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Time-delay equation governing electron motion

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A previously proposed differential-difference equation governing the motion of the classical radiating electron is considered further. A set of three assumptions is offered, under which the proposed equation yields asymptotically stable acceleration.

I. INTRODUCTION

Presently, the Lorentz and Lorentz-Dirac equations are considered by some as probably giving the correct classical description of the motion of a nonrelativistic or relativistic radiating electron. This is so despite the fact that, firstly, the various derivations of these equations are not rigorous (depending on questionable expansions about singular points, or on assumptions concerning the interior of the electron, or on *ad hoc* phenomenological assumptions concerning the electron holistically), and that, secondly, the equations are plagued by runaway solutions (eliminable by positing suitable boundary conditions) and the phenomenon of preacceleration.

These difficulties seem to be related to the (presently unmeasurable) quantity $\tau_0 = 2e^2/3mc^3$ entering the equations. The possibility then suggests itself that, perhaps, one can alter (albeit phenomenologically) the Lorentz equation by quantities of the order of τ_0 with the hope of leaving the *actually* observable consequences intact while removing the problem of runaway solutions and preacceleration difficulties.

In this spirit, in a previous paper by this author,[1] a differential-difference (or time-delay) equation was tentatively postulated as governing the motion of a classical electron in arbitrary nonrelativistic motion. It was suggested that this equation replace the well-known Lorentz equation of the classical electron, since the former equation has only retarded solutions and does not have runaway solutions under zero external force—two features that plague the Lorentz equation, as mentioned.

The proposed equation has the form

$$\tau_0 \dot{\vec{a}}(t) - \vec{a}(t-\lambda) = -\frac{1}{m}\vec{F}^{\,\mathrm{ext}}(t)\ , \tag{1.1}$$

where $\tau_0 = 2e^2/3mc^3$, the dot signifies a time derivative, and λ is an unspecified positive delay (expected to be of the order of τ_0). Further, to ensure that the equation is uniquely solvable the boundary assumption was made that

$$\vec{a}(t) = \frac{1}{m}\vec{F}^{\,\mathrm{ext}}(t) \tag{1.2}$$

for $0 \leq t \leq \lambda$ (or $t_0 \leq t \leq t_0+\lambda$), and where it is also assumed that $F^{\mathrm{ext}}(t) = 0$ for all $t < 0$ (or $t < t_0$).

Although the differential-difference equation under discussion does not suffer some of the defects of the Lorentz equation, an important aspect of the equation which was not investigated at the time is whether or not this equation has solutions for which the acceleration goes to zero for very large times (asymptotic stability).

It is the purpose of the present work to investigate this problem to some extent. It will result that, if (i) λ is limited to not exceed τ_0, (ii) $F^{\mathrm{ext}}(t) = 0$ for $t >$ some time T_0, and (iii) a certain, rather mild—but not altogether physical—condition holds for $F^{\mathrm{ext}}(t)$, then the equation does have asymptotic stability. However, in the present work we only discuss very briefly the degree to which the assumptions allow or preclude *physically* realizable external forces. These assumptions will appear at various places in the following discussion where their significance will be apparent.

In the next sections, we discuss these matters via the Laplace transform approach, which we take essentially from Bellman and Cooke.[2] Many of the statements to be used here are found in one place or another in this text (though sometimes in altered form), and as such will be utilized without proof.

Finally, for simplicity, we only explicitly make one-dimensional considerations in the following discussion; however, all the results can be directly generalized to the three-dimensional case.

II. SERIES FORM FOR ACCELERATION

In this section, we apply Laplace transform theory to the equation of interest to find under what conditions the solutions to the equation can be expressed as a series of simple exponentials in the time. We will learn that a single assumption is sufficient for this purpose.

We consider then the one-dimensional time-delay equation,

$$\tau_0 u'(t) - u(t-\lambda) = f(t) \text{ for } t > \lambda\,, \tag{2.1a}$$

with boundary condition

$$u(t) = -f(t) \text{ for } 0 \le t \le \lambda, \tag{2.1b}$$

where the prime denotes differentiation with respect to time, u denotes the single component of the acceleration, f denotes $-(1/m)F^{\text{ext}}$, and $\tau_0 = 2e^2/3mc^3$ as before.

