Quantum vacuum energy in a closed universe*

L. H. Ford
Department of Physics, University of Wisconsin—Milwaukee, Milwaukee, Wisconsin 53201
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The problem of defining the vacuum energy and pressure of quantized fields in the static Einstein universe is considered. A regularization procedure which utilizes a wavelength cutoff in the mode sum is discussed and applied to the cases of the massive conformally coupled scalar field, the electromagnetic field, and the neutrino field. In all cases a positive vacuum energy density and pressure are obtained. In the case of the massive scalar field it is possible for the vacuum pressure to exceed the vacuum energy density, thus violating the dominant energy condition. For the electromagnetic and neutrino fields the energy density and pressure are of the form $\rho = \gamma \hbar c/a^4$ and $P = \rho/3$, respectively, where $\gamma = 11/240\pi^2$ for the electromagnetic field and $\gamma = 17/1920\pi^2$ for the neutrino field, and where a is the radius of the universe.

I. INTRODUCTION

There has been considerable interest recently in the problem of regularizing the energy-momentum tensor for a quantized field in a curved spacetime,[1-13] although no general solution to this problem has yet been found. In a previous paper[9] (henceforth referred to as I), the cases of a massless scalar field in the closed Robertson-Walker metric and in the background of a gravitational plane wave were discussed. The method used there consisted of removing the divergences of the energy-momentum tensor by a cutoff in the mode sum and then isolating and subtracting the cutoff-dependent terms. In the present paper this method will be discussed further and applied to the cases of the massive conformal scalar field, the electromagnetic field, and the neutrino field in the Einstein universe.

In Sec. II a general discussion of the use of cutoff functions for regularizing the energy-momentum tensor will be given. In Sec. III the vacuum energy density and pressure of a massive conformal scalar field are calculated. The case of the electromagnetic field is taken up in Sec. IV, and that of the neutrino field in Sec. V. Finally, in Sec. VI a discussion of the case of the closed, expanding Robertson-Walker metric is given.

II. THE REGULARIZATION PROCEDURE

In this paper, as in I, a regularization method based on a wavelength cutoff will be utilized. By insertion of a cutoff function in the expression for $\langle 0|T_{\mu\nu}|0\rangle$ as a sum over modes, the ultraviolet divergences are removed. One must then extract a finite, cutoff-independent expression to be identified as the physical energy-momentum tensor of the vacuum state. The fact that the vacuum energy and momentum in flat space must be zero puts a constraint on the subtractions which are to be made to obtain the finite result. One may take Casimir's beautiful calculation[14] of the vacuum energy of the quantized electromagnetic field in the presence of a pair of conducting plates as a model for the calculations in the case of a quantized field in a curved spacetime.

Let us consider momentarily the case of an asymptotically flat spacetime. If we can compare the cutoff-dependent expression for $\langle T_{\mu\nu}\rangle$ at an arbitrary point with that at a point in the asymptotic region, the correct subtraction may be suggested. Suppose that there exists a mode decomposition so that $T_{\mu\nu}(k,x)$ is the value of the energy-momentum tensor due to mode k at point x. The divergent vacuum energy-momentum tensor is

$$\langle 0|T_{\mu\nu}(x)|0\rangle = \sum_k T_{\mu\nu}(k,x). \tag{1}$$

Consider a cutoff function $f(\alpha,k)$ which is such that

$$(1)\ \lim_{\alpha\to 0} f(\alpha,k) = 1 \tag{2a}$$

and

$$(2)\ \langle T_{\mu\nu}(x)\rangle_c \equiv \sum_k f(\alpha,k) T_{\mu\nu}(k,x) < \infty. \tag{2b}$$

Let x' be a point in the asymptotically flat region of spacetime and let x be an arbitrary point connected to x' by a geodesic G. Parallel transport $\langle T_{\mu\nu}(x')\rangle_c$ to x along G and call the result $\langle T'_{\mu\nu}(x)\rangle_c$. Since this is now a tensor at x, it may be compared directly with $\langle T_{\mu\nu}(x)\rangle_c$. Define the physical energy-momentum tensor to be

$$\langle T_{\mu\nu}(x)\rangle \equiv \lim_{\alpha\to 0}[\langle T_{\mu\nu}(x)\rangle_c - \langle T'_{\mu\nu}(x)\rangle_c]. \tag{3}$$

If the limit exists, this procedure defines a finite, cutoff-independent tensor which satisfies the requirement of vanishing in the asymptotic region (i.e., at x').

