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Gauge theory of Poincaré symmetry*

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The Einstein-Cartan theory of gravitation is shown to be the unique gauge theory of Poincaré symmetry as far as one chooses the Lagrangian to be the lowest possible combination in field strengths. Kibble's derivation of the theory is reformulated and refined in the fiber-bundle picture. The gauge potentials of the Lorentz subgroup are identified as the local affine-connection coefficients which in general allow torsion, and the field strengths of this subgroup are identified as the curvature tensor of the corresponding Riemann-Cartan geometry. The spin current of fermion fields creates the torsion of the geometry.

I. INTRODUCTION

Recently it has been shown[1] that the gauge theory of the 4-dimensional translation group T_4 is unique and becomes precisely the vierbein formalism of Einstein's theory of gravitation. On the other hand it was also pointed out[1] that in the presence of spinor source fields the consistency condition of the theory (that it should not depend upon a choice of a local orthonormal basis) naturally leads us to introduce gauge fields for the Lorentz group, and that one can modify Einstein's theory if one has spinor source fields in the theory. In this case the translational gauge group alone is not enough and one has to include the Lorentz group in one's gauge group.

In this paper we show that with the 4-dimensional Poincaré group P_4 as one's gauge group one can derive the Einstein-Cartan theory[2] of gravitation as the unique gauge theory of P_4 if one chooses one's Lagrangian to be the lowest-order possible combination, linear or quadratic, in field strengths (in this paper the uniqueness should always be understood with this criterion). Kibble[3] was the first to suggest that the Einstein-Cartan theory may be viewed as a gauge theory of Poincaré symmetry. But in his argument the gauge potentials of the translation and those of the Lorentz group seem not to be treated on the same footing, and there appear questions of how these two groups are related and to what extent the theory is unique. The theory was also derived by Sciama[4] as a gauge theory of the Lorentz group alone, but in an already curved space-time, i.e., with the Riemannian metric as an input. In the following we will reformulate and refine the works of Kibble and Sciama in the fiber-bundle picture and will argue that the Einstein-Cartan theory of gravitation is indeed the unique gauge theory of P_4.

The geometry of gauge theories has been known as that of a principal fiber bundle,[5] but there seem to have been few applications of bundle structure to specific problems in physics until recently. The power of the bundle formalism has been appreciated by Cho and Freund[6,7] in unifying non-Abelian gauge theories with gravitation. The relevance of the bundle structure in physics has also been emphasized by Wu and Yang.[8] Throughout this paper we will use the bundle formalism to exclude any possible confusion in the argument although it is not necessary to use the bundle language. But in this bundle formalism one does not need matter fields, neither does one have to make local "phase" transformations, to introduce gauge fields. Rather, gauge potentials are created, without ever mentioning the matter fields, from the fact that one can choose a 4-dimensional horizontal tangent subspace in the tangent space of the bundle which does not necessarily form a closed Lie algebra, and the field strengths are given by the vertical components of the commutator of two horizontal vector fields. For the details about the bundle formalism we refer the reader to Ref. 6.

We will first prove our claim by constructing the 14-dimensional bundle of the Poincaré group P_4 over space-time and then discuss the theory in the presence of spinor source fields. After this the gauge potentials and the field strengths of the Lorentz subgroup are respectively identified as the connection coefficients that allow torsion in general and as the curvature tensor of the corresponding Riemann-Cartan geometry of the theory.

II. GAUGE THEORY OF THE POINCARÉ GROUP

To derive the unique gauge theory of the Poincaré group P_4 it is relevant to understand how Einstein's theory comes about as the unique gauge

$$\partial^*_{ij}B_k^\alpha=-\kappa^{-1}f_{ij,\beta}^{\ \ \alpha}B_k^\beta\ ,$$
$$\partial^*_\alpha C_k^{\ ij}=0\ ,$$
$$\partial^*_{ij}C_k^{\ mn}=-\tfrac{1}{2}\kappa^{-1}f_{ij,pq}^{\ \ mn}C_k^{\ pq}\ ,$$

where ∂^*_α and ∂^*_{ij} are the directional derivatives to the vertical direction of ξ^*_α and ξ^*_{ij}, respectively, one easily obtains

