Generalized Einstein-Cartan field equations

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Field equations are studied for generalized Einstein-Cartan-Sciama-Kibble (EC) theories in which the connection is not necessarily compatible with the metric and the Lagrangian is not necessarily the curvature scalar. The condition that the Euler-Lagrange equations for a general Lagrangian density $L(g,\partial g,\partial\partial g,\Gamma,\partial\Gamma)$ involve no third- or higher-order derivatives of the metric requires that the gravitational field equations be equivalent to those of general relativity with modified sources. The divergence of the symmetric "energy-momentum" tensor $\delta L_{\text{matter}}/\delta g_{(ij)}$ evaluated with $\delta(\Gamma^l{}_{mn}-\{^{\,l}_{mn}\})=0$ for a generalized EC theory does not vanish in the presence of spin. The general form of the spin field equation linear in the defect $\lambda^i{}_{jk}\equiv\Gamma^i{}_{jk}-\{^{\,i}_{jk}\}$ is derived.

I. INTRODUCTION

In Einstein's theory of gravitation, the mass distribution alone determines the metric tensor g_{ij} (in the notation of Ref. 1) in terms of which the geometric properties of the pseudo-Riemannian space-time of general relativity can be described. However, spin is an intrinsic property of matter as significant as mass,[3] so it seems natural to modify general relativity to reflect the fundamental importance of spin. One such modification is achieved in the Einstein-Cartan-Sciama-Kibble theory[4] [often abbreviated Einstein-Cartan (EC) theory] in which the material spin density acts as the source of the geometric torsion $Q^i{}_{jk}$ of space-time.

The linear connection $\Gamma^i{}_{jk}$ for the pseudo-Riemannian geometry of general relativity

$$\Gamma^i{}_{jk}(\text{Einstein})=\{^{\,i}_{jk}\}=\tfrac{1}{2}g^{il}(g_{jl,k}+g_{kl,j}-g_{jk,l}) \qquad (1.1)$$

is symmetric, so the torsion, defined by

$$Q^i{}_{jk}=\Gamma^i{}_{kj}-\Gamma^i{}_{jk}\,, \qquad (1.2)$$

is zero in that geometry. On the other hand, the linear connection of the EC theory is determined by both torsion and metric, and these alone, through the assumption that the connection is compatible with the metric tensor:

$$g_{ij;k}=0\,. \qquad (1.3)$$

[A space-time for which (1.3) is satisfied is said to be metric.] The solution of (1.2) and (1.3) defines the defect[5]

$$\lambda^i{}_{jk}\equiv\Gamma^i{}_{jk}-\{^{\,i}_{jk}\} \qquad (1.4)$$

for the EC theory:

$$\lambda^i{}_{jk}(\text{EC})=-\tfrac{1}{2}(Q^i{}_{jk}+Q_{jk}{}^i+Q_{kj}{}^i)\,. \qquad (1.5)$$

Thus, the metric and the torsion describe the geometric properties of the EC space-time.

The metric condition generalizes the case of special relativity for which the Minkowski metric η_{ij} satisfies the condition $\eta_{ij,k}=0$ sufficient for a flat space. It ensures that space-time is locally Minkowskian with respect to both metric and connection.[6] Furthermore, while the existence of a metric allows one to introduce spinors, the metric condition defines natural covariant derivatives of spinors.[2]

These arguments for space-time being metric are not so strong, however, that nonmetric space-times must be excluded from consideration. Therefore, in the following we consider geometries in which the defect need not be restricted by the metric condition.

Similar considerations apply to the possibility of generalizing the field equations of the EC theory. A field equation for a geometrical object in a physics theory decomposes naturally into two sides, one involving purely geometric variables and the other involving the material or physical field sources. The geometric sides of the field equations for the EC theory are a straightforward generalization of the Einstein equations, as they are derived from the geometric Lagrangian density $L_G(\text{EC})$ given by

$$L_G(\text{Einstein})=(-g)^{1/2}R[\{\;\}]\to(-g)^{1/2}R[\Gamma(\text{EC})]=L_G(\text{EC})\,, \qquad (1.6)$$

where, for example, $R[\Gamma(\text{EC})]$ is the curvature scalar formed from the connection

$$\Gamma^i{}_{jk}(\text{EC})=\{^{\,i}_{jk}\}+\lambda^i{}_{jk}(\text{EC})\,. \qquad (1.7)$$

This Lagrangian density may be rewritten

$$L_G(\text{EC})=(-g)^{1/2}R[\{\;\}]+L_s+\text{divergence}\,, \qquad (1.8)$$

where L_s is a scalar density that is a homogeneous polynomial of degree two in the torsion. Thus, an effective Lagrangian density for the EC theory is the Einstein Lagrangian density plus a homogeneous quadratic in the torsion. In other words, the EC theory may be considered as general relativity with modified sources.[7]

Generalized geometric Lagrangian densities L_G have been considered recently by von der Heyde.[8] He assumed that the metric condition is satisfied, so the geometry of space-time is determined by the metric g_{ij} and the torsion $Q^i{}_{jk}$, and that the geometric Lagrangian density contains no fundamental constants except c and the gravitational constant G and is quadratic in the torsion. He found that, under these conditions, the general form of the effective geometric Lagrangian density is

$$L_G = (-g)^{1/2}(aR[\{\ \}] + b) + L_s , \tag{1.9}$$

where a and b are numbers and L_s is homogeneous quadratic in the torsion and is completely determined up to three arbitrary constants. By assuming that there exists a basis, anholonomic in the presence of torsion, in the tangent plane at any single point in space-time such that the total Lagrangian can be reduced at that point to the matter Lagrangian of special relativity, von der Heyde obtained the Lagrangian density (1.6) of the EC theory.

