PHYSICAL REVIEW D VOLUME 14, NUMBER 12 15 DECEMBER 1976

Gauge theory, gravitation, and symmetry*

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(Received 2 February 1976; revised manuscript received 22 June 1976)

The coupled Einstein-Yang-Mills (or Einstein-Maxwell in the case of an Abelian internal symmetry) theory is shown to be the unique gauge theory of $T_4 \otimes G$, where T_4 is the 4-dimensional translation group and G is an internal-symmetry group. In the case of $P_4 \otimes G$ where P_4 is the 4-dimensional Poincaré group one obtains Einstein-Cartan theory coupled with the internal gauge fields. As in the Abelian case internal Yang-Mills fields do not create any extra torsion, owing to the gauge invariance of the internal symmetry. The arguments are given in terms of the conventional gauge formalism, without using the bundle language.

Recently it was shown[1] that the gauge theory of the 4-dimensional translation group T_4 becomes precisely Einstein's theory of gravitation. Also, in an accompanying paper,[2] the Einstein-Cartan[3] theory of gravitation was derived as the gauge theory of the 4-dimensional Poincaré group P_4. The theory in each case is shown to be unique[1,2] if one decides to choose the Lagrangian to be the lowest-order possible combination, linear or quadratic, in gauge-field strengths (in this paper the uniqueness should always be understood with this criterion). Then, combining an internal symmetry G with the space-time symmetry T_4 or P_4, one can easily construct the gauge theories of $T_4 \otimes G$ and $P_4 \otimes G$. As shown in the following one obtains the coupled Einstein-Yang-Mills (or Einstein-Maxwell in the case of an Abelian internal symmetry) theory for the $T_4 \otimes G$ gauge group and the Einstein-Cartan-Yang-Mills (or Einstein-Cartan-Maxwell) theory for $P_4 \otimes G$. This way of combining the gravitation with an internal gauge theory is more economical and has fewer degrees of freedom than the higher-dimensional unification proposed earlier by Cho and Freund,[4,5] in that here we do not have the additional scalar fields[4,5,6] of the "internal" gravitation.

In the previous papers[1,2] we have used the bundle language to obtain the desired results, and accordingly the arguments were quite mathematical, but in this paper we will use the conventional gauge formalism to make the physics more transparent. However, it remains crucial to interpret the space-time symmetries T_4 and P_4 as the symmetries among the source fields and of local orthonormal frames at each *physical* space-time point, but not of a global Minkowskian coordinate transformation, in order to obtain the desired result. In other words, the generators of T_4 are *not* identified as the derivatives in a coordinate basis, as has been sometimes suggested.[7] Rather, the generators of T_4 annihilate the fields,[1] and the fields remain invariant at each physical space-time point under T_4 since under a translation of a coordinate

$$x \to x' = x + \delta x$$

one has

$$\delta\phi = \phi'(x') - \phi(x) = 0$$

as the translational invariance. Notice that the translational invariance described here serves as a good global symmetry *independent* of the space-time curvature, whereas there does not exist a translational invariance as a global coordinate translation, in general, in a curved space.

For P_4 the interpretation goes the same way,[2] and we have

$$\delta\phi = -\tfrac{1}{2}\xi_{ij}\,\theta^{ij}\phi$$

not only under the Lorentz subgroup *but also under the whole Poincaré group P_4*, where $\xi_{ij} = -\xi_{ji}$ $(i,j = 0,1,2,3)$ are the representations of the generators of the Lorentz subgroup for ϕ and $\theta^{ij} = -\theta^{ji}$ are the six infinitesimal parameters.

With this interpretation of the space-time symmetries as the symmetries among the source fields and of local orthonormal frames at each physical space-time point we will now construct in the conventional gauge formalism the unique gauge theories of the space-time symmetries combined with an internal-symmetry group G. Only the mainstream of the argument will be given since the whole argument is a straightforward generalization of the previous works.[1,2]