$$\begin{aligned}X_{ij}^{\ \alpha}&=\partial_iB_j^\alpha-\partial_jB_i^\alpha+\delta_k^\alpha\eta_{l\mu}(B_i^\mu C_j^{\ kl}-B_j^\mu C_i^{\ kl})\\&\quad-T_{ij}^{\ \ k}B_k^\alpha\ ,\\R_{ij}^{\ \ kl}&=\partial_iC_j^{\ kl}-\partial_jC_i^{\ kl}+C_i^{\ km}C_j^{\ lm}-C_j^{\ km}C_i^{\ lm}\\&\quad-T_{ij}^{\ \ m}C_m^{\ kl}\ .\end{aligned}\tag{9}$$

Now gauge transformations in this bundle picture are changes of bundle cross section.[5,6,7] However, since we want to view here the Poincaré symmetry as a space-time symmetry, not just as an internal symmetry, *the Lorentz subgroup should also act on the local orthonormal basis e_i as well as on the vertical fiber space*. For the translational subgroup we interpret as in Ref. 1 that it does not act on the local orthonormal basis e_i, but on the coordinate basis e_μ, generating a general coordinate transformation. So in this gauge theory of space-time symmetry the gauge transformation is made of two parts, one from the change of the cross section, the other from the corresponding change of the basis. Thus under an infinitesimal gauge transformation of P_4, one has

$$\begin{aligned}\delta B_i^\alpha&=\tfrac{1}{2}f_{\beta,jk}^{\ \alpha}B_i^\beta\theta^{jk}+\tfrac{1}{2}f_{jk,\beta}^{\ \ \alpha}C_i^{\ jk}\theta^\beta+\partial_i\theta^\alpha+\theta_{ik}B_k^\alpha\ ,\\\delta C_i^{\ jk}&=\tfrac{1}{4}f_{mn,pq}^{\ \ jk}C_i^{\ mn}\theta^{pq}+\partial_i\theta^{jk}+\theta_{il}C_l^{\ jk}\end{aligned}\tag{10}$$

so that

$$\begin{aligned}\delta X_{ij}^{\ \alpha}&=\tfrac{1}{2}f_{\beta,kl}^{\ \alpha}X_{ij}^{\ \beta}\theta^{kl}+\tfrac{1}{2}f_{kl,\beta}^{\ \ \alpha}R_{ij}^{\ \ kl}\theta^\beta\\&\quad-\tfrac{1}{4}f_{ij,pq}^{\ \ mn}X_{mn}^{\ \ \alpha}\theta^{pq}\ ,\\\delta R_{ij}^{\ \ kl}&=\tfrac{1}{4}f_{mn,pq}^{\ \ kl}R_{ij}^{\ \ mn}\theta^{pq}-\tfrac{1}{4}f_{ij,pq}^{\ \ mn}R_{mn}^{\ \ kl}\theta^{pq}\ ,\end{aligned}\tag{11}$$

where θ^α and $\theta^{ik}=-\theta^{ki}$ are the 10 infinitesimal functions corresponding to the P_4 gauge transformation. Notice that the last term in each of the 4 equations above comes from the rotation of the local orthonormal basis

$$\partial_i\to\partial_i'=\partial_i+\theta_{ik}\partial_k\ ,$$

which one has to take into account in addition to the "internal" part of the gauge transformation of the space-time symmetry P_4.

As for the action integral, Kibble's[3]