Sandberg[9] recently has considered the case in which the linear connection $\Gamma^i{}_{jk}$ satisfies no *a priori* conditions, such as the metric condition, under the assumption that the geometric Lagrangian density is the generalized Einstein density

$$L_G = (-g)^{1/2} R[\Gamma] . \tag{1.10}$$

He further allowed the spin density

$$S^{kij} = \frac{\delta L_m}{\delta \Gamma_{ijk}} = \frac{\delta L_m}{\delta \lambda_{ijk}}, \tag{1.11}$$

defined in terms of the matter Lagrangian density L_m, to be arbitrary and thus not restricted by the skew property

$$S^{kij} = -S^{kji} \tag{1.12}$$

usually assumed to provide a physical interpretation for $\delta L_m/\delta\lambda_{ijk}$ as the physical spin.[10] Sandberg showed that as a result of the projective invariance of the Lagrangian density (1.10) the matter Lagrangian density was restricted by the condition $S_{ki}{}^i = 0$. By modifying the usual comma-to-semicolon rule ($\partial/\partial x^i$ in special relativity $\rightarrow \nabla_i$ in general relativity for determining L_m from the Lagrangian in special relativity) he found a way to satisfy this restriction automatically, and in this manner obtained field equations from (1.10) equivalent to Einstein's with a modified source Lagrangian.

The Einstein-Cartan (and the Sandberg) Lagrangian density is the simplest generalization of that of Einstein [see Eq. (1.6)] and thus is a very natural choice, but it is not necessary; because there is an additional geometric field, the defect tensor, in principle L_G(Einstein) can be generalized at least by adding to it any scalar density that vanishes for zero defect. In the present paper, we consider the general case in which (i) the geometric Lagrangian density L_G is dependent in an arbitrary way on the metric tensor and its first and second derivatives and on the connection, subject to no *a priori* conditions, together with its first derivatives,

$$L_G = {}'L_G(g, \partial g, \partial\partial g, \Gamma, \partial\Gamma) , \tag{1.13}$$

and (ii) the Euler-Lagrange equations involve no derivatives higher than the second. We show in Sec. II A that all such L_G do reduce to L_G(Einstein) for zero torsion, generalizing results of von der Heyde,[8] Sandberg,[9] and Nester.[7] In Sec. II B we derive the geometric side of the gravitational field equation in terms of the spin and show that L_G must satisfy conditions in order that the Euler-Lagrange equations do not involve third- or fourth-order derivatives of the metric tensor.

More detailed consideration is given to special cases in which the geometric side of the field equation for the defect is (i) derivative-free and linear in the defect and (ii) linear in the defect and its first two derivatives. Case (ii) allows for the possibilities that the defect is nonzero in regions of zero spin density and that defect waves exist,[11] contrary to case (i) and the EC theory where the defect is related algebraically to the spin density.[12]

II. GENERAL CASE

A. Generalized form of the Lagrangian density

The difference between two connections is a tensor, so, in a space-time whose geometry is determined by a linear connection $\Gamma^i{}_{jk}$ and a metric g_{ij}, with its corresponding Riemannian connection $\{{}^{\,i}_{jk}\}$, this geometry is determined also by the defect tensor $\lambda^i{}_{jk}$ of (1.4) and the metric g_{ij}. Thus, the general geometric Lagrangian density (1.13) assumed in this section has the form

$$L_G = L_G(g, \partial g, \partial\partial g, \lambda, \partial\lambda) . \tag{2.1}$$

We assume that L_G is a scalar density for all tensors λ_{ijk} and all symmetric tensors g_{ij} with nonvanishing determinant (so the inverse tensor g^{ij} exists). Thus, in particular, the functions

$$L_E \equiv L_G(g, \partial g, \partial\partial g, 0, 0) \tag{2.2}$$

and

$$L_S \equiv L_G - L_E \tag{2.3}$$

are scalar densities. Because L_E is a scalar density dependent only on the symmetric second-order tensor g_{ij} and its derivatives, it satisfies the Rosenfeld identity,[13] which remains valid for Lagrangian densities dependent on second derivatives,

$$\left(\frac{\delta L_E}{\delta g_{(ij)}}\sigma^m{}_{ij:n}{}^{kl}g_{kl}\right)_{,m}+\frac{\delta L_E}{\delta g_{(ij)}}g_{ij,n}\equiv 0 \qquad (2.4)$$

for all symmetric second-order tensors g_{ij}. The notation in this equation is as follows. Firstly,

$$\frac{\delta L_E}{\delta g_{(ij)}}=\frac{\partial L_E}{\partial g_{(ij)}}-\left(\frac{\partial L_E}{\partial g_{(ij),k}}\right)_{,k}+\left(\frac{\partial L_E}{\partial g_{(ij),kl}}\right)_{,kl}. \qquad (2.5)$$