However, it is not clear whether this tensor is unique. Since the result of parallel transport between two points generally depends upon the path, a different choice of G might yield a different result. Another source of ambiguity is in the choice of $f(\alpha, k)$. It is possible to find different cutoff functions which all yield finite but inequivalent results. It is therefore necessary to define the regularization procedure more precisely to attempt to remove the ambiguities.

In this paper we will not attempt to give a general prescription for a large class of spacetimes, but will rather restrict attention to the case of the static Einstein universe. Since this is not an asymptotically flat spacetime, one must first decide how to make the comparison with Minkowski space. This may be done by considering the case of a closed, expanding Robertson-Walker metric

$$ds^2 = dt^2 - A^2(t)d\sigma^2, \tag{4}$$

where $d\sigma^2$ is the metric of a 3-sphere of unit radius. The Einstein universe is the case when $A = a$, independent of t. If $A(t)$ satisfies the conditions

$$\lim_{t\to -\infty} A(t) = a, \tag{5a}$$

$$\lim_{t\to\infty} A(t) = \infty, \tag{5b}$$

$$\lim_{t\to\infty} \dot{A}(t) = \lim_{t\to\infty} \ddot{A}(t) = 0, \tag{5c}$$

then it represents a spacetime which in the distant past was an Einstein universe of radius a and in the distant future is Minkowski space. We now have an asymptotically flat spacetime and may follow the method outlined above, taking x' to be a point in the region where $A = a$. We will also require that the rate of expansion be very small, that is, that $\dot{A} \ll 1$. This ensures that there is no particle creation so that the vacuum state at $t = -\infty$ (in-vacuum) and that at $t = +\infty$ (out-vacuum) coincide. For a conformally invariant field this requirement may not be necessary since there is no particle creation in any case.

The regularization procedure used here is closely related to that of adiabatic regularization[5,6] in that both involve a comparison with Minkowski space. For an expanding flat Robertson-Walker metric, adiabatic regularization involves the subtraction of the leading terms which arise in an expansion in powers of a slowness parameter. This ensures that in the limit in which the expansion rate approaches zero (which is Minkowski space) the regularized energy-momentum tensor will vanish.

One may specify the energy-momentum tensor by giving its eigenvalues. The frame in which it is diagonal is just that given by the coordinates in Eq. (4), provided that the spatial metric $d\sigma^2$ is itself diagonal. In this frame the energy-momentum tensor takes the form

$$T^{\mu}{}_{\nu} = \mathrm{diag}(\rho, -P, -P, -P),$$

where the eigenvalues ρ and P are the energy density and pressure, respectively. These two quantities are scalars, so we may replace the problem of regularizing a tensor by that of regularizing scalars. The advantage of this is that the regularization procedure is now clearly independent of the choice of path G because the result of transporting a scalar between two points is independent of the path.

We must still place some restrictions on the allowable cutoff functions f. As was mentioned above, we are interested in a theory based on a wavelength cutoff. In a closed universe, the wave number is a discrete rather than a continuous variable and is of the form $k = n/a$ where n is an integer or half-integer. The function f must then approach zero as $k \to \infty$ rapidly enough to remove the divergences. The motivation for the use of a wavelength cutoff is that the modes of large k, which are responsible for the divergences, can be thought of as being less affected by the spacetime curvature than those of small k. That is, we may establish a local inertial frame which is valid in a region whose spatial dimensions are of the order of $l \ll a$. Those modes for which $k \gtrsim l^{-1}$ are unaffected by the presence of the curvature and take the same form as in Minkowski space. Those modes for which $k \lesssim l^{-1}$, however, are affected by the curvature. It is these long-wavelength modes (i.e., n of the order of 1) which contribute to the vacuum energy. (See Fulling and Parker[6] for a discussion of a wavelength cutoff from another point of view.)