Multiplying Eq. (2.1a) through by e^{-st}, integrating over t, and then taking Laplace transforms, it can be shown[3] that one obtains the relation

$$u(t) = \frac{1}{2\pi i}\int_{(c)} e^{st} h(s)^{-1}[p_0(s) + q(s)]\, ds \quad \text{for } t > \lambda\,, \tag{2.2a}$$

where

$$p_0(s) = -\left[\tau_0 f(\lambda)e^{-\lambda s} + e^{-\lambda s}\int_0^{\lambda} f(t_1)e^{-st_1}\, dt_1\right], \tag{2.2b}$$

$$q(s) = \int_{\lambda}^{\infty} f(t_1)e^{-st_1}\, dt_1\,, \tag{2.2c}$$

$$h(s) = \tau_0 s - e^{-\lambda s}\,, \tag{2.2d}$$

and where $\int_{(c)}$ signifies integration over any infinite vertical line (symmetric about the real axis) in the complex s plane which is to the right of all zeros of the function, $h(s)$.

Zeros of $h(s)$

Here we briefly consider some essential properties of the zeros of $h(s)$. The zeros of $h(s)$ are given by solutions to the equation

$$s = \frac{1}{\tau_0}e^{-\lambda s}\,. \tag{2.3}$$

In general, this equation allows roots with positive and negative (and zero) real parts. An important root which has a positive real part is $s_0 = (x_0, 0)$, where x_0 is the *unique* solution of the equation

$$x_0 = \frac{1}{\tau_0}e^{-\lambda x_0}\,. \tag{2.4}$$

We see that no root can have a real part which is greater than x_0, since as x increases beyond x_0, $e^{-\lambda x}$ decreases below $e^{-\lambda x_0}$. We also see that if $s = x + iy = (x, y)$ is a root, so then is $(x, -y)$, so the zeros are symmetrically located about the real axis, as shown in Fig. 1.

Finally, we note that all the roots of $h(s) = 0$ must be simple—since, if some root s_i were not, we would have, at least, both $h(s_i) = 0$ and $h'(s_i) = 0$. Thus, we would have the relations

$$s_i = \frac{1}{\tau_0}e^{-\lambda s_i} \tag{2.5}$$

and

$$\tau_0 + \lambda e^{-\lambda s_i} = 0\,, \tag{2.6}$$

which yield the relation

$$\tau_0(1 + \lambda s_i) = 0\,, \tag{2.7}$$

giving $s_i = -\lambda^{-1}$. However, when this result is inserted into Eq. (2.6), it gives the result $\tau_0/\lambda = -e$, which cannot be, as both λ and τ_0 are positive.

We now consider under what circumstances the limited integration in Eq. (2.2a) can be extended to the full Bromwhich (closed) contour, so that $u(t)$ can be expressed as a series (sum of residues).

Implementation of Bromwhich contour

In order to see that the full Bromwhich contour can be used for describing $u(t)$, we note that, over any large circular path in the complex s plane, we have that $|h(s)^{-1}|$ is of the order of magnitude $|s|^{-1}$ (at most), if the path is so selected that it passes only *near* (and not through) any zeros of $h(s)$.[4]

Further, we can be assured of being able to select such paths (i.e., a countably infinite set of nested curves with ever-increasing $|s|$, intersecting no zeros of h) since the zeros of the analytic function $h(s)$ cannot have a condensation point.

Thus, in Fig. 2 we indicate a typical Bromwhich contour, for large $|s|$, which is so selected that no zeros of $h(s)$ lie on it. Now concerning integrations over the paths indicated in Fig. 2 we have,

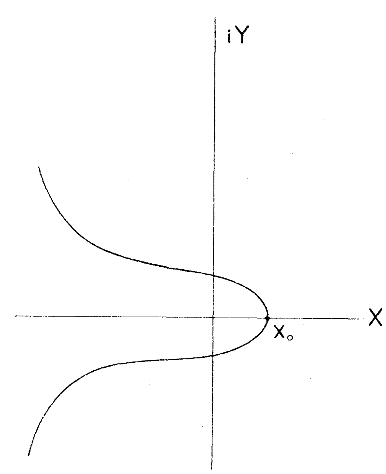

FIG. 1. Curve of zeros of $h(s)$.