Secondly, $\sigma^m{}_{A:n}{}^B$ denotes the generators of the infinitesimal transformations of a tensor t_A; if under the vector transformation

$$V^i(x)\to V^{i'}(x')=\frac{\partial x^{i'}}{\partial x^i}V^i(x)\equiv a^{i'}{}_iV^i(x) \qquad (2.6)$$

the tensor $t_A(x)$ transforms as

$$t_A(x)\to t_{A'}(x')=\sigma_{A'}{}^B(a)t_B(x)\,, \qquad (2.7)$$

then under an infinitesimal transformation $a^{i'}{}_i=\delta^{i'}{}_i+\epsilon^{i'}{}_i$, with all $|\epsilon^{i'}{}_i|\ll 1$, $t_A(x)$ transforms by

$$t_{A'}(x')-t_A(x)=\epsilon^n{}_m\sigma^m{}_{A:n}{}^Bt_B(x)\,, \qquad (2.8)$$

with

$$\sigma^m{}_{A:n}{}^B=\frac{\partial\sigma_A{}^B(a)}{\partial a^n{}_m}\bigg|_{a^i{}_j=\delta^i{}_j}. \qquad (2.9)$$

The generators for an arbitrary covariant second-order tensor g_{ij} are given by

$$\sigma^m{}_{ij:n}{}^{kl}g_{kl}=-(\delta^m{}_ig_{nj}+\delta^m{}_jg_{in})\,. \qquad (2.10)$$

Substituting this relation into the Rosenfeld identity (2.4) and using (1.1) show that any scalar density L_E, dependent only on a second-order symmetric tensor, with nonzero determinant, and its first and second derivatives, satisfies the relation

$$\left[(-g)^{-1/2}\frac{\delta L_E}{\delta g_{(ij)}}\right]_{;j}\equiv 0\,. \qquad (2.11)$$

Thus, under our assumptions the second-order tensor components

$$E_1{}^{ij}=(-g)^{-1/2}\frac{\delta L_E}{\delta g_{(ij)}} \qquad (2.12)$$

are (a) concomitants of the metric tensor and its first two derivatives [assumption (ii) of Sec. I] and (b) divergence-free [Eq. (2.11)]. One of Lovelock's theorems[14] therefore applies to $E_1{}^{ij}$; it necessarily has the form

$$E_1{}^{ij}=aG^{ij}[\{\ \}]+bg^{ij}\,, \qquad (2.13)$$

where G^{ij} is the Einstein tensor

$$G^{ij}=R^{ij}-\tfrac{1}{2}g^{ij}R\,, \qquad (2.14)$$

and can be derived from a Lagrangian density of the form

$$L_E=(-g)^{1/2}(\alpha R[\{\ \}]+\beta)\,, \qquad (2.15)$$

where α and β are constants, as well as others.[15]

The residual Lagrangian density $L_s\equiv L_G-L_E$ depends crucially on the defect tensor $\lambda^i{}_{jk}$ in the sense that L_s vanishes when the defect vanishes. Thus, we arrive at the following conclusion:

The most general Euler-Lagrange field equations dependent on the metric tensor and its first and second derivatives and the defect tensor and its derivatives are the Euler-Lagrange equations of the total Lagrangian density

$$L\equiv(-g)^{1/2}(\alpha R[\{\ \}]+\beta)+L_s+L_m\,, \qquad (2.16)$$

where L_s depends crucially on the defect tensor and L_m is the source Lagrangian density of the matter and physical fields.

This result generalizes von der Heyde's theorem[8] which applies in the special case in which (i) the connection is metric and (ii) the Lagrangian density L_s is homogeneous quadratic in the torsion. The form (2.16) for the total Lagrangian density also generalizes previously published results, described above, for the EC theory[7] and for its generalization with $L_G=R[\Gamma]$ for unconstrained defect tensors.[9]

B. Geometric side of the gravitational field equations

The variational derivative

$$E_2{}^{ij}\equiv\frac{\delta L_s}{\delta g_{(ij)}}, \qquad (2.17)$$

which appears with the usual Einstein tensor $E_1{}^{ij}$ (2.13) on the geometric side of the gravitational field equations, is determined in part by the variational derivative

$$C^{ijk}\equiv\frac{\delta L_G}{\delta\lambda_{ijk}}\equiv\frac{\delta L_s}{\delta\lambda_{ijk}}, \qquad (2.18)$$

which is the geometric side of the spin field equations, because the particular combination of g_{ij} and λ_{ijk} forming L_s is a scalar density. The relation between $E_2{}^{ij}$ and C^{ijk} allows one to simplify the gravitational field equations by replacing the variational derivative $\delta L_G/\delta\lambda_{ijk}$ with the spin and also to determine conditions under which $E_2{}^{ij}$ does not depend on third- or fourth-order derivatives [through terms like $(\partial L_s/\partial g_{ij,kl})_{,kl}$] of the metric tensor. In this subsection we calculate the relation between $E_2{}^{ij}$ and C^{ijk}.

We assume that L_s has the general form

$$L_s = L_1(g, \partial g, \partial\partial g, \lambda, \partial\lambda)\,. \tag{2.19}$$

The ordinary derivative $\lambda_{ijk,l} \equiv \lambda_{A,l}$ is not a tensor, but the combination

$$\lambda_{A|l} \equiv \lambda_{A,l} + \sigma^{m}{}_{A:n}{}^{B}\left\{{}^{n}_{lm}\right\}\lambda_B \tag{2.20}$$

is. Similarly, the second derivatives $g_{ij,kl}$ of the metric tensor are not tensors, but they can be written at any point, in an orthonormal Riemann normal coordinate system,[16] as linear combinations of the curvature tensor

$$\tilde{R}_{ijkl} \equiv R_{ijkl}[\,\{\,\}\,]\,. \tag{2.21}$$

In such a coordinate system the first derivatives $g_{ij,k}$ vanish at that point. Thus, in that coordinate system at that point, Equation (2.19) can be written as

$$L_s = L_s(g_{ij}; \tilde{R}_{ijkl}; \lambda_A; \lambda_{A|i})\,, \tag{2.22}$$

a tensor relation in one coordinate system that is therefore valid in all coordinate systems.

