_r^{(3)}$ gives disconnected contributions, which either match the above expression or vanish as a consequence of (7.6) [(7.6) can, of course, also be deduced in this way], plus connected contributions which must vanish in order to

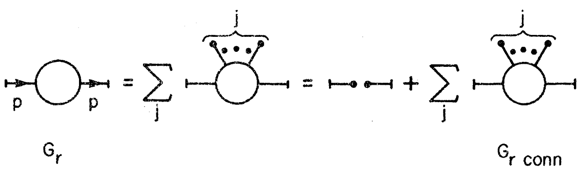

FIG. 5. The two-point Green's function in the unsymmetric theory expressed as in Fig. 4. A line which disappears into the vacuum at a blob represents the field vacuum expectation value v. The final form is the decomposition into connected and disconnected contributions. There are no vacuum-to-vacuum contributions.

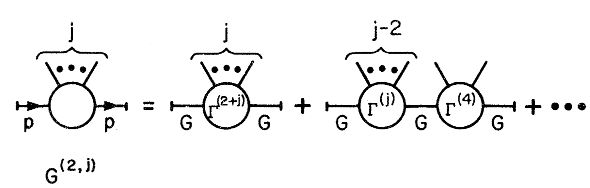

FIG. 6. Decomposition of $G^{(2,j)}$ defined by Figs. 4 and 5 into a sum of products of G's and $\Gamma^{(2+k)}$'s for $k = j$, $j-2, \ldots$.

have agreement with the above expression. We obtain in this way new zero-momentum theorems for the vertex functions,

$$\Gamma^{(4+2k)}(p_1p_2 0\cdots 0)=0,\quad k\geq 0 \tag{7.7}$$

where there are $2k+1$ zeros in the argument of $\Gamma^{(4+2k)}$. Continuing in this manner, we eventually obtain the strongest theorems

$$\Gamma^{(n)}(p_1\cdots p_{n-2}0)=0,\quad n>2. \tag{7.8}$$

We have in particular that

$$\Gamma^{(n)}(0\cdots 0)=0,\quad n>2 \tag{7.9}$$

so that (4.16) becomes

$$\mathcal{V}(\chi)=\tfrac{1}{2}\chi^2\Gamma^{(2)}(0), \tag{7.10}$$

and the inconsistent behavior (4.19) is avoided because (4.18) vanishes. Because of the theorems (7.9), we cannot normalize at $P=0$. Equation (4.19) is the correct behavior only if $\bar{g}<0$. If $\bar{g}=0$, then (7.10) gives the appropriate solution to (4.17). The theory is thus seen to provide its own redemption from the fate of instability.

To see what is happening, it is useful to consider again a deviative-coupling model such as (2.7). Each coupling of ϕ involves an external momentum factor and so the theorems (7.5)–(7.9) can be maintained in each order of (nonrenormalizable) perturbation theory (for vanishing ϕ momenta). This corresponds to the fact that the R invariance ($\phi\to\phi+r$, $\psi\to\psi$) is only broken by mass terms. Here one, of course, cannot define G by $\Gamma^{(3)}_{\bar{\psi}\psi\phi}(p, -p,0)=0$, but one is forced to normalize away from vanishing momentum even though all the particles in the theory are massive.

Let us now explore some of the further consequences of (7.5)–(7.9). We note first that since these theorems are *not* satisfied order by order, the perturbation expansions for the Γ's *cannot* be asymptotic expansions when some momenta vanish. Because the theory is not infrared free, this is not alarming, but since we assumed the contrary at various stages in our arguments, we must note that such strong assumptions are not necessary to deduce (7.5)–(7.10). We need only assume that the expansions for the Γ's are asymptotic for $P\notin\mathcal{E}'$, where $\mathcal{E}'$ is the set of momenta at least one of which vanishes ($\mathcal{E}\subset\mathcal{E}'$). (We can actually get by if the expansions are only asymptotic for e.g. $|p^2|>m^2$ or even $p\to\infty$.) Then Z_1, Z_3, $\bar{g}=g_0$, β, γ, and η can still be calculated as above, but not necessarily Δ_2, and (7.10) follows as above if the field equation (1.6), known to be finite, is a valid exact equation, and the contradiction with positivity is avoided.