This suggests a further restriction on the form of f; it should not affect these long-wavelength modes, so $f \to 1$ as $k \to 0$. In general, f must be a function of a as well as k and α, so this requirement is

$$\lim_{k\to 0} f(\alpha, k, a) = 1. \tag{6}$$

We take $f(\alpha, k, a)$ to be defined for all real, positive values of k so that the limit is meaningful.

It is often possible to express the quantity to be subtracted as an integral over k. The remainder is then the difference between a sum and an integral of the same formal expression. In order to do this, it will become necessary to further restrict f to depend only on certain combinations of k and a. Thus we will require that f be of the form

$$f = f(\alpha, \Omega(k, a)), \tag{7}$$

where Ω is a function which is chosen to facilitate the expression of subtraction as an integral.

A final requirement which is convenient to impose is that f be sufficiently smooth that

$$\lim_{\alpha\to 0}\left(\frac{\partial^n f}{\partial\Omega^n}\right)=0, \quad n=1,2,\ldots,L \tag{8}$$

where L is some integer. This condition will be of use in Sec. IV.

The restrictions which f must satisfy are then those of Eqs. (2), (6), (7), and (8). The last two requirements may not be essential, but are used in the derivation of the results for the electromagnetic and neutrino fields. In this case it may be seen that any function which satisfies the above conditions (with a particular Ω) may be used to obtain the same result. Similarly, if one has $\Omega=k/a$, any cutoff function satisfying these conditions will yield the result obtained in I for the massless conformal scalar field. In the case of the massive scalar field, the explicit choice of $f=e^{-\alpha k}$ will be used, which satisfies all the conditions stated above. It is highly probable (but not proven) that any other allowed choice of f will lead to the same results.

III. THE MASSIVE SCALAR FIELD

In this section we will calculate the vacuum energy of the massive, conformally coupled scalar in the Einstein universe. This field satisfies the equation[15]

$$\Box\psi+\tfrac{1}{6}R\psi+\mu^2\psi=0, \tag{9}$$

where $\Box=\nabla_\alpha\nabla^\alpha$, R is the scalar curvature ($6\,a^{-2}$ for the Einstein universe), and μ is the mass of field. We will consider the case of a Hermitian field ψ, corresponding to uncharged scalar particles.

If the spatial metric is written in the usual form

$$d\sigma^2=d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\,d\phi^2), \tag{10}$$

a set of positive-frequency solutions of Eq. (9) is

$$F_{nlm}=A_{nlm}\sin^l\chi\, C_{n-l}^{l+1}(\cos\chi)Y_{lm}(\theta,\phi)e^{-i\omega_n t}. \tag{11}$$

Here C_{n-l}^{l+1} are Gegenbauer functions and Y_{lm} are the usual spherical harmonics. The index n takes on the values $n=0,1,\ldots,\infty$, and for fixed n the values of are $l=0,1,\ldots,n$. The eigenfrequencies are

$$\omega_n=a^{-1}[(n+1)^2+\mu^2a^2]^{1/2}, \tag{12}$$

with a degeneracy of $(n+1)^2$.

Let $\langle,\rangle$ denote the Klein-Gordon inner product:

$$\langle f,h\rangle=i\int f^*\overleftrightarrow{\partial}^0 h\sqrt{-g}\;d^3x. \tag{13}$$

Then the solutions F_λ are orthonormal with an appropriate choice of the coefficients A_λ:

$$\langle F_\lambda,F_{\lambda'}\rangle=\delta_{\lambda\lambda'}, \tag{14}$$

where λ denotes n, l, and m. Likewise, the F_λ^* form an orthonormal set of solutions of Eq. (9) of negative norm. The field operator may be expanded as

$$\psi=\sum_\lambda\,(a_\lambda F_\lambda+a_\lambda^\dagger F_\lambda^*), \tag{15}$$

where

$$[a_\lambda,a_{\lambda'}^\dagger]=\delta_{\lambda\lambda'}. \tag{16}$$

The vacuum state $|0\rangle$ is defined by

$$a_\lambda|0\rangle=0 \quad\text{for all }\lambda. \tag{17}$$

The energy-momentum tensor for the conformal scalar field is[16]