To begin with it is very convenient to introduce a local orthonormal basis at each space-time point. Let us start with a flat space-time and let

e_i ($i=0,1,2,3$) be a local orthonormal basis at each point, i.e., four orthonormal vector fields, with the commutation relations

$$[e_i, e_j] = T_{ij}{}^k e_k . \tag{1}$$

In a commuting coordinate basis e_μ ($\mu=0,1,2,3$) e_i can be decomposed in terms of e_μ and vice versa;

$$e_i = h_i^\mu e_\mu , \quad e_\mu = h_\mu^i e_i , \tag{2}$$

where h_i^μ and h_μ^i are the vierbein fields and the inverse vierbein fields so that

$$h_i^\mu h_\mu^k = \delta_i^k , \quad h_\mu^i h_i^\nu = \delta_\mu^\nu .$$

From Eqs. (1) and (2) one easily finds

$$\begin{aligned} T_{ij}{}^k &= (\partial_i h_j^\mu - \partial_j h_i^\mu) h_\mu^k \\ &= -h_i^\mu h_j^\nu (\partial_\mu h_\nu^k - \partial_\nu h_\mu^k) , \end{aligned} \tag{3}$$

where ∂_i and ∂_μ are the directional derivatives of the basis e_i and e_μ, respectively. Then the action integral, say, for a set of scalar multiplets ϕ^a ($a=1,2,\ldots,n$) of an internal symmetry G can *formally* be written as

$$\begin{aligned} I &= \int (\tfrac{1}{2}\eta^{ik}\partial_i\phi^a\partial_k\phi^a - \tfrac{1}{2}m^2\phi^a\phi^a) dV \\ &= \int (\tfrac{1}{2}\eta^{ik}\partial_i\phi^a\partial_k\phi^a - \tfrac{1}{2}m^2\phi^a\phi^a)\sqrt{-g}\, d^4x , \end{aligned} \tag{4}$$

where dV is the volume element made out of four orthonormal vectors e_i, and

$$\begin{aligned} \sqrt{-g} &= \det(h_\mu^i) , \\ \eta_{ik} &= \mathrm{diag}(+,-,-,-) . \end{aligned} \tag{5}$$

Notice that all the above expressions are just a matter of a formality and we have not assumed that our space-time is curved; one can have local orthonormal frames in a flat space-time as well as in a curved one. We will create a curvature by requiring local T_4 invariance, but we start from a flat space-time and so far our h_i^μ remain trivial.

Let us consider in this basis the gauge theory of an internal-symmetry group G first, whose generators ξ_a ($a=1,2,\ldots,n$) have the following commutation relations:

$$[\xi_a, \xi_b] = f_{ab}{}^c \xi_c . \tag{6}$$

Then clearly we have

$$[\partial_i, \xi_a] = 0 , \tag{7}$$

and the G-covariant derivative D_i can be introduced as usual;

$$D_i = \partial_i + eA_i^a \xi_a , \tag{8}$$

where

$$A_i^a = h_i^\mu A_\mu^a$$

are the gauge potentials of the internal symmetry in this basis, and e is a dimensionless constant which will serve as the coupling constant for the gauge group G.

Observe that

$$\begin{aligned} [D_i, D_j] &= [\partial_i + eA_i^a\xi_a, \partial_j + eA_j^b\xi_b] \\ &= T_{ij}{}^k D_k + eF_{ij}{}^a \xi_a , \end{aligned} \tag{9}$$

where

$$\begin{aligned} F_{ij}{}^a &= \partial_i A_j^a - \partial_j A_i^a + ef_{bc}^a A_i^b A_j^c - T_{ij}{}^k A_k^a \\ &= h_i^\mu h_j^\nu (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ef_{bc}^a A_\mu^b A_\nu^c) \\ &= h_i^\mu h_j^\nu F_{\mu\nu}{}^a . \end{aligned}$$

Thus *the gauge field strengths are identified as the commutation coefficients for the group generators of two covariant derivatives.*

The gauge-invariant Lagrangian for ϕ^a is written as usual:

$$\mathcal{L} = \sqrt{-g}\,(\tfrac{1}{2}\eta^{ik} D_i\phi^a D_k\phi^a - \tfrac{1}{2}m^2\phi^a\phi^a - \tfrac{1}{4}\eta^{ik}\eta^{jl} F_{ij}^a F_{kl}^a) . \tag{10}$$

Again all these expressions are formal in a flat space-time, but of course they hold true in a curved space-time with nontrivial h_i^μ as well.