The desired formula for $\delta L_s/\delta g_{ij}$ can be obtained from a Rosenfeld-type identity, expressing the fact that L_s is a scalar density function of the variables listed in (2.22). We obtain this identity in obviously covariant form in the following way. Let $F(x)$ denote the totality of field variables, with all indices and labels suppressed. The substantive variation of $F(x)$, under the infinitesimal coordinate transformation

$$x^i \to \bar{x}^i = x^i + \xi^i(x)\,, \tag{2.23}$$

is given by [see (2.8)]

$$\delta F = \bar{F}(\bar{x}) - F(x) = \sigma^j{}_i F \xi^i{}_{,j}\,, \tag{2.24}$$

while the local variation of F is

$$\begin{aligned}\delta^* F &= \bar{F}(x) - F(x) = \delta F - F_{,j}\xi^j \\ &= \sigma^j{}_i F \xi^i{}_{|j} - F_{|j}\xi^j\,.\end{aligned} \tag{2.25}$$

Some useful local variations are

$$\delta^* \tilde{R}^i{}_{jkl} = -\left(\delta^*\left\{{}^{i}_{jk}\right\}\right)_{|l} + \left(\delta^*\left\{{}^{i}_{jl}\right\}\right)_{|k}\,, \tag{2.26}$$

$$\delta^*\left\{{}^{i}_{jk}\right\} = \tfrac{1}{2} g^{il}\left[(\delta^* g_{jl})_{|k} + (\delta^* g_{kl})_{|j} - (\delta^* g_{jk})_{|l}\right], \tag{2.27}$$

and the following. Each type of variation of $\lambda_{A|i}$ depends on the corresponding variations of the λ_B's and the g_{ij}'s:

$$\delta\lambda_{A|i} = \delta\lambda_{A|i}\big|_{\delta g=0} + \delta\lambda_{A|i}\big|_{\delta\lambda=0}\,. \tag{2.28}$$

However, while $\delta^* F_{,i} = (\delta^* F)_{,i}$, the similar result for Riemannian covariant derivatives is

$$\delta^* F_{|i}\big|_{\delta g=0} = (\delta^* F)_{|i}\,, \tag{2.29}$$

so

$$\begin{aligned}\delta^*\lambda_{A|i} = {} & (\delta^*\lambda_A)_{|i} \\ & + \tfrac{1}{2}\sigma^m{}_{A:n}{}^B\lambda_B g^{nl} \\ & \times\left[(\delta^* g_{lm})_{|i} + (\delta^* g_{li})_{|m} - (\delta^* g_{im})_{|l}\right].\end{aligned} \tag{2.30}$$

The condition that L_s be a scalar density is that

$$\int[\delta^* L_s + (L_s\xi^i)_{|i}]\,d^4x \equiv 0 \tag{2.31}$$

for arbitrary infinitesimal coordinate transformations, i.e., for arbitrary ξ^i. Application of the formulas above and the integration-by-parts formulas (valid for δ and δ^*)

$$M^i(\delta F)_{|i} = (M^i\delta F)_{|i} - M^i{}_{|i}\delta F \tag{2.32}$$

and

$$N^{ij}(\delta F)_{|ij} = [N^{ij}(\delta F)_{|i} - N^{ji}{}_{|i}\delta F]_{|j} + N^{ij}{}_{|ji}\delta F \tag{2.33}$$

gives, for arbitrary $\xi^i(x)$,

$$\begin{aligned}\int d^4x \Big\{ & \xi^i[2E_2{}^j{}_{i|j} - C^A\lambda_{A|i} - (C^A\sigma^j{}_{A:i}{}^B\lambda_B)_{|j}] \\ & + \Big[\xi^i\Big(L_s\delta^j{}_i - 2E_2{}^j{}_i + C^A\sigma^j{}_{A:i}{}^B\lambda_B - \frac{\partial L_s}{\partial\lambda_{A|j}}\lambda_{A|i}\Big) + \xi^i{}_{|k}\Big(P^{ljk}{}_{i|l} + \frac{\partial L_s}{\partial\lambda_{A|j}}\sigma^k{}_{A:i}{}^B\lambda_B - U^{j(k}{}_{i)}\Big) - \xi^i{}_{|kl}P^{ljk}{}_i\Big]_{|j}\Big\} \equiv 0\,,\end{aligned} \tag{2.34}$$

where

$$\begin{aligned}U^{i(j}{}_{k)} \equiv {} & \frac{\partial L_s}{\partial\lambda_{A|i}}\sigma^{(j}{}_{A:k)}{}^B\lambda_B + \frac{\partial L_s}{\partial\lambda_{A|(j}}\sigma^i{}_{A:k)}{}^B\lambda_B \\ & - \frac{\partial L_s}{\partial\lambda_{A|(j}}\sigma_{k)A:}{}^{iB}\lambda_B\,,\end{aligned} \tag{2.35}$$

$$P^{ijkl} \equiv P^{(ij)(kl)} \equiv P^{klij} \equiv 2\frac{\partial L_s}{\partial\tilde{R}_{i(kl)j}}\,, \tag{2.36}$$

$$\begin{aligned}E_2{}^{ij} \equiv E_2{}^{(ij)} & \equiv \frac{\delta L_s}{\delta g_{(ij)}} \\ & \equiv \frac{\partial L_s}{\partial g_{(ij)}}\bigg|_{\Gamma,R} - \tfrac{1}{2}U^{k(ij)}{}_{|k} + \tfrac{1}{2}P^{klij}{}_{|kl}\,,\end{aligned} \tag{2.37}$$

and

$$C^A \equiv \frac{\delta L_s}{\delta\lambda_A} \equiv \frac{\partial L_s}{\partial\lambda_A} - \left(\frac{\partial L_s}{\partial\lambda_{A|i}}\right)_{|i}\,. \tag{2.38}$$