$$T_{\alpha\beta}=\psi_{,\alpha}\psi_{,\beta}-\tfrac{1}{2}g_{\alpha\beta}\psi_{,\rho}\psi^{,\rho}-\tfrac{1}{6}\nabla_\alpha(\psi^2)_{,\beta}+\tfrac{1}{6}g_{\alpha\beta}\Box(\psi^2)-\tfrac{1}{6}G_{\alpha\beta}\psi^2+\tfrac{1}{2}g_{\alpha\beta}\mu^2\psi^2, \tag{18}$$

where $G_{\alpha\beta}$ is the Einstein tensor. The vacuum expectation value of $T_{\alpha\beta}$ is

$$\rho_0=(4\pi^2a^3)^{-1}\sum_{n=0}^\infty\,(n+1)^2\omega_n, \tag{19a}$$

$$P_0=(4\pi^2a^3)^{-1}\sum_{n=0}^\infty\,(n+1)^2\left(\frac{\omega_n^{\,2}-\mu^2}{3\omega_n}\right), \tag{19b}$$

where ρ_0 denotes $\langle 0|T_0^0|0\rangle$ and P_0 denotes $-\langle 0|T_1^1|0\rangle=-\langle 0|T_2^2|0\rangle=-\langle 0|T_3^3|0\rangle$, the formal divergent quantities. The finite cutoff-dependent energy density and pressure are $\tilde\rho$ and $\tilde P$, respectively; $\bar\rho$ and $\bar P$ are the subtractions which are to be made. The final, physical energy density and pressure are denoted by ρ and P, respectively.

These divergent expressions must now be regularized by insertion of a cutoff function in the mode sums. A convenient choice is

$$f(\alpha,k_n)=e^{-\alpha k_n}, \tag{20}$$

where $k_n=n/a$. This yields

$$\tilde\rho=(4\pi^2a^3)^{-1}\sum_{m=1}^\infty\,m^2(k_m^{\,2}+\mu^2)^{1/2}e^{-\alpha k_m} \tag{21a}$$

and

$$\tilde P=(12\pi^2a^3)^{-1}\sum_{m=1}^\infty\,m^4(k_m^{\,2}+\mu^2)^{-1/2}e^{-\alpha k_m}. \tag{21b}$$

We must now determine the subtractions which are to be made from $\tilde\rho$ and $\tilde P$. The appropriate subtractions are obtained by letting $a\to\infty$, in which case the sums in Eqs. (21) are replaced by integrals. Let

$$\bar\rho=\lim_{a\to\infty}\tilde\rho=(4\pi^2)^{-1}\int_0^\infty k^2(k^2+\mu^2)^{1/2}e^{-\alpha k}dk \tag{22a}$$

and

$$\bar{P}=\lim_{a\to\infty}\tilde{P}=(12\pi^2)^{-1}\int_0^\infty k^4(k^2+\mu^2)^{-1/2}e^{-\alpha k}dk. \quad (22b)$$

As was discussed in Sec. II, these quantities are to be thought of as being the eigenvalues $\langle T_{\mu\nu}\rangle$ in the limit $t\to\infty$ for an adiabatically expanding universe. The quantities $\bar{\rho}$ and $\bar{P}$ are the cutoff-dependent vacuum energy density and pressure in Minkowski space, where k is the magnitude of the 3-momentum.

The physical energy density and pressure for the Einstein universe are defined to be

$$\rho=\lim_{\alpha\to 0}(\tilde{\rho}-\bar{\rho}) \quad (23a)$$

and

$$P=\lim_{\alpha\to 0}(\tilde{P}-\bar{P}). \quad (23b)$$

If the summations in Eq. (21) are split into two portions and the square roots are expanded in a power series, the result is

$$\tilde{\rho}=(4\pi^2a^4)^{-1}\left[\sum_{m=1}^{M-1} m^2(m^2+r^2)^{1/2}e^{-\alpha m}+\sum_{n=3}^{\infty} b_n r^{2n}\sum_{m=M}^{\infty} m^{3-2n}e^{-\alpha m}+\sum_{m=M}^{\infty}\left(m^3+\frac{1}{2}mr^2-\frac{1}{8}\frac{r^4}{m}\right)e^{-\alpha m}\right] \quad (24a)$$