Another possible implication of our theorems can be obtained from consideration of the expression

$$\Gamma^{(2)}(p^2)=-\tfrac{1}{6}gZ_1\tilde{\Pi}(p^2)-Z_3\Delta-Z_3(p^2-m^2) \tag{7.11}$$

for the renormalized $\Gamma^{(2)}$ in terms of (4.7). We have

$$\tilde{\Pi}(p^2)=-\frac{6Z_3}{gZ_1}\Delta_1+\tilde{\Pi}_2(p^2) \tag{7.12}$$

(see Fig. 3), and (7.7) for $k=0$ gives $\tilde{\Pi}_2(0)=0$ if the $p\to 0$ limit commutes with the cutoff removal limit $\Lambda\to\infty$ implied in (7.11). Then

$$\Gamma^{(2)}(0)=-Z_3(\Delta_2-m^2), \tag{7.13}$$

which is divergent if Δ_2 is. The same result is given by the relation[19] $C_r=-v\Gamma(0)$ if it is valid in the exact theory. Then we would have $\mathcal{V}(\chi)=+\infty$, which is strange, but not necessarily inconsistent. Of course either (7.13) or (5.24) could be wrong, and so there seems to be no need for immediate concern.

If the perturbation expansion only determines the Green's functions for momenta larger than the mass-shell values, the conditions (4.1) may fail to be valid for the exact theory obtained by continuation from the larger momenta, even though they hold in each order. Then the particle structure of the theory would be uncertain. Independently of the presence of a mass m particle, the theorems (7.5)–(7.9) actually suggest the presence of a massless bound state in the exact theory. Then

$$\Gamma(0)=0 \tag{7.14}$$

and

$$\mathcal{V}(\chi)\equiv 0, \tag{7.15}$$

and the failure of perturbation theory at $p=0$ could be understood. If this were the case, then $\mathcal{E}'$ could be the set of exceptional momenta for the exact theory.

In perturbation theory, where nothing is special about vanishing momenta, one can use such momenta as renormalization points at which normalization conditions can be specified. Because of our theorems, normalizing at vanishing momenta is probably not consistent for the exact theory. This *may* be understandable as a consequence of the presence of a massless bound state, or just as a consequence of the partial R invariance. [Recall our discussion above of the derivative-coupling model (2.7).] Clearly the condition $\Gamma^{(4)}(p,q,0)=g$ cannot be used if the interacting theory is consistent unless the condition fails to be valid in the exact theory. The condition (7.14), on the contrary, is perhaps usable as a normalization condition. If it remained valid in the exact theory, (7.15) would result, and also (5.21) and (7.7) would then give $\Delta_2 = 0$, so that the perturbative expansion for Δ_2 would not be asymptotic, thus invalidating the calculations of Sec. V B. Furthermore, if the usual RG relations between theories normalized at different points remain valid in the exact theory, this would imply that the old Δ_2 (4.5) is at least finite.

Because of the uncertainties just described (some further discussion will be given in Sec. VIII), we cannot claim to fully understand the theory at this point. However, we hope to have shown that the previous difficulty with the effective potential is overcome by the theory in a natural way.

VIII. DISCUSSION

Unstable or not, much has been learned about $g\phi^4$ theory with $g<0$ in recent years. The bare Lagrangian (1.1) appears to have no ground state, but the fact the it was ill defined in perturbation theory, where order by order no inconsistency arose, meant that the fate of the renormalized theory was uncertain. The situation remained unclear until the modern era, which began with Symanzik's discovery of AFD. The AFD enabled Symanzik to sum the perturbation series to conclude that the exact bare coupling constant vanished, thus rendering irrelevant the formal argument for inconsistency. The same AFD subsequently enabled Coleman to conclude that $\mathcal{V}(\infty) = -\infty$ if $\tilde{g} \equiv \mathcal{V}^{(4)}(0)$ is negative, and thus establish a new and convincing argument for inconsistency. We have here explored further the implications of AFD and argued that in fact $\tilde{g}=0$ so that $\mathcal{V}(\chi) \propto \chi^2$, and the theory is perhaps not yet dead. It seems to want to stay alive.