$$I=\frac{1}{\kappa^2}\int\sqrt{-g}\,R_{ij}^{\ \ ij}d^4x\tag{12}$$

with

$$\sqrt{-g}=\det(h_\mu^i)\tag{13}$$

is linear in the field strength $R_{ij}^{\ \ kl}$ and clearly a possible candidate; it comes from the Lorentz part of the field strengths alone. To ensure the P_4 gauge invariance of the action integral (12) one has only to notice that the volume element $\sqrt{-g}\,d^4x$ can be made invariant under the P_4 gauge transformation.[11] Kibble has chosen this form for simplicity, but notice that the Killing form of the invariant metric for P_4 is singular since the group is not semisimple so that, on the basis of the simplicity alone, there seems *a priori* no reason why one should exclude a quadratic combination in $X_{ij}^{\ \alpha}$ in one's Lagrangian, this being the lowest-order combination of the translational part of the field strengths. However, one can easily argue that the action integral (12) is indeed unique since any quadratic combination in $X_{ij}^{\ \alpha}$ is not invariant under the P_4 gauge transformation (or more specifically under the T_4 subgroup gauge transformation). This is so because in Eq. (11) $\delta X_{ij}^{\ \alpha}$ necessarily contains the term $\frac{1}{2}f_{kl,\beta}^{\ \ \alpha}R_{ij}^{\ \ kl}\theta^\beta$ under the gauge transformation, which will spoil the gauge invariance for any quadratic combination in $X_{ij}^{\ \alpha}$. Thus the action integral (12) for P_4 gauge fields is indeed unique if one keeps the lowest-order possible combination of field strengths in one's Lagrangian.

Of course gauge invariance alone cannot exclude a quadratic term and/or higher-order combinations in $R_{ij}^{\ \ kl}$ from one's Lagrangian, and this kind of arbitrariness is common to any gauge theory. Now since the action integral (12) is linear in $R_{ij}^{\ \ kl}$ one might like to include the "kinetic" term $\mathcal{L}'$ for $C_i^{\ jk}$,

$$\mathcal{L}'=-\tfrac{1}{4}\times\tfrac{1}{2}\sqrt{-g}\,R_{ij}^{\ \ kl}R_{ij}^{\ \ kl}\ ,$$

in one's Lagrangian. But there seems no compelling motivation for this since we have already seen that the gauge invariance of the theory has excluded the kinetic term for B_i^α also.

In the absence of matter fields the action integral (12) gives the following equations of motion:

$$\begin{aligned}&R_{ik}^{\ \ jk}-\tfrac{1}{2}\delta_i^jR_{kl}^{\ \ kl}=0\ ,\\&C_{ijk}-C_{jik}+\eta_{ik}C_{llj}-\eta_{jk}C_{lli}=T_{ijk}+\eta_{ik}T_{jll}-\eta_{jk}T_{ill}\ ,\end{aligned}\tag{14}$$

where

$$C_{ijk}=C_i^{\ mn}\eta_{jm}\eta_{kn}\ ,$$
$$T_{ijk}=T_{ij}^{\ \ m}\eta_{km}\ .$$

One can solve the second equation and obtain

$$C_{ijk}=\tfrac{1}{2}(T_{ijk}-T_{ikj}-T_{jki})\ .\tag{15}$$

Using Eq. (15) the integral (12) can be rewritten in terms of B_i^α alone and one easily finds that the theory reduces to Einstein's vacuum theory. Notice here that had one included the term $\mathcal{L}'$ in one's Lagrangian, one would not have had Eq. (15)

theory of translation because some of the arguments apply the same way here. We will not repeat these arguments and for the details of the derivation of Einstein's theory as the unique gauge theory of translation we refer the reader to Ref. 1.

Let us first start with a 4-dimensional flat space-time and introduce a local orthonormal basis at each point, i.e., four orthonormal vector fields e_i $(i=0,1,2,3)$, with the following commutation relations:

$$[e_i, e_j] = T_{ij}{}^k e_k . \tag{1}$$

If we introduce a commuting coordinate basis e_μ $(\mu=0,1,2,3)$, e_i can be written in terms of the vierbein fields h_i^μ:

$$e_i = h_i^\mu e_\mu, \quad e_\mu = h_\mu^i e_i , \tag{2}$$

where h_μ^i are the inverse vierbein fields, so that

$$h_i^\mu h_\mu^k = \delta_i^k, \quad h_\mu^i h_i^\nu = \delta_\mu^\nu .$$

From Eqs. (1) and (2) one easily finds

$$\begin{aligned} T_{ij}{}^k &= (\partial_i h_j^\mu - \partial_j h_i^\mu) h_\mu^k \\ &= -h_i^\mu h_j^\nu (\partial_\mu h_\nu^k - \partial_\nu h_\mu^k) , \end{aligned} \tag{3}$$

where ∂_i and ∂_μ are the directional derivatives with respect to e_i and e_μ. Notice that we start from a flat space-time and so far our h_i^μ remain trivial although they become nontrivial when one requires local T_4 invariance.[1]