For a vector density V^i, $V^i{}_{|i}=V^i{}_{,i}$, so by choosing ξ^i's that vanish at the boundary of the region of integration but are otherwise arbitrary we obtain

$$E^j{}_{i|j}\equiv E_1{}^j{}_{i|j}+E_2{}^j{}_{i|j}\equiv\tfrac{1}{2}C^A\lambda_{A|i}+\tfrac{1}{2}(C^A\sigma^j{}_A:{}_i{}^B\lambda_B)_{|j}\,; \tag{2.39}$$

the divergence of the geometric side of the gravitational field equations depends solely on the spin and the defect. This divergence is not zero, as it is in general relativity, so the "energy-momentum" tensor $\sim\delta L_{\text{matter}}/\delta g_{(ij)}|_{\delta\lambda=0}$ of the EC theory or its simple generalizations is not conserved in the presence of spin.

The condition that L_s be a scalar density implies both (2.39) and that the remaining coefficients of ξ^i and its symmetrized covariant derivatives under the integral in (2.34) be identically zero. The symmetrization is necessary in order that only an independent set of arbitrary functions be considered; the relation

$$\phi_{A|ij}-\phi_{A|ji}=-\tilde{R}^n{}_{mij}\sigma^m{}_{A:n}{}^B\phi_B \tag{2.40}$$

can be used to eliminate skew pairs of adjacent covariant-derivative indices. For example, with the symmetries of the curvature tensor and the use of (2.40), we obtain

$$-\xi^i{}_{|klj}P^{ljk}{}_i$$
$$=\tfrac{2}{3}P^{ijk}{}_l\tilde{R}^l{}_{pki|j}\xi^p+\tfrac{1}{3}\xi^p{}_{|q}(3P^{qij}{}_k\tilde{R}^k{}_{pji}+P^{ijk}{}_p\tilde{R}^q{}_{ijk})\,. \tag{2.41}$$

The coefficient of $\xi^i{}_{|(jk)}$ in the integrand of (2.34) vanishes identically. The coefficient of ξ^i from the divergence term can be shown to vanish as a consequence of the coefficients of $\xi^i{}_{|j}$ and $\xi^i{}_{|(jk)}$ being zero. The coefficient of $\xi^{(i}{}_{|j)}$ gives the contribution from L_s to the geometric side of the gravitational field equation:

$$E_2{}^i{}_j\equiv E_2{}^{(i}{}_{j)}$$
$$\equiv\frac{1}{2}\Big[L_s\delta^i{}_j+C^A\sigma^{(i}{}_{A:j)}{}^B\lambda_B-\frac{\partial L_s}{\partial\lambda_{A|(i}}\lambda_{A|j)}$$
$$+\Big(P^{kli}{}_{j|k}+\frac{\partial L_s}{\partial\lambda_{A|l}}\sigma^{(i}{}_{A:j)}{}^B\lambda_B-U^{l(i}{}_{j)}\Big)_{|l}\Big]$$
$$-\tfrac{1}{3}P^{klm(i}\tilde{R}_{j)klm}\,. \tag{2.42}$$

This can involve third and fourth derivatives of the metric in the $(\partial L_s/\partial\lambda_{A|l})_{|l}$ and $P^{kl(i}{}_{j)|kl}$ terms. These higher-order derivatives are absent if both $\partial L_s/\partial\lambda_{A|i}$ and $\partial L_s/\partial\tilde{R}_{k(ij)l}$ are independent of $\tilde{R}_{abcd}$. The latter case obtains, for example, if L_s is linear in the curvature tensor.

For example, if L_s consists of a term that is homogeneous quadratic in the first Riemann covariant derivatives of the defect λ_A with only metric tensor coefficients and a term like $\tilde{R}_{ijkl}\lambda^{mij}\lambda^{kl}{}_m$, then $E^i{}_j$ does not depend on derivatives of the metric tensor g_{ij} higher than the second. Moreover, in this case the spin field equations *in vacuo* have the form

$$g^{--}\lambda_{---|--}+g^{--}g^{--}\tilde{R}_{----}\lambda_{---}=0\,, \tag{2.43}$$

with appropriate contractions on the indices. This field equation describes waves of the defect (called contortional waves for the special case of a metric space-time in speculations by von der Heyde and Hehl[6]). Here the curvature tensor acts like inertia to modify the magnitude of the propagation vector of the waves.

III. SPECIAL CASES

This section is concerned with the special cases in which the spin field equations are linear and homogeneous in the defect λ_A and its first and second derivatives, thus generalizing some results of von der Heyde.[8] We derive the general forms of these field equations, deduce the number of arbitrary constants that appear in them, and find the conditions among these constants that must be satisfied if the spin field equations are the Euler-Lagrange equations of some Lagrangian.