Now, for the T_4 gauge group, let ξ_μ ($\mu=0,1,2,3$) be four commuting generators of T_4,

$$[\xi_\mu, \xi_\nu] = 0 , \tag{11}$$

and let B_i^μ be the corresponding gauge potentials. The covariant derivative D_i in this case will be given by

$$D_i = \partial_i + B_i^\mu \xi_\mu . \tag{12}$$

The reason we do not introduce a coupling constant for T_4 in Eq. (12) will become clear soon. With Eq. (12) one finds

$$[D_i, D_j] = T_{ij}{}^k D_k + G_{ij}{}^\mu \xi_\mu , \tag{13}$$

where

$$G_{ij}{}^\mu = \partial_i B_j^\mu - \partial_j B_i^\mu - T_{ij}{}^k B_k^\mu$$

are the gauge field strengths for T_4.

One can identify the gauge potentials B_i^μ of T_4 as the nontrivial part of h_i^μ (Ref. 1):

$$h_i^\mu = \delta_i^\mu + \kappa B_i^\mu , \tag{14}$$

where we have introduced a dimensional parameter κ (of dimension of a length) to give the correct dimension to h_i^μ. Clearly this κ can serve as the coupling constant for the gauge group T_4 and one does not have to introduce an extra coupling in Eq. (12). Notice that *we now have created the curvature of space-time by erecting the gauge potentials B_i^μ for T_4 and having nontrivial vierbein fields h_i^μ given by Eq.* (14). Now, interpreting the T_4 symmetry as a symmetry among the source fields at each physical space-time point as before[1,2] we have

$$[\partial_i, \xi_\mu]=0 \qquad (15)$$

and

$$\xi_\mu\phi=0\,. \qquad (16)$$

Notice that, had one identified the coordinate derivatives ∂_μ as one's generators for T_4, one would not have had Eqs. (15) and (16), and one would have lost the complete parallelism between a gauge theory of an internal symmetry and that of the T_4 symmetry. At this point one may ask how the gauge fields of T_4 can couple to the source fields if they are singlets under the group. The way they couple is dictated by Eq. (14):

$$\begin{aligned} D_i\phi &=(\partial_i+B_i^\mu\xi_\mu)\phi \\ &=\partial_i\phi=(\delta_i^\mu+\kappa B_i^\mu)\partial_\mu\phi\,. \end{aligned} \qquad (17)$$

Thus *the gauge fields* B_i^μ *couple to the source fields as if the generators of* T_4 *were the coordinate derivatives* ∂_μ, which one can interpret *a posteriori* but not *a priori*, owing to Eq. (14)

With Eq. (14) one can write down the gauge field strengths ${G_{ij}}^\mu$ of T_4 in terms of ${T_{ij}}^k$, and one is led to have the following unique Lagrangian quadratic in the field strengths and independent of the choice of a local orthonormal basis:

$$\mathfrak{L}=\frac{1}{\kappa^2}\sqrt{-g}\,(\tfrac14 T_{ijk}T_{ijk}+\tfrac12 T_{ijk}T_{ikj}-T_{ijj}T_{ikk}), \qquad (18)$$

as has been shown in detail in Ref. 1.

Now it is straightforward to construct the unique gauge theory of $T_4\otimes G$. The covariant derivative D_i in this case will be

$$D_i=\partial_i+eA_i^a\xi_a+B_i^\mu\xi_\mu\,, \qquad (19)$$

and one obtains

$$[D_i,D_j]={T_{ij}}^k D_k+e{F_{ij}}^a\xi_a+{G_{ij}}^\mu\xi_\mu\,,$$

with ${F_{ij}}^a$ and ${G_{ij}}^\mu$ given as before.

The uniqueness argument of the Lagrangian for $T_4\otimes G$ goes the same way as in the T_4 case[1] [notice that the Lagrangian (10) is explicitly independent of the choice of a local orthonormal basis], and one obtains the following Lagrangian for the $T_4\otimes G$ gauge theory in the case of the scalar-multiplet source fields:

$$\mathfrak{L}=\sqrt{-g}\left[\frac{1}{\kappa^2}(\tfrac14 T_{ijk}T_{ijk}+\tfrac12 T_{ijk}T_{ikj}-T_{ijj}T_{ikk}) -\tfrac14 {F_{ij}}^a{F_{ij}}^a+\tfrac12 D_i\phi^a D_i\phi^a-\tfrac12 m^2\phi^a\phi^a\right], \qquad (20)$$

where D_i is the $T_4\otimes G$ covariant derivative (19). Clearly the Lagrangian (20) describes the coupled Einstein-Yang-Mills theory with the scalar multiplets provided that

$$\kappa^2=16\pi G \qquad (21)$$

where G is the gravitational constant.