For the group space of P_4, let ξ_α $(\alpha=0,1,2,3)$ and $\xi_{ik} = -\xi_{ki}$ $(i,k=0,1,2,3)$ be the 10 generators of P_4 with the familiar commutation relations

$$\begin{aligned} &[\xi_\alpha, \xi_\beta] = 0, \\ &[\xi_{ik}, \xi_\alpha] = (\eta_{i\alpha}\delta_k^\beta - \eta_{k\alpha}\delta_i^\beta)\xi_\beta = f_{ik,\alpha}{}^\beta \xi_\beta, \\ &[\xi_{ij}, \xi_{kl}] = \eta_{ik}\xi_{jl} + \eta_{jl}\xi_{ik} - \eta_{il}\xi_{jk} - \eta_{jk}\xi_{il} \\ &\qquad = \tfrac{1}{2} f_{ij,kl}{}^{mn} \xi_{mn} , \end{aligned} \tag{4}$$

where

$$\eta_{ik} = \text{diag}(+,-,-,-) .$$

Here we have used and will continue to use the Greek indices $\alpha, \beta, \ldots$ for the translational subgroup T_4 and the Latin indices $i, j, k, \ldots$ for the Lorentz subgroup for the convenience of the physical interpretation of the group actions, since the translational subgroup can be related to the transformation of the coordinate frame e_μ as in Ref. 1 and the Lorentz subgroup will be related to the transformation of the local orthonormal frame e_i.

Now let us construct the 14-dimensional bundle space whose base manifold is the 4-dimensional space-time and whose fiber space (the vertical space) is isomorphic to the 10-dimensional Poincaré group P_4, and introduce a connection[6]

$$\omega = \omega^\alpha \xi_\alpha + \tfrac{1}{2}\omega^{ik}\xi_{ik}, \quad \omega^{ik} = -\omega^{ki}$$

in this bundle. Then the gauge potentials B_i^α and $C_i{}^{jk}$ are as usual given by the connection coefficients of $\tilde{e}_i$ $(i=0,1,2,3)$, the lift of e_i into a gauge-defining 4-dimensional submanifold σ of the bundle:

$$\begin{aligned} \kappa B_i^\alpha &= \omega^\alpha(\tilde{e}_i) , \\ \kappa C_i{}^{jk} &= \omega^{jk}(\tilde{e}_i) , \end{aligned} \tag{5}$$

where we have introduced a dimensional constant κ (of dimension of a length) to give the canonical dimension[9] to the gauge potentials B_i^α and $C_i{}^{jk}$. This κ will serve as the coupling constant for the gauge group P_4 and will be related to the gravitational constant later on. As in Ref. 1 we will interpret the gauge potentials B_i^α as the nontrivial part of the vierbein fields,

$$h_i^\mu = \delta_i^\mu + \kappa B_i^\mu . \tag{6}$$

This means that *we create the curvature for our space-time by erecting the gauge potentials* B_i^μ *and making the vierbein fields* h_i^μ *nontrivial*. This identification was a crucial ingredient in deriving Einstein's theory of gravitation as the unique gauge theory of the translation group T_4. A justification of this identification can easily be given as before[1,3] and we will not repeat the argument here.

In the bundle let $\hat{e}_i$ be the horizontal lift of e_i, and let ξ_α^* and $\xi_{ik}^* = -\xi_{ki}^*$ be the vertical fundamental vector fields corresponding to ξ_α and ξ_{ik}. Clearly these 14 vector fields $\hat{e}_i$, ξ_α^*, and ξ_{ik}^* can serve as a basis for the bundle with the following commutation relations:

$$\begin{aligned} &[\xi_\alpha^*, \xi_\beta^*] = 0, \\ &[\xi_{ij}^*, \xi_\alpha^*] = \kappa^{-1} f_{ij,\alpha}{}^\beta \xi_\beta^*, \\ &[\xi_{ij}^*, \xi_{kl}^*] = \tfrac{1}{2}\kappa^{-1} f_{ij,kl}{}^{mn} \xi_{mn}^*, \\ &[\hat{e}_i, \xi_\alpha^*] = [\hat{e}_i, \xi_{jk}^*] = 0, \\ &[\hat{e}_i, \hat{e}_j] = T_{ij}{}^k \hat{e}_k - \kappa X_{ij}{}^\alpha \xi_\alpha^* - \tfrac{1}{2}\kappa R_{ij}{}^{kl} \xi_{kl}^* . \end{aligned} \tag{7}$$