We conclude from our analysis that there is at present no argument that the model is inconsistent. Let us therefore assume that the model is consistent and summarize its properties. What we know about it depends on the assumptions we are willing to make. We first state the strongest reasonable assumptions which are not known to be inconsistent, and recall the consequences of these assumptions. We will afterwards discuss to what extent they can be weakened without losing much of the predictive power of the model.

Assumption 1. The perturbation expansions for the $\Gamma(P)$'s [normalized by (4.1)] are asymptotic power series in g for $P \not\in \mathcal{E}'$. (Also, $|g|$ is sufficiently small and the mass insertion term, negligible in each order, remains negligible in the exact theory.)

Consequences. The $\Gamma(\lambda P)$ are exactly computable for $\lambda \to \infty$, $P \notin \mathcal{E}'$. The renormalization constants Z_1, Z_3, ζ, and Δ_1 and the RG functions $\bar{g} = g_0$, β, γ, and η, which are all determined by Γ's at large momenta, are also computable. However, Δ_2 is not necessarily computable.

Assumption 2. The field equation (5.40), finite, local, and valid in each order, remains valid in the exact theory. (It follows from assumption 1 that it is finite and local in the exact theory.) [Also, the shifted field equation (6.9) has a unique solution which converges to the solution of (5.40) for $r \to 0$.]

Consequences. The zero-momentum theorems $\Gamma^{(n)}|_{p_i=0}=0$ are satisfied for $n>2$ so that $\mathcal{V}(\chi) = \frac{1}{2}\Gamma(0)\chi^2$. The perturbation series is therefore not asymptotically convergent for $P \in \mathcal{E}'$. If $\Gamma(0) = -\Delta_2$, and if $\Delta_2 = -\infty$, then $\mathcal{V}(\chi) = +\infty$.

Assumption 3. The renormalization conditions (4.1) remains valid for the exact theory.

Consequences. The theory has a particle interpretation.

Assumption 4. $\Gamma^{(2)}(p)$ can also be normalized at $p=0$, and the resulting theory is connected in the usual way to the above one normalized at $p^2 = m^2$.

Consequences: The effective potential is identically zero. Δ_2 probably also vanishes and then its perturbation expansion is not asymptotically convergent. Then the Δ_2 in the above theory is finite. One can also choose $\Gamma^{(2)}(0) = m^2$, and then $\mathcal{V}(\chi)$ is the free field function $\frac{1}{2}m^2\chi^2$.

If the above assumptions are all correct, the model is the one in four dimensions about which the most is exactly known. The vertex functions are known in three places: $p \to \infty$, $p^2 \to m^2$, and $p \to 0$. However, the desired theorems (7.6) are derivable from much weaker assumptions. If assumption 3 is wrong, only the particle interpretation is lost and the situation would be like that of the AF NAGT's. Independent of this, the consequences of assumption 2 suggest the presence of a zero-mass composite excitation in the exact theory, as we discussed in Sec. VII, so that $\mathcal{V}(\chi) \equiv 0$. This vanishing is also a consequence of assumption 4, and if that assumption is wrong, one would at worse have $\mathcal{V}(\chi) = +\infty$.