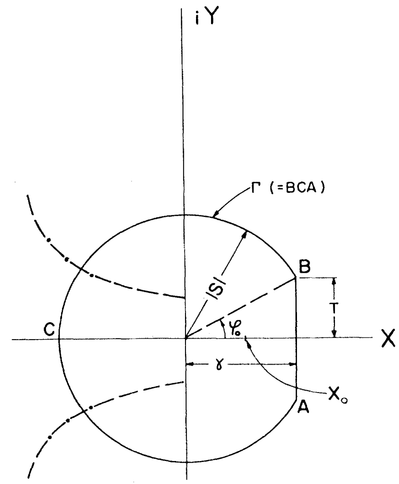

FIG. 2. Typical Bromwhich contour.

symbolically,

$$\int_{\gamma - iT}^{\gamma + iT} + \int_{\Gamma} \equiv \oint , \tag{2.8a}$$

$$\lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} \equiv \int_{(c)} , \tag{2.8b}$$

and

$$\lim_{T\to\infty} \oint = 2\pi i \sum \text{residues} . \tag{2.8c}$$

We will now show that

$$\lim_{T\to\infty} \int_{\Gamma} = 0 \tag{2.9}$$

for the integrals relevant to our problem, and therefore, that

$$\int_{(c)} = 2\pi i \sum \text{residues} \tag{2.10}$$

for the integrands involved in our problem.

For this, we return to the expression for $u(t)$ as given in Eq. (2.2a). Explicitly, we now wish to show that

$$\lim_{T\to\infty} \int e^{st} \left[\frac{p_0(s)+q(s)}{h(s)}\right] ds = 0 , \tag{2.11}$$

where $p_0(s)$ and $q(s)$ are given in Eqs. (2.2b) and (2.2c).

Considering the expression for $p_0(s)$, the first contribution to the above integral is

$$\tau_0 f(\lambda) I_1(t;T) \equiv - \int_{\Gamma} \tau_0 f(\lambda) e^{-\lambda s} e^{st} h(s)^{-1} ds, \tag{2.12}$$

and we will show that

$$\lim_{T\to\infty} I_1 = 0 .$$

Now

$$|I|_1 \leq \tau_0 |f(\lambda)| \int_{\Gamma} \frac{|e^{(t-\lambda)s}|}{|h|} |ds| \text{ for } t>\lambda. \tag{2.13}$$

Further, from the preceding considerations, we have that

$$\int_{\Gamma} \frac{|e^{(t-\lambda)s}|}{|h|} |ds| \leq \int_{\Gamma} \frac{|e^{(t-\lambda)s}|}{|s|} |ds| \quad \text{for } t>\lambda \tag{2.14}$$

and that the latter integral can be directly shown to vanish as $|s| \to \infty$. Thus, the next contribution to the integral in Eq. (2.11) is the term

$$I_2(t;T) = -\int_0^{\lambda} f(t_1) \left[\int_{\Gamma} \frac{e^{(t-t_1-\lambda)s}}{h(s)} ds\right] dt_1$$

$$= \int_0^{\lambda} f(t_1) I_1(t-t_1; T) dt_1 , \tag{2.15}$$

and we then have that

$$\lim_{T\to\infty} I_2(t;T) = \int_0^{\lambda} f(t_1) \left(\lim_{T\to\infty} I_1\right) dt_1$$

$$= 0 \quad \text{for } t > 2\lambda. \tag{2.16}$$

The last contribution to the integral in Eq. (2.11) can be handled in the same way. We have

$$I_3(t;T) = \int_{\lambda}^{\infty} f(t_1) \left[\int_{\Gamma} e^{s(t-t_1)} h(s)^{-1} ds\right] dt_1. \tag{2.17}$$

Assumption (A). In order to utilize the same device here as with I_2, it is now *assumed* that $f(t)$ is such that $f(t) = 0$, for all $t >$ some time T_0. Then

$$I_3(t;T) = \int_{\lambda}^{T_0} f(t_1) \left[\int_{\Gamma} e^{s(t-t_1)} h(s)^{-1} ds\right] dt_1 , \tag{2.18}$$

and, therefore, as in the preceding case,

$$\lim_{T\to\infty} I_3(t;T) = 0 \text{ for } t > T_0 + \lambda . \tag{2.19}$$

Thus, with assumption (A) we have that Eq. (2.11) is verified, so that, by Eqs. (2.10) and (2.11a) we have that, finally,

$$u(t) = \sum_i \left[\frac{p_0(s_i)+q(s_i)}{h'(s_i)}\right] e^{s_i t} \quad \text{for } t > T_0 + \lambda , \tag{2.20}$$

where the sum is over all zeros of $h(s)$. Thus, $u(t)$ has been expressed as a series, as was the intent in this section.