Here again κ is inserted to give the canonical dimension to the vector fields. To determine $X_{ij}{}^\alpha$ and $R_{ij}{}^{kl}$ in terms of potentials, observe that $\hat{e}_i$ can be written as

$$\hat{e}_i = \tilde{e}_i - \kappa B_i^\alpha \xi_\alpha^* - \tfrac{1}{2}\kappa C_i{}^{jk} \xi_{jk}^* , \tag{8}$$

which comes from the definition of ω (Ref. 6)

$$\begin{aligned} &\omega^\alpha(\hat{e}_k) = 0, \quad \omega^{ij}(\hat{e}_k) = 0 , \\ &\omega^\alpha(\xi_\beta^*) = \delta_\beta^\alpha, \quad \omega^{ij}(\xi_\alpha^*) = 0 , \\ &\omega^\alpha(\xi_{ij}^*) = 0, \quad \omega^{ij}(\xi_{kl}^*) = \delta_k^i\delta_l^j - \delta_l^i\delta_k^j . \end{aligned}$$

Using Eqs. (7) and (8), and noticing that[10]

$$\partial_\alpha^* B_k^\beta = -\tfrac{1}{2}\kappa^{-1} f_{\alpha,}{}^\beta{}_{ij} C_k{}^{ij} ,$$

and there would have been no way to interpret the gauge potentials B_i^α as dynamically independent variables.

III. SPINOR SOURCE FIELDS

Now let us consider the theory in the presence of spinor source fields. For simplicity let us consider a Dirac spinor ψ as the source field and start with the following action integral for the spinor written in the local orthonormal basis (1),

$$I=\int\sqrt{-g}\,[\tfrac{1}{2}i(\bar\psi\gamma^i\partial_i\psi-\partial_i\bar\psi\gamma^i\psi)-m\bar\psi\psi]d^4x\ , \tag{16}$$

where γ^i $(i=0,1,2,3)$ are the Dirac matrices. Now notice that, independent of whether the space-time is curved or not, the action integral (16) as it stands cannot describe a meaningful theory simply because it is *not* independent of the choice of a local orthonormal basis. To take care of this defect and also to create a Riemannian metric for the space-time let us require Poincaré gauge invariance on the action integral (16), and introduce the P_4-gauge-covariant derivative D_i for the spinor;

$$D_i=\partial_i+B_i^\mu\xi_\mu+\tfrac{1}{2}C_i{}^{jk}S_{jk}\ , \tag{17}$$

where B_i^μ, $C_i{}^{jk}$, and ξ_μ are as before and

$$S_{ij}=-\tfrac{1}{4}[\gamma_i,\gamma_j] \tag{18}$$

are the spinor representations of the generators ξ_{ij} of the Lorentz subgroup. Now as in Ref. 1 one can interpret that the field ψ remains invariant under the translational subgroup T_4 in this bundle formalism so that

$$\xi_\mu\psi=0\ , \tag{19}$$

and one has

$$\delta\psi=-\tfrac{1}{2}S_{ij}\theta^{ij}\psi \tag{20}$$

under the *whole* infinitesimal P_4 gauge transformation. With the covariant derivative (17) one can easily write down the P_4-gauge-invariant Lagrangian for ψ:

$$\begin{aligned}\mathcal{L}&=\sqrt{-g}\,[\tfrac{1}{2}i\bar\psi\gamma^iD_i\psi+(\text{c.c.})-m\bar\psi\psi]\\&=\sqrt{-g}\,[\tfrac{1}{2}i(\bar\psi\gamma^i\partial_i\psi-\partial_i\bar\psi\gamma^i\psi)-m\bar\psi\psi\\&\qquad+\tfrac{1}{4}iC_i{}^{jk}\bar\psi(\gamma^iS_{jk}+S_{jk}\gamma^i)\psi]\ .\end{aligned} \tag{21}$$