and

$$\tilde{P}=(12\pi^2a^4)^{-1}\left[\sum_{m=1}^{M-1} m^4(m^2+r^2)^{-1/2}e^{-\alpha m}+\sum_{n=3}^{\infty} c_n r^{2n}\sum_{m=M}^{\infty} m^{3-2n}e^{-\alpha m}+\sum_{m=M}^{\infty}\left(m^3-\frac{1}{2}mr^2+\frac{3}{8}\frac{r^4}{m}\right)e^{-\alpha m}\right], \quad (24b)$$

where

$$b_n=(-1)^{n-1}(2n-3)!!/2^n n! \quad (25a)$$

and

$$c_n=(-1)^n(2n-1)!!/2^n n!, \quad (25b)$$

and where M is any integer greater than $r\equiv\mu a$. The final term in these expressions contain all of the divergent parts. The sums which appear there, together with their asymptotic forms as $\alpha\to 0$, are

$$\sum_1^\infty m^3e^{-\alpha m}=(e^\alpha-1)^{-4}(e^{3\alpha}+4e^{2\alpha}+e^\alpha)\sim\frac{6}{\alpha^4}+\frac{1}{120}+O(\alpha), \quad (26a)$$

$$\sum_1^\infty me^{-\alpha m}=e^\alpha(e^\alpha-1)^{-2}\sim\frac{1}{\alpha^2}-\frac{1}{12}+O(\alpha), \quad (26b)$$

and

$$\sum_1^\infty\frac{e^{-\alpha m}}{m}=\alpha-\ln(e^\alpha-1)\sim-\ln\alpha+O(\alpha). \quad (26c)$$

The integrals which appear in Eq. (22) may be evaluated explicitly. The result is[17]

$$\bar{\rho}=(8\pi a^4)^{-1}r\frac{d^2}{d\alpha^2}\{\alpha[H_1(r\alpha)-N_1(r\alpha)]\}$$

$$\sim(4\pi^2a^4)^{-1}\left[\frac{6}{\alpha^4}+\frac{r^2}{2\alpha^2}+\frac{1}{8}r^4\ln\left(\frac{r\alpha}{2}\right)+\left(\frac{1+4\gamma}{32}\right)r^4\right] \quad (27a)$$

and

$$\bar{P}=(24\pi a^4)^{-1}r\frac{d^3}{d\alpha^3}[1+N_1(r\alpha)-H_1(r\alpha)]$$

$$\sim(12\pi^2a^4)^{-1}\left[\frac{6}{\alpha^4}-\frac{r^2}{2\alpha^2}-\frac{3}{8}r^4\ln\left(\frac{r\alpha}{2}\right)-\left(\frac{7+12\gamma}{32}\right)r^4\right], \quad (27b)$$

where N_1 is a Bessel function of the second kind, H_1 is a Struve function, and $\gamma=0.577\ldots$ is Euler's constant.

If we now combine Eqs. (24), (26), and (27) we obtain the final result

$$\rho=(4\pi^2a^4)^{-1}\left\{\frac{1}{120}-\frac{1}{24}r^2-\frac{1}{8}r^4\left(\ln\frac{r}{2}+\gamma+\frac{1}{4}\right)+\sum_{m=1}^{M-1}\left[m^2(m^2+r^2)^{1/2}-m^3-\frac{1}{2}r^2m+\frac{r^4}{8m}\right]+\sum_{n=3}^{\infty}b_nr^{2n}\sum_{m=M}^{\infty}m^{3-2n}\right\} \tag{28a}$$

and

$$P=(12\pi^2a^4)^{-1}\left\{\frac{1}{120}+\frac{1}{24}r^2+\frac{1}{8}r^4\left(3\ln\frac{r}{2}+3\gamma+\frac{7}{4}\right)+\sum_{m=1}^{M-1}\left[m^4(m^2+r^2)^{-1/2}-m^3+\frac{1}{2}r^2m-\frac{3r^4}{8m}\right]+\sum_{n=3}^{\infty}c_nr^{2n}\sum_{m=M}^{\infty}m^{3-2n}\right\}. \tag{28b}$$

These finite, cutoff-independent expressions are independent of the choice of M provided that $r<M$. These values of ρ and P are for the case of the neutral scalar field; those for the charged scalar field are twice as large.