The field equations are assumed to have the form

$$C^{ijk}(g,\partial g,\partial\partial g,\lambda,\partial\lambda,\partial\partial\lambda)=S^{kij}\,, \tag{3.1}$$

where S^{kij} represents the spin density. We consider initially only those cases in which C^{ijk} can be expressed in a form with no explicit dependence on the curvature tensor; a brief discussion is given at the end of the section on those cases in which the curvature tensor must appear explicitly.

A. Spin field equations linear in λ_A

We consider here the case in which the geometric side of the spin field equations does not depend on the curvature tensor and is linear in the defect λ_A, and thus necessarily homogeneous unless another odd-order tensor is involved; we treat the case that involves derivatives of the λ_A's in the next section.

Since there does not exist a tensor that can be formed from the metric tensor and its first derivatives, the tensor coefficients D^{AB} appearing in

$$C^{ijk}=(-g)^{1/2}D^{ijklmn}\lambda_{lmn} \tag{3.2}$$

depend only on the metric tensor g_{rs}. These coefficients must have the form

$$D^{i_1i_2i_3i_4i_5i_6} = \sum_{\pi'} A_{\pi'(1,2,3,4,5,6)}\, g^{\pi'(i_1,i_2,i_3,i_4,i_5,i_6)}, \tag{3.3}$$

where $\pi'(n_1, n_2, \ldots, n_{2m})$ is a permutation of $n_1, n_2, \ldots, n_{2m}$ formed from the product of m transpositions of the $2m$ arguments, $\{A_{\pi'(1,2,\ldots,2m)}\}$ is a set of constants whose individual values depend on the particular permutation $\pi'(1, 2, \ldots, 2m)$ but not otherwise on the indices i_s, and $g^{\pi'(i_1,\ldots,i_{2m})}$ is the product of the m g^{ij}'s whose indices are the elements of the transpositions in $\pi'(i_1, \ldots, i_{2m})$, for example $g^{(i_1i_3)(i_2i_6)(i_4i_5)} \equiv g^{i_1i_3}g^{i_2i_6}g^{i_4i_5}$. The general spin field equation (3.2) contains fifteen arbitrary constants.

Assume now that C^A is the variational derivative of some Lagrangian density L_G:

$$C^A = \frac{\delta L_G}{\delta\lambda_A} \equiv \frac{\partial L_s}{\partial\lambda_A}. \tag{3.4}$$

The integrability condition for this set of equations is

$$D^{AB} = D^{BA}, \tag{3.5}$$

reducing the number of arbitrary constants by four to eleven. We have

$$L_s = \tfrac{1}{2}(-g)^{1/2} D^{AB}\lambda_A\lambda_B. \tag{3.6}$$

An explicit form for $E_2{}^{ij} \equiv \delta L_s/\delta g_{(ij)}$ can be obtained from the following considerations which exemplify the fact that the $\partial L_s/\partial g_{(ij)}|_{\Gamma,R}$ terms in (2.42) arise from the factors g^{ij} and $(-g)^{1/2}$ required to form, with the other tensors, a scalar density. The tensor transformation law (2.7) involves the same function σ as the transformation

$$\tau^{i_1\cdots i_N} = \sigma^{i_1\cdots i_N j_1\cdots j_N}(g^{ij})\tau_{j_1\cdots j_N} \tag{3.7}$$

from covariant to contravariant components. Therefore, since $g^i{}_j = \delta^i{}_j$, varying g_{ij} keeping $\tau_{j_1\cdots j_N}$ fixed gives

$$\delta\tau^{i_1\cdots i_N} = \sigma^{(m|i_1\cdots i_N:|n)j_1\cdots j_N}\tau_{j_1\cdots j_N}\delta g_{(mn)}, \tag{3.8}$$

with $\sigma^m{}_{A:n}{}^B$ given by (2.9). Thus, the variations in L_s, (3.6), due to the variations in g_{ij} only yield

$$2\frac{\delta L_s}{\delta g_{(mn)}} = (-g)^{1/2}\sum_{\pi'} A_{\pi'(1,\ldots,6)}\lambda_{i_1i_2i_3} \times \sigma^{(mn)\pi'(i_1,\ldots,i_6)}\lambda_{i_4i_5i_6} + \tfrac{1}{2}g^{mn}C^{ijk}\lambda_{ijk}, \tag{3.9}$$

where the last term comes from the variation of $(-g)^{1/2}$ and the notation for $\sigma^m{}_{A:n}{}^B$ has been modified. This variational derivative may be written as

$$2\frac{\delta L_s}{\delta g_{(mn)}} = Lg^{mn} + \frac{\delta L_s}{\delta\lambda_A}\sigma^{(m}{}_{A:}{}^{n)B}\lambda_B, \tag{3.10}$$

the form to which (2.42) reduces in the special case under consideration here.

We now examine the spin field equations

$$C^{ijk} = S^{kij} \tag{3.11}$$

and the conditions for a metric theory under the usual assumption of a skew spin density

$$S^{kij} = -S^{kji}. \tag{3.12}$$

Introduce the collective symbols $a, b, \ldots$ for the combinations $[ij]k$, $a', b', \ldots$ for the corresponding $k[ij]$, and $\alpha, \beta, \ldots$ for $(ij)k$:

$$D^{ab}\lambda_b + D^{a\alpha}\lambda_\alpha = (-g)^{-1/2}S^{a'}, \tag{3.13a}$$

$$D^{\alpha a}\lambda_a + D^{\alpha\beta}\lambda_\beta = 0. \tag{3.13b}$$

The condition that the connection be compatible with the metric is

$$\lambda_\alpha = 0. \tag{3.14}$$

If this is satisfied and the λ_a can be determined from (3.13a), then $\det(D^{ab}) \neq 0$. This restriction can be evaluated[8] by direct computation of the spin equations for the Lagrangian density

$$L_{s[]} \equiv \tfrac{1}{2}(-g)^{1/2}D^{ab}\lambda_a\lambda_b = \tfrac{1}{2}(-g)^{1/2}(A_1\lambda_{[ij]k}\lambda^{[ij]k} + A_2\lambda_{[ij]k}\lambda^{[ki]j} + A_3\lambda_{[ij]}{}^j\lambda^{[ik]}{}_k), \tag{3.15}$$

$$A_1 + A_2 \neq 0, \quad 2A_1 - A_2 \neq 0, \quad 2A_1 - A_2 + 3A_3 \neq 0. \tag{3.16}$$