For the $P_4\otimes G$ case let us start with a set of spinor multiplets ψ^a $(a=1,2,\ldots,n)$. The covariant derivative D_i in this case will be

$$D_i=\partial_i+eA_i^a\xi_a+B_i^\mu\xi_\mu+\tfrac12 {C_i}^{jk}S_{jk}\,, \qquad (22)$$

where A_i^a, ξ_a, and ξ_μ are as before, and

$$S_{jk}=-\tfrac14[\gamma_j,\gamma_k]$$

are the spinor representations of the six generators of the Lorentz subgroup of P_4. Here we have used the Greek indices for the translational subgroup T_4 and the Latin ones for the Lorentz subgroup for the convenience of the physical interpretation of the group actions, since we relate, as before,[1,2] the translation subgroup to a transformation of the coordinate frame e_μ and the Lorentz subgroup to the local orthonormal frame e_i. Notice that we have not introduced the coupling constant for P_4 in Eq. (22) for the same reason as before: It will be introduced by Eq. (14). Remember that our gauge group P_4 is not a direct-product group and we need only one coupling parameter κ for P_4.

Clearly

$$[\partial_i,\xi_\mu]=[\partial_i,S_{jk}]=0\,,$$

and one has

$$[D_i,D_j]={T_{ij}}^k D_k+e{F_{ij}}^a\xi_a+{X_{ij}}^\mu\xi_\mu+\tfrac12 {R_{ij}}^{kl}\xi_{kl}\,, \qquad (23)$$

where ${F_{ij}}^a$ are as before, and

$$\begin{aligned} {X_{ij}}^\mu &={G_{ij}}^\mu+\delta_k^\mu\eta_{l\alpha}(B_i^\alpha {C_j}^{kl}-B_j^\alpha {C_i}^{kl})\,, \\ {R_{ij}}^{kl} &=\partial_i {C_j}^{kl}-\partial_j {C_i}^{kl}+{C_i}^{km}{C_j}^{lm}-{C_j}^{km}{C_i}^{lm} \\ &\quad -{T_{ij}}^m {C_m}^{kl}\,. \end{aligned}$$

For the Lagrangian for the Poincaré gauge fields one can show that P_4 gauge invariance of the theory prevents any quadratic form that can be made of ${X_{ij}}^\mu$ in one's Lagrangian and, for the spinor-multiplet source fields ψ^a, one is led to choose the following Einstein-Cartan-Yang-Mills Lagrangian as the unique Lagrangian for the gauge group $P_4\otimes G$:

$$\mathfrak{L}=\sqrt{-g}\left[\tfrac12 i(\bar\psi\gamma^i\partial_i\psi-\partial_i\bar\psi\gamma^i\psi)-m\bar\psi\psi+\tfrac12 ieA_i^a\bar\psi\gamma^i(\xi_a+\xi_a^\dagger)\psi-\tfrac14 i{C_i}^{jk}\bar\psi(\gamma^i S_{jk}+S_{jk}\gamma^i)\psi-\tfrac14 {F_{ij}}^a{F_{ij}}^a+\frac{1}{\kappa^2}{R_{ij}}^{ij}\right]. \qquad (24)$$

Notice that with the Lagrangian (24) the gauge fields of internal symmetry do not create additional torsion. That is, the spin density of internal Yang-Mills fields does not couple directly to the gauge fields of the Lorentz subgroup C_i^{jk} (as the current density of the subgroup) as one would naively expect. Clearly this is due to the requirement of the gauge invariance of the internal symmetry which dictates the above Lagrangian.

Thus, interpreting the space-time symmetries T_4 and P_4 as the symmetries among the source fields and of the local orthonormal frames at each physical space-time point, we have shown in the conventional gauge formalism that the gauge theory of $T_4 \otimes G$ is indeed the coupled Einstein-Yang-Mills (or Einstein-Maxwell) theory and the gauge theory of $P_4 \otimes G$ is the coupled Einstein-Cartan-Yang-Mills (or Einstein-Cartan-Maxwell) theory. If one keeps the lowest-order possible combination, linear or quadratic in gauge field strengths in one's Lagrangian, the theory in each case becomes unique.

It is a great pleasure to thank Professor P. G. O. Freund and Professor Y. Nambu for illuminating discussions. I am also grateful to S. O. Ahn for encouragement.

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*Work supported in part by the NSF under Contract No. PHYS74-08833 A01.

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