In the case that $\mu=0$, we have

$$\rho=3P=\frac{1}{480\pi^2a^4}, \tag{29}$$

which is the result obtained in I. This result was rederived by Dowker and Critchley[13] by a method which involves removing a certain term in the expansion of the Feynman propagator and which agrees with the result of the method used here.

The expressions for ρ and P in the general case may be evaluated numerically. With increasing radius a, the vacuum energy density and pressure for nonzero μ decrease more rapidly to zero than in the case when $\mu=0$. Also, for fixed a both ρ and P decrease rapidly with increasing μ (after an initial increase in P). Figure 1 illustrates the behavior of ρ and P.

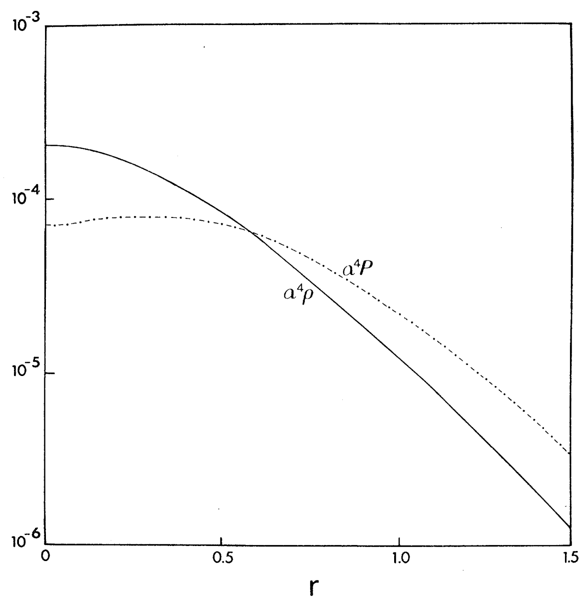

FIG. 1. The behavior of the vacuum energy density ρ and pressure P as a function of the scalar particle mass μ. The radius of the universe is a, and $r=\mu a$.

Of particular interest is the fact that $P>\rho$ for certain values of a and μ. This violates the dominant energy condition (DEC), which states that if t^α is a timelike vector, $T_{\alpha\beta}t^\alpha t^\beta \geq 0$ and $T^\alpha{}_\beta t^\beta$ is also a timelike vector. In an orthonormal frame, this statement says that $\rho \geq |P_i|$ where P_i is any of the principal pressures. For classical matter this is a reasonable requirement which ensures that the speed of sound is always less than that of light.

For quantized matter fields, however, the dominant energy condition fails. Hawking[18] has shown that DEC, along with the conservation law $T^{\mu\nu}{}_{;\nu}=0$, implies that if $T_{\mu\nu}=0$ on an initial spacelike hypersurface of a spacetime, then it is zero on any subsequent spacelike hypersurface. This would preclude the possibility of particle creation by the gravitational field. However, quantum field theory unambiguously predicts that such creation must occur,[19] so we are forced to conclude that DEC is incompatible with field theory. Zel'dovich and Pitaevsky[20] have also argued that a small perturbation $h_{\mu\nu}$ of the Minkowski metric will induce a vacuum energy density and pressure for which $\rho \sim h^2$ and $P \sim h$ and hence DEC is violated. Parker and Fulling[21] have constructed an explicit cosmological model in which the pressure is both negative and larger in magnitude than the energy density. This makes it possible to avoid the singularity. In fact, DEC is violated in flat-space quantum field theories. Epstein, Glaser and Jaffe[22] have proven under very general assumptions that in any local field theory there exist state vectors $|\Psi\rangle$ such that $\langle\Psi|:T_{00}:|\Psi\rangle<0$ somewhere. (The colons denote normal ordering.) A negative energy density is not only incompatible with DEC, but also violates the other energy conditions (i.e., the weak and strong conditions) assumed in the proof of various singularity theorems.[23]

It is thus not particularly surprising that the vacuum energy density and pressure of the massive scalar field in the Einstein universe also violates DEC.