If the conditions (3.16) are satisfied, then in order for the theory to be metric it is necessary that $D^{\alpha a} = 0$, in which case L_s involves at most eight arbitrary constants.

If, moreover, $\det(D^{\alpha\beta}) \neq 0$ then the theory with skew spin is necessarily metric. This restriction can be evaluated from

$$L_{s()} = (-g)^{1/2}(A_4\lambda_{(ij)}{}^j\lambda^{(ik)}{}_k + A_5\lambda_{(ij)k}\lambda^{(ij)k} + A_6\lambda_{(ij)k}\lambda^{(ik)j} + A_7\lambda^i{}_{ij}\lambda_k{}^{kj} + A_8\lambda^i{}_{ij}\lambda^{(jk)}{}_k) \tag{3.17}$$

as follows:

$$(5A_4 + 2A_5 + A_6 + A_8)(A_5 + 8A_7 + A_8) \neq (A_4 + A_6 + 2A_8)(A_6 + 4A_7 + 5A_8), \tag{3.18}$$

$$2A_5 - A_6 \neq 0,$$

$$A_5 + A_6 \neq 0.$$

If these inequalities are not satisfied, as they are not in the EC theory, then one can generalize Trautman's Theorem 3:[2] Among the defects satisfying a spin field equation with skew spin and $D^{\alpha a}=0$, there is exactly one such that the theory is metric.

If the theory is metric, i.e., $\lambda_\alpha=0$, either as a consequence of the spin field equations or by decree, then one can solve the equations

$$\Gamma_{ijk}=[jk,i]+\lambda_{[ij]k}, \tag{3.19a}$$

$$\Gamma_{ijk}-\Gamma_{ikj}=Q_{ikj} \tag{3.19b}$$

for the defect $\lambda_{ijk}\equiv\lambda_{[ij]k}$ in terms of the torsion or for the torsion in terms of the defect:

$$\lambda_{ijk}=\tfrac{1}{2}(Q_{ikj}-Q_{kji}+Q_{jik}), \tag{3.20a}$$

$$Q_{ijk}=\lambda_{ikj}-\lambda_{ijk}. \tag{3.20b}$$

In this case the theory is equivalent to one in which the geometry is determined by the metric tensor g_{ij} and the torsion tensor Q_{ijk}. This is the case considered by von der Heyde.[8]

B. Spin field equations linear in the derivatives of λ_A

Tensor field equations linear in λ_A and its derivatives and involving at most second-order derivatives of the metric divide into two groups, those containing even-order Riemann covariant derivatives and those containing odd-order Riemann covariant derivatives. This result follows since contraction with the tensors g_{ij} and $\tilde{R}_{ijkl}$ of even order cannot change the character of oddness or evenness of a tensor's order. We consider only those terms linear and homogeneous in the second-order Riemann covariant derivatives since these could appear in the spin field equations $C^{ijk}=S^{kij}$ combined with the terms linear in the λ_A. Similar considerations as those given directly below apply to field equations in $\lambda_{A|i}$, which need not be homogeneous.

If the Riemann curvature tensor does not appear explicitly in the terms that are linear and homogeneous in $\lambda_{A|ij}$, the general form of these terms in the spin field equation is

$$C^{AN_1}=(-g)^{1/2}\sum_{\pi'}A_{\pi'(N_1,N_2,n_1,n_2)}\times g^{\pi'\{N_1,N_2,n_1,n_2\}}\lambda_{A_{N_2}|n_2n_1}. \tag{3.21}$$

The general expression of this type involves 105 arbitrary constants $A_{\pi'}$. This high degree of arbitrariness can be reduced to a manageable level by, firstly, imposing the condition that C^{AN_1} be the Euler-Lagrange expression derived from a Lagrangian density. Since

$$(-g)^{1/2}g^{\pi'\{ABij\}}\lambda_{A|ij}\lambda_B=-(-g)^{1/2}g^{\pi'\{ABij\}}\lambda_{A|i}\lambda_{B|j}+\text{divergence}, \tag{3.22}$$

the typical term in the Lagrangian density $L_s^{(\text{deriv})}$ can be taken to be

$$\Delta L_s=A(-g)^{1/2}g^{\pi'\{ABij\}}\lambda_{A|i}\lambda_{B|j}. \tag{3.23}$$

The integrability condition for C^{A_N} to be the variational derivative $-(\partial L_s^{(\text{deriv})}/\partial\lambda_{A|i})_{|i}$ of a sum of terms of the form of ΔL_s is that

$$A_{\pi'(N_1N_2n_1n_2)}=A_{\pi'(N_2N_1n_2n_1)}; \tag{3.24}$$

this reduces the number of arbitrary $A_{\pi'}$'s to sixty. Secondly, the number of arbitrary $A_{\pi'}$'s can be reduced further by assuming that the theory is metric, so $\lambda_{ijk}=-\lambda_{[ij]k}$. Under these conditions the number of $A_{\pi'}$'s is sixteen.