III. STABILITY OF $u(t)$

Here we consider sufficient conditions such that $u(t) \to 0$ as $t \to \infty$. From Eq. (2.20) for $u(t)$, we see that $u(t)$ will be asymptotically stable if only roots with *negative* real parts enter the series. We shall guarantee this by first ensuring that there can only be one root with positive real part, and then by requiring that its coefficient in the series expansion for $u(t)$ vanishes. The former requirement will also be seen to prevent any roots from having vanishing real part.

Consider then the possibility of a zero of $h(s)$, other than $s_0 = (x_0, 0)$, with positive real part. Then, for $s = x + iy$, for some $x > 0$ and $y \neq 0$, we must have, by Eqs. (2.3), that

$$|y| = -\frac{1}{\tau_0} e^{-\lambda x} \sin\lambda |y|. \tag{3.1}$$

It follows from this equation and the imaginary part of Eq. (2.3) that $\cos\lambda y > 0$, but that $\sin\lambda |y| < 0$. Therefore, $\lambda |y|$ must be in the fourth quadrant. Now, in order for a solution to Eq. (3.1) to exist, the 45° line, $\xi |y|$, must intersect the sine curve, $\xi = -(1/\tau_0) e^{-\lambda x} \sin\lambda |y|$. Now $\sin\lambda |y|$ is convex (upward) in the neighborhood of the origin, and in this region the magnitude of the right-hand side of Eq. (3.1) has the form, $(\lambda/\tau_0) e^{-\lambda x} |y|$. Thus, if $\lambda/\tau_0 \leq 1$, its slope will be less than 45° (since $x > 0$ and $\lambda > 0$) in the neighborhood of the origin, and Eq. (3.1) will have no solution (other than s_0). The situation is depicted in Figs. 3 and 4. Therefore, we make the next assumption.

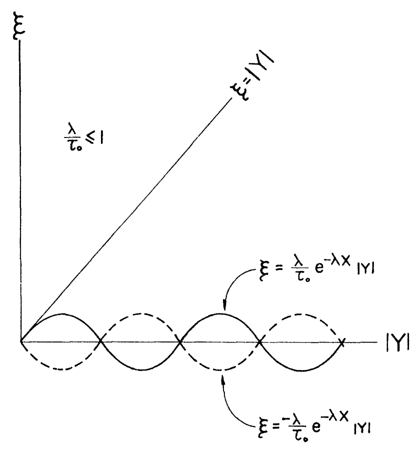

FIG. 3. Nonsolution case for $\gamma/\tau_0 \leq 1$.

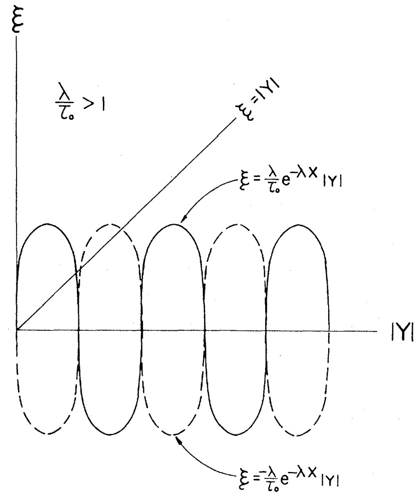

FIG. 4. Solution case for $\lambda/\tau_0 > 1$.

Assumption (B). We have

$$\lambda \leq \tau_0,$$

and we then have only one root with positive real part to contend with.