IV. THE ELECTROMAGNETIC FIELD

The solution and quantization of Maxwell's equations in a closed Robertson-Walker metric (and hence for the Einstein universe) have been carried out by Mashhoon.[24] The eigenfrequencies of the electromagnetic field in the Einstein universe are

$$\omega_n = k_n = n/a, \quad n = 2, 3, \ldots \tag{30}$$

with a degeneracy of $2(n^2-1)$. The vacuum expectation value of the energy-momentum tensor is

$$\rho_0 = 3P_0 = (2\pi^2 a^3)^{-1} \sum_{n=2}^{\infty} (n^2-1)\omega_n . \tag{31}$$

If we introduce a cutoff function as before, we have the cutoff-dependent quantities

$$\tilde{\rho} = 3\tilde{P} = (2\pi^2 a^4)^{-1} \sum_{n=0}^{\infty} n(n^2-1) f(\alpha, \omega, a). \tag{32}$$

We will require that f depend on ω and a only through the combination

$$\Omega(\omega) = \tfrac{1}{4}(\omega^4 - 2\omega^2 a^2). \tag{33}$$

We may use the Euler-MacLaurin formula to write Eq. (32) as

$$\rho = (2\pi^2)^{-1}\left[\int_0^\infty f(\alpha,\Omega)d\Omega - \sum_{l=1}^{N} \frac{B_{2l}}{(2l)!} F^{(2l-1)}(0) + R_N\right], \tag{34}$$

where B_{2l} is a Bernoulli number, R_N is a remainder term, and $F^{(n)}(x)$ is defined by

$$F^{(n)}(x) = \frac{d^n F}{dx^n} = \frac{d^n}{dx^n}[(x^3 - x)a^{-4} f(\alpha, \Omega(x))]. \tag{35}$$

Here N may be any integer greater than 2. Thus

$$F^{(1)}(0) = -a^{-4}, \tag{36a}$$

$$F^{(3)}(0) = 6a^{-4}, \tag{36b}$$

and

$$\lim_{\alpha\to 0} F^{(n)}(0) = 0, \quad 3 < n \le 2N \tag{36c}$$

if we impose the smoothness condition, Eq. (8). The explicit expression for the remainder is[25]

$$R_N = \int_0^\infty F^{(2N)}(x) P_{2N}(x) dx, \tag{37}$$

where $P_{2N}(x)$ is the Bernoulli periodic function. We are, in particular, requiring that $F^{(2N)}(x) \to 0$ as $\alpha \to 0$ in such a way that

$$\lim_{\alpha\to 0} R_N = 0. \tag{38}$$

The form of $\tilde{\rho}$ for infinite a is

$$\bar{\rho} = (2\pi^2)^{-1}\int_0^\infty f(\alpha,\Omega)d\Omega, \tag{39}$$

so that

$$\rho = \lim_{\alpha\to 0}(\tilde{\rho} - \bar{\rho}) = \frac{11}{240\pi^2 a^4} \tag{40a}$$

and

$$P = \tfrac{1}{3}\rho. \tag{40b}$$

Thus one obtains a vacuum energy density and pressure which are 22 times as large as those corresponding to the massless conformal scalar field. One might expect that the factor should be 2, corresponding to the doubling of the number of polarization states. A comparison of Eq. (30) with Eq. (12) for $\mu = 0$ shows, however, that only for the high-frequency modes is the contribution to the zero-point energy of the electromagnetic field twice that for the scalar field. It is the low-frequency modes which contribute significantly to the physical vacuum energy, and for these modes no such simple relationship exists.

V. THE NEUTRINO FIELD

A. The Dirac equation in the Einstein universe

The formulation of the Dirac equation in a general curved spacetime has been discussed by numerous authors, among them Schrödinger,[26] Bargmann,[27] and Brill and Wheeler.[28] The solutions of the Dirac equation in the Einstein universe have been considered by Schrödinger[29] and by Unruh.[30] In this section a brief review of the general formalism will be given and the solutions for neutrinos will be presented as a prelude to the formal quantization and calculation of the vacuum energy of the neutrino field.