The second derivative terms $\lambda_{A|ij}$ may involve the curvature tensor implicitly:

$$\lambda_{A|ij}=\lambda_{A|(ij)}-\tfrac{1}{2}\tilde{R}^k{}_{lij}\sigma^l{}_{A:k}{}^B\lambda_B. \tag{3.25}$$

These gravitation-spin coupling terms are absent only if the $A_{\pi'}$'s satisfy the condition

$$A_{\pi'(N_1N_2n_1n_2)}=A_{\pi'(N_1N_2n_2n_1)}; \tag{3.26}$$

this reduces the number of arbitrary constants to nine.

The contribution from the Lagrangian density $L_s^{(\text{deriv})}$ to the gravitational field equations is given in (2.42). The contribution of $L_s^{(\text{deriv})}$ arising from the g^{ij} coefficients in (3.23) occurs in the second, third, and fifth terms in the same manner as the first term in (3.9) because

$$\sigma^i{}_{Ak:j}{}^{Bl}=\sigma^i{}_{A:j}{}^B\delta^l{}_k-\delta^i{}_j\delta^l{}_k\delta^B{}_A. \tag{3.27}$$

The terms involving $\partial L_s/\partial\lambda_{A|i}$ in $U^{i(j}{}_{k)}$ arise because $\lambda_{A|i}$ involves the metric tensor and its derivatives in the connection.

Derivative terms $\lambda_{A|ij}$ can appear linearly with λ_A or direct gravitation-defect coupling can exist only if some constant l defining a fundamental length is involved; the general form of a linear spin field equation is

$$l^2(g^{--}\lambda_{---|--}+g^{--}g^{--}R_{----}\lambda_{---})+\lambda_{---}=S_{---}. \tag{3.28}$$

Von der Heyde[8] used the fact that there is no natural fundamental length for such a theory as an argument to discard the terms in parentheses in (3.28) from the field equations, but this argument can be reversed. Evidence for derivative or direct gravitation-defect coupling terms in the spin field equations would provide evidence for the existence of a fundamental length in the geometry of space-time. However, at present there is no experimental evidence even for torsion, so the possibility of determining a fundamental length in this way seems remote.

If one accepts the arguments of von der Heyde and Hehl[6] that the gravitational theory required for elementary-particle considerations must involve at least torsion in space-time, one could combine Planck's constant with c and the gravitational constant G to form the Planck length $(G\hbar/c^3)^{1/2} \sim 10^{-35}$ m. Defect waves in flat space would then correspond to the Planck mass $(c\hbar/G)^{1/2} \sim 10^{19}$ GeV, a large value but one already being considered in unified models of weak and electromagnetic interactions. Furthermore, acceptance of von der Heyde and Hehl's arguments could lead one to a particular choice[17] for L_s that corresponds to a definite spin state for the defect tensor λ_{ijk}.

IV. CONCLUSIONS

Any simple generalizations of the Einstein-Cartan-Sciama-Kibble theory of gravitation obtained by introducing a general connection into the geometry is equivalent to general relativity. The defect would manifest itself, however, in the covariant derivatives of the matter and field variables, although one could interpret such cross terms as just another coupling.

The Euler-Lagrange equations, given in part by (2.13) and (2.42), of generalized EC theories do not involve derivatives higher than second order if, as one might suspect, $\partial L_G/\partial\lambda_{A|i}$ is independent of the curvature and L_G involves the curvature only linearly. These conditions do not eliminate the possibility of direct gravitation-defect coupling.

The divergence of the symmetric energy-momentum tensor density $T^{ij} \equiv \delta L_{\text{matter}}/\delta g_{(ij)}|_{\delta\lambda=0}$ does not vanish in the presence of spin. There is a local balance between that energy-momentum tensor and the spin given by

$$\tfrac{1}{2}C^A\lambda_{A|j} + \tfrac{1}{2}(C^A\sigma^i{}_{A:j}{}^B\lambda_B)_{|i} = T^i{}_{j|i}\,. \tag{4.1}$$

Metric generalized EC spin field equations linear in the defect and with no direct gravitation-defect coupling can involve up to seventeen arbitrary constants, nine of these before terms linear in second derivatives of the defect. These latter terms have the appealing feature of allowing the torsion to extend into space-time beyond the spin sources and to undergo wave motions. However, this feature would require the introduction of a fundamental length. Direct gravitation-defect coupling also requires the existence of a fundamental length.

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[1]Tensors are described by their components relative to a coordinate basis. The notation is that of Refs. 2 and 5 with one exception (Trautman defines E^{ij} and T^{ij} using $\partial/\partial g_{(ij)}$, whereas we use $\delta/\delta g_{(ij)}$) and the following additions: The covariant derivative with respect to the Riemannian connection is denoted by a bar, as in $g_{ij|k}=0$. Indices inside parentheses () or square brackets [], save those denoted collectively by a capital Latin letter or excluded by vertical bars, are to be symmetrized or antisymmetrized, respectively, as in $\lambda_{[i|j|k]} = \frac{1}{2}(\lambda_{ijk} - \lambda_{kji})$. The set of first and second derivatives of a tensor T_A are denoted by $\partial T = \{T_{A,i}\}$ and $\partial\partial T = \{T_{A,ij}\}$, respectively.

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