Notice that under the infinitesimal P_4 gauge transformation, one has

$$\begin{aligned}&\psi\to\psi'=U\psi\ ,\\&\bar\psi\to\bar\psi'=\bar\psi\gamma^0U^\dagger\gamma^0=\bar\psi U^{-1}\ ,\\&\tfrac{1}{2}C_i{}^{jk}S_{jk}\to\tfrac{1}{2}C_i'{}^{jk}S_{jk}\\&\qquad=\tfrac{1}{2}C_i{}^{jk}US_{jk}U^{-1}+U\partial_iU^{-1}+\tfrac{1}{2}\theta_{im}C_m{}^{jk}S_{jk}\ ,\end{aligned} \tag{22}$$

where

$$U=1-\tfrac{1}{2}S_{jk}\theta^{jk}\ ,$$

so that

$$\begin{aligned}&\bar\psi\gamma^i(\partial_i+\tfrac{1}{2}C_i{}^{jk}S_{jk})\psi\\&\quad\to\bar\psi'\gamma^i(\partial_i'+\tfrac{1}{2}C_i'{}^{jk}S_{jk})\psi'\\&\qquad=\bar\psi U^{-1}\gamma^iU(\partial_i'+\tfrac{1}{2}C_i'{}^{jk}U^{-1}S_{jk}U+U^{-1}\partial_i'U)\psi\\&\qquad=\bar\psi U^{-1}\gamma^iU(\partial_i'+\tfrac{1}{2}C_i{}^{jk}S_{jk}+\theta_{im}C_m{}^{jk}S_{jk})\psi\\&\qquad=\bar\psi\gamma^i(\partial_i+\tfrac{1}{2}C_i{}^{jk}S_{jk})\psi\ .\end{aligned}$$

Thus, the Lagrangian (21) is indeed P_4 gauge invariant. Now including the Lagrangian for the gauge fields of P_4 in Eq. (21) one obtains the following Lagrangian:

$$\begin{aligned}\mathcal{L}=\sqrt{-g}\,[&\tfrac{1}{2}i(\bar\psi\gamma^i\partial_i\psi-\partial_i\bar\psi\gamma^i\psi)-m\bar\psi\psi\\&+\tfrac{1}{4}iC_i{}^{jk}\bar\psi(\gamma^iS_{jk}+S_{jk}\gamma^i)\psi+(1/\kappa^2)R_{ij}{}^{ij}]\ .\end{aligned} \tag{23}$$

As before, $C_i{}^{jk}$ are not dynamically independent since the Lagrangian (23) is linear in $R_{ij}{}^{kl}$. From the Euler-Lagrange equations of motion $C_i{}^{jk}$ can be written as

$$C_{ijk}=C_{ijk}^{(0)}+C_{ijk}^{(1)}\ , \tag{24}$$

where[12]

$$\begin{aligned}&C_{ijk}^{(0)}=\tfrac{1}{2}(T_{ijk}-T_{ikj}-T_{jki})\ ,\\&C_{ijk}^{(1)}=\tfrac{1}{8}\kappa^2\epsilon_{ijkl}\bar\psi\gamma^l\gamma^5\psi\ .\end{aligned}$$

Thus, in general, spinor source fields make an additional contribution to $C_i{}^{jk}$. Geometrically, this additional term (i.e., $C_{ijk}^{(1)}$) is known as the contortion created by the spinors. The geometric meaning of the theory will be discussed in detail soon.

Using Eq. (24) the Lagrangian (23) can be rewritten in terms of dynamically independent variables alone:

$$\begin{aligned}\mathcal{L}=\sqrt{-g}\,\Big[&\frac{1}{\kappa^2}(\tfrac{1}{4}T_{ijk}T_{ijk}+\tfrac{1}{2}T_{ijk}T_{ikj}-T_{iij}T_{ikk})\\&+\tfrac{1}{2}i(\bar\psi\gamma^i\partial_i\psi-\partial_i\bar\psi\gamma^i\psi)-m\bar\psi\psi\\&-\tfrac{1}{8}\epsilon^{ijkl}T_{ijk}\bar\psi\gamma_l\gamma^5\psi+\tfrac{3}{32}\kappa^2(\bar\psi\gamma^i\gamma^5\psi)^2\Big]\\&+\text{a total divergence}\ .\end{aligned} \tag{23$'$}$$