Assumption 1 can also be considerably weakened with no adverse affects. For example, the perturbative expansions of the $\Gamma(P)$'s could only be asymptotic for all $p_i^2 > m^2$. This could come about because for $p_i^2 < m^2$ the Feynman diagrams tend to add up constructively; e.g., in two dimensions, the perturbation series is divergent for $p_i^2 < m^2$.[34] Another possibility is that one has an asymptotic expansion in g only for the $p_i \to \infty$. More precisely, one could have

$$\Gamma(P) = \Gamma_0(P) + \Gamma_1(P) \tag{8.1}$$

with

$$\Gamma_0(\lambda P)/\Gamma_1(\lambda P) \underset{\lambda\to\infty}{\longrightarrow} 0, \quad (8.2)$$

and with $\Gamma_1(P)$ having an asymptotic expansion but not $\Gamma_0(P)$. We cannot have $\Gamma_1(P)=\Gamma_{as}(P)$ if

$$\Gamma(\lambda P) \underset{\lambda\to\infty}{\longrightarrow} \Gamma_{as}(\lambda P)$$

for $P\in\mathcal{E}'$, but if this is given up, then the expansions for the $\Gamma_{as}(P)$ could be asymptotic or even convergent for $P\notin\mathcal{E}$. The effective exceptional momenta set [the P's for which $\Gamma(\lambda P)\not\to\Gamma_{as}(\lambda P)$] for the exact theory would then be $\mathcal{E}'$. In any case, we would still have $\Gamma(\lambda P)$ computable for $\lambda\to\infty$ and retain $\mathcal{V}\sim\chi^2$.

Still another possibility is that only the massless ϕ^4 theory (which *must* be normalized in part away from $P=0$) is consistent. Because of the zero-momentum theorems, this is not unreasonable. In this case, the $\Gamma(0)\sim\Delta_2\sim\infty$ difficulty could be avoided and the set $\mathcal{E}'$ would be more significant. However, the particle interpretation would be unclear.

If assumption 1 were wrong in any form, we would loose our result $\mathcal{V}\sim\chi^2$, but (4.19) could also not be deduced, and so there would still be no indication that the model is unstable. The model would unfortunately then not be interesting because its exact computability would be lost.

If assumptions 1 and 2 are essentially correct and the theory is consistent, the failure of perturbation theory to provide an asymptotic expansion $P\notin\mathcal{E}'$ is very interesting and suggests that four-dimensional quantum field theories are substantially more complicated than are the constructed two-dimensional ones.[15]

Our analysis here is similar to our previous treatment of AF NAGT's.[9-11] In all such theories, one has $Z=Z_1/Z_3=0$, just as we have deduced here [Eq. (5.7)]. Also, the analog of the result $Z\phi^2=c$ number deduced here [Eq. (6.8)] is true in many AF NAGT's in the form $ZA^a_\mu A^b_\nu=c$ number.[35] In these theories, if the assumption that the R transformation[11] (or Abelian gauge transformation[10]) commutes with the gauge-invariant regularization removal is correct, then one has strict R invariance and the consequent zero-momentum theorems for all n. Then the effective vector-meson potential $\mathcal{V}(A)$ vanishes identically, but the gauge dependence of $\mathcal{V}$ makes this of unclear significance. Similarly, the implication of $\Gamma_A(0)=0$ is unclear, unlike the case of quantum electrodynamics, where the perturbative R invariance implies the existence of the massless physical photon.[14] The treatment given in this paper of the consequences of partial R invariance actually serves to strengthen the conclusion of Refs. 9–11. The zero-momentum theorems for $n>2$ are now seen to hold even in the presence of mass counterterms which are absent if a gauge-invariant regularization is used but not necessarily[36] if the more relevant point-separating regularization is used.

One can now consider many combined AF theories involving non-Abelian vector mesons and scalar mesons. The partial R invariance theorems will be valid, and perhaps such hoped for properties as confinement or spontaneous symmetry breaking can be shown to occur. But even if the AF ϕ^4 theory plays no role in nature, it will remain as an interesting and most simple field theory. If it is consistent, it is the field theory about which most is known, and, if it is eventually shown not to be consistent, it will at least be the only nontrivial four-dimensional model which is known to be inconsistent, and even that is interesting.

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†Permanent address.

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