Before considering the coefficient of this root in the expression (2.20), we presently dispose of the possibility that any root has vanishing real part. If such a root existed, its imaginary part would be described by Eq. (2.3), which yields then

$$0 = \frac{1}{\tau_0} \cos \lambda y \text{ and } y = -\frac{1}{\tau_0} \sin\lambda y. \tag{3.2}$$

Thus, $\lambda y = n\pi/2$, with n odd, follows from the first equation and, therefore, $n\pi/2\lambda = \mp 1/\tau_0$ follows from the second equation above, where the minus sign goes with $n = 1, 5, 9, 13, \ldots$, and the plus sign goes with $n = 3, 7, 11, 15, \ldots$. Now $\lambda/\tau_0 = \mp n\pi/2$, so, if $n = 1$, then $\lambda/\tau_0 = -\pi/2$, which cannot be as both λ and τ_0 are positive. Again, if $n = 3$, then $\lambda/\tau_0 = 3\pi/2 > 1$, which violates assumption (B), etc. Thus, all zeros of $h(s)$ have definite signs for their real parts.

Returning now to the series, Eq. (2.20), we see that if the coefficient depending on the single root with positive real part vanishes, then $u(t) \to 0$ as $t \to \infty$. Thus, we make the following final assumption.

Assumption (C). The coefficient of $e^{s_0 t}$ in the expansion (2.20) vanishes. This means that $p_0(s_0) + q(s_0) = 0$, or that $f(t)$ is such that the relation

$$\int_{\lambda}^{T_0} f(t)e^{-s_0 t}\,dt = \tau_0 f(\lambda)e^{-\lambda s_0} + e^{-\lambda s_0}\int_0^{\lambda} f(t)e^{-s_0 t}dt \qquad (3.3)$$

is satisfied—for given λ and τ_0—where the defining Eqs. (2.2) have been used for p_0 and q, and we are reminded that, by assumption (A), $f(t)=0$ for $t>$ some time T_0.

Assumption (C) seems to this author, to be rather artificial, as it allows the possibility that, of two possible forces $f_1(t)$ and $f_2(t)$, the more rapidly (asymptotically) decaying one (in time) might not be the one satisfying Eq. (3.3) and thus yielding asymptotic stability for the acceleration. However, in favor of the assumption, we note that (if T_0 is very large) *most* $f(t)$ *approximately* satisfy Eq. (3.3). This can be seen by assuming that f is constant over the interval $(0, \lambda)$, by assuming that $f(t)$ (for $t>\lambda$) changes slowly enough that $f(t)\approx f(\lambda)$ can be removed from the integral on the left-hand side of the equation, and finally by using Eq. (2.3).

IV. SOLUTIONS FOR $u(t)$

Here we consider very briefly the possibility of certain forces yielding solutions to the series, Eq. (2.20)—for the acceleration $u(t)$.

An interesting case to consider would be the solution for the force, $f(t)=$ constant, for all $t>0$. However, this force violates assumption (A) and so is excluded.

A related (but not very interesting) case is where $f(t)=0$, for all $t>0$. Here the three assumptions can all hold. Indeed, in this case (as for the case where $f=$ constant, $p_0(s_i)+q(s_i)=0$ for *all* roots s_i, as is seen by putting the condition $f(t)=0$ into Eq. (3.3). So we obtain $u(t)=0$ for all $t>0$, since here, we can take $T_0=0$.

A more interesting case to consider is where $f(t)=A\delta(t-t_0)$, where A is a constant. However, when this expression for f is inserted into $p_0(s)+q(s)$ one finds that

$$p_0(s)+q(s)\propto e^{-st_0} \;\; (\text{or } e^{-s(t_0+\lambda)}), \qquad (4.4)$$

depending on whether $t_0>\lambda$ or $t_0<\lambda$, respectively. Thus, in this case, assumption (C) is violated, and this force also is excluded from consideration.

To what extent one can find *physically* realizable forces that satisfy all three assumptions is a very interesting question which, however, will not be pursued further in this work.

V. SUMMARY

It has been demonstrated that three assumptions are sufficient to ensure that the proposed differential-difference equation yield an acceleration which is asymptotically stable. The first two assumptions are somewhat physical, but the third one seems artificial by comparison. The last assumption actually requires a rather sensitive restriction on the external force as a function of time, and is found to be inconsistent with a δ-function type of force. Of course, physical forces are almost always given in nature directly as a function of location rather than time, and so the question remains as to what extent the proposed equation—together with the three assumptions—will allow meaningful solutions for physical situations.

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[1]J. Cohn, Nuovo Cimento 26B, 47 (1975).
[2]R. Bellman and K. L. Cooke, *Differential-Difference Equations* (Academic, New York, 1963).
[3]Reference 2, p. 64.
[4]Reference 2, p. 102.