Clearly the Lagrangian (23′) describes the Einstein-Cartan theory of gravitation[2,3,4] in the presence of the spinor source field provided that

$$\kappa^2=16\pi G\ , \tag{25}$$

where G is the gravitational constant. The 4-point fermion interaction in the Lagrangian (23′) measures precisely the difference between Einstein's theory and the Einstein-Cartan theory in the presence of spin-$\frac{1}{2}$ source fields. Observe that this modification was naturally introduced to make the theory basis independent in the presence of the spinor source fields and, from the point of the gauge principle, Einstein-Cartan theory seems the natural one for the theory of gravitation. Of course this modification is obtained within the context of the classical field theory and it is an open question[13] whether or not a quantum correction would induce new terms not contained in the Lagrangian (23′) into the theory. At this point it would be nice to recall how Einstein's theory accommodates itself with the local Lorentz invariance in the presence of spinor source fields. Observe that $C^{(0)}_{ijk}$ itself satisfies the transformation law (10) under the Lorentz gauge transformation so that one can make the Lagrangian (23) gauge invariant just by replacing C_{ijk} by $C^{(0)}_{ijk}$ in it. Indeed, this is the way Einstein's theory satisfies the local Lorentz invariance.

It is not the purpose of this paper to study the detailed properties of the Lagrangian (23) and for this we simply refer the reader to the existing literature.[14] Rather we will discuss the geometrical meaning of our P_4 gauge theory in the following.

IV. GEOMETRICAL INTERPRETATION

Thus we have constructed the unique gauge theory of the Poincaré group P_4. Now, at this point there is no doubt that[3,4] the geometrical meaning of the gauge theory should be given in terms of Riemann-Cartan geometry. In fact one can easily identify the gauge potentials and field strengths of the Lorentz subgroup as the connection coefficients and curvature tensor of the corresponding Riemann-Cartan geometry. To see this let us first recall some basic quantities in Riemann-Cartan geometry.

Let $C_i(\tau)$ be the integral curve of the vector field e_i with a scalar parameter τ and $\phi_i(\tau)$ be the parallel transport of the tangent space at the point $C_i(0)$ onto the one at $C_i(\tau)$ that leaves the metric invariant. Then the affine connection coefficients $\Gamma_{ij}{}^k$ are defined as the components of $\nabla_i e_j$, i.e., the "covariant" derivative of e_j with respect to e_i:

$$\nabla_i e_j = \Gamma_{ij}{}^k e_k = \lim_{\tau\to 0} \frac{1}{\tau}\left[\phi_i^{-1}(\tau) e_j(\tau) - e_j(0)\right] . \tag{26}$$

Here, by "covariant" it is meant that one is comparing the two vectors $\phi_i^{-1}(\tau)e_j(\tau)$ and $e_j(0)$ at the same point $C_i(0)$ and the "covariant" derivative ∇_i has nothing to do with the P_4-gauge-covariant derivative (17). Now in Riemannian geometry, of course, the connection coefficients $\Gamma_{ij}{}^k$ are determined by the metric alone. But in Riemann-Cartan geometry this is not so since in this case one has the torsion $t_{ij}{}^k$ given by

$$\nabla_i e_j - \nabla_j e_i = [e_i, e_j] + t_{ij}{}^k e_k . \tag{27}$$

So in general the connection coefficients $\Gamma_{ij}{}^k$ are given in terms of the metric *and* the torsion. In the local orthonormal basis one easily obtains[15]

$$\Gamma_{ijk} = \Gamma^{(R)}_{ijk} + \Gamma^{(C)}_{ijk} , \tag{28}$$

where

$$\Gamma^{(R)}_{ijk} = \tfrac{1}{2}(T_{ijk} - T_{ikj} - T_{jki}) ,$$
$$\Gamma^{(C)}_{ijk} = \tfrac{1}{2}(t_{ijk} - t_{ikj} - t_{jki})$$

and

$$\Gamma_{ijk} = \Gamma_{ij}{}^m \eta_{mk} ,$$
$$t_{ijk} = t_{ij}{}^m \eta_{mk} .$$

Observe that Γ_{ijk} has a new contribution $\Gamma^{(C)}_{ijk}$ from the torsion in addition to the good old Riemannian part $\Gamma^{(R)}_{ijk}$. This new term $\Gamma^{(C)}_{ijk}$ is characteristic of Riemann-Cartan geometry and called by definition the contortion. Also the curvature tensor $R_{ijk}{}^l$ is defined in this basis as

$$R_{ijk}{}^l e_l = [\nabla_i, \nabla_j] e_k - \nabla_{[e_i, e_j]} e_k ,$$

so that

$$R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l - \partial_j \Gamma_{ik}{}^l + \Gamma_{im}{}^l \Gamma_{jk}{}^m - \Gamma_{jm}{}^l \Gamma_{ik}{}^m - T_{ij}{}^m \Gamma_{mk}{}^l . \tag{29}$$

Now it is clear from Eqs. (24) and (28) that the gauge potentials C_{ijk} are nothing more than the connection coefficients Γ_{ijk}. In the presence of spinor source fields the spin current of the fermions creates the controtion $C^{(1)}_{ijk}$ and one has the torsion

$$t_{ijk} = \Gamma^{(C)}_{ijk} - \Gamma^{(C)}_{jik} = C^{(1)}_{ijk} - C^{(1)}_{jik} = \tfrac{1}{4}\kappa^2 \epsilon_{ijkl} \bar{\psi}\gamma^l \gamma^5 \psi \tag{30}$$

in one's geometry which is in addition to the metric and makes the geometry Riemann-Cartan. Also, from Eqs. (9) and (29) one identifies the gauge field strengths $R_{ij}{}^{kl}$ precisely as the curvature tensor of the corresponding geometry.

The gauge potentials of the translation subgroup can be interpreted as usual as the nontrivial part of the vierbein fields,[1] and they create a Riemannian metric to the geometry.

V. CONCLUSION

It has been shown that the Einstein-Cartan theory of gravitation is indeed the unique gauge theory of the Poincaré group if one chooses the Lagrangian to be the lowest-order possible combinations in field strengths. The gauge potentials of the Lorentz subgroup are identified as the connection coefficients that allow torsion in general and the field strengths of the subgroup as the curvature tensor of the corresponding Riemann-Cartan geometry. The gauge potentials of the translational subgroup are interpreted as the nontrivial part of the vierbein fields, and they create a Riemannian metric to the geometry.

Note added in proof. Recently F. W. Hehl, P. Heyde, G. D. Kerlick, and J. M. Nester [Rev. Mod. Phys. 48, 393 (1976)] also tried to derive Einstein-Cartan theory as the gauge theory of the Poincaré group. However, we would like to make clear that our approach is *different* from theirs. These authors admit that the commutation relation of their gauge group, in particular Eq. (4.28), does *not* yield the algebra of the Poincaré group, and thus their derivation of Einstein-Cartan theory as the gauge theory of Poincaré symmetry appears to be controversial and incomplete.

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[9] Our units in this paper will be $c=\hbar=1$.

[10] These relations can easily be obtained from the general structure of the bundle as is shown in detail in Ref. 5, Eq. (10).

[11] A simple way to keep the volume element invariant is to require

$$\delta h_i^\mu = \theta_{ik} h_k^\mu = \theta_{ik} h_k^\mu + \partial_i \theta^\mu .$$

According to Eq. (6) this amounts to having

$$\delta_i^\mu \to \delta_i^\mu + \theta_{ik}\delta_k^\mu - \tfrac{1}{2} f_{\alpha,jk}^{\ \mu} B_i^\alpha \theta^{jk} - \tfrac{1}{2}\kappa f_{jk,\alpha}^{\ \ \mu} C_i^{\ jk}\theta^\alpha$$

under the P_4 gauge transformation. The appearance of the last two terms in the second equation can be understood if one remembers that the decomposition of the vierbein field by Eq. (6) is after all gauge dependent under the Lorentz subgroup.

[12] Our notation here is $\epsilon_{0123}=+1$ and $\gamma^5=\frac{1}{4}i\epsilon_{ijkl}\gamma^i\gamma^j\gamma^k\gamma^l = i\gamma^0\gamma^1\gamma^2\gamma^3$.

[13] For the present situation on the renormalizability of gravitation, see S. Deser, invited paper, GTMFT Conference, 1975 (unpublished).

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