If $g_{\mu\nu}$ is the metric of the spacetime, let γ_μ be a set of four 4×4 matrices which satisfy the anticommutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} I, \tag{41}$$

where I is the unit matrix. The spinor connections Γ_μ are defined by the relation

$$\gamma_{\alpha,\mu} - \Gamma^\rho_{\alpha\mu}\gamma_\rho = \Gamma_\mu\gamma_\alpha - \gamma_\alpha\Gamma_\mu, \tag{42}$$

where $\Gamma^\rho_{\alpha\mu}$ are the usual Christoffel symbols formed from $g_{\mu\nu}$. If in addition to Eq. (42) one requires that the traces of the Γ_μ vanish, they are uniquely defined. The Dirac equation for a massless particle is

$$\gamma^\alpha \nabla_\alpha \psi = 0, \tag{43}$$

where $\nabla_\alpha = \partial_\alpha - \Gamma_\alpha$ is the spinor covariant derivative. In addition to Eq. (43), the spinor for a neutrino must satisfy an addition constrain that ensures that it be an eigenstate of helicity. Let

$$\gamma_5=(-g)^{-1/2}\gamma_0\gamma_1\gamma_2\gamma_3$$
$$=\frac{1}{4!}(-g)^{-1/2}\epsilon^{\mu\nu\sigma\rho}\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho\,, \tag{44}$$

where $\epsilon^{\mu\nu\sigma\rho}$ is totally antisymmetric and $\epsilon^{0123}=1$. Then the spinor for a neutrino (negative helicity) must satisfy

$$(1-i\gamma_5)\psi=0, \tag{45a}$$

and that for an antineutrino (positive helicity) must satisfy

$$(1+i\gamma_5)\psi=0. \tag{45b}$$

It is convenient to introduce an orthonormal tetrad of vectors λ_a^μ at each point in spacetime which satisfy the relations

$$\eta^{ab}\lambda_a^\mu\lambda_b^\nu=g^{\mu\nu}, \tag{46a}$$

$$g_{\mu\nu}\lambda_a^\mu\lambda_b^\nu=\eta_{ab}, \tag{46b}$$

where η_{ab} is the Minkowski metric. If the $\tilde{\gamma}_a$ are a set of matrices which satisfy

$$\{\tilde{\gamma}_a,\tilde{\gamma}_b\}=2\eta_{ab}I, \tag{47}$$

then

$$\gamma_\mu=\lambda_\mu^a\tilde{\gamma}_a \tag{48}$$

satisfy Eq. (41). Latin indices are raised and lowered by η_{ab} and Greek indices by $g_{\mu\nu}$. The spinor connections may be expressed explicitly as

$$\Gamma_\alpha=\tfrac{1}{2}g_{\mu\nu}(\lambda_a^\mu\lambda_{\rho,\alpha}^a-\Gamma_{\rho\alpha}^\mu)S^{\nu\rho}, \tag{49}$$

where

$$S^{\nu\rho}=\tfrac{1}{2}[\gamma^\nu,\gamma^\rho]. \tag{50}$$

We now turn to the specific case of the Einstein universe. If the metric is given as in Eqs. (4) and (10) with $A=a$, a natural choice for the λ_μ^a is

$$\lambda_0^0=1,\ \lambda_1^1=a,\ \lambda_2^2=a\sin\chi,\ \text{and}\ \lambda_3^3=a\sin\chi\sin\theta. \tag{51}$$

The connections Γ_μ then become

$$\Gamma_0=0,\ \Gamma_1=0,\ \Gamma_2=\tfrac{1}{8}\cos\chi[\tilde{\gamma}^2,\tilde{\gamma}^1],$$

and (52)

$$\Gamma_3=\tfrac{1}{8}\{\sin\theta\cos\chi[\tilde{\gamma}^3,\tilde{\gamma}^1]+\cos\theta[\tilde{\gamma}^3,\tilde{\gamma}^2]\}.$$

If we let

$$\Psi=\psi\sin\chi(\sin\theta)^{1/2}, \tag{53}$$

Eq. (43) may be written as

$$H\Psi\equiv\frac{i}{a}\left(\tilde{\gamma}^1\Psi_{,1}+\frac{\tilde{\gamma}^2}{\sin\chi}\Psi_{,2}+\frac{\tilde{\gamma}^3}{\sin\chi\sin\theta}\Psi_{,3}\right)\tilde{\gamma}^0=i\Psi_{,0}. \tag{54}$$

Let us now choose a representation in which

$$\tilde{\gamma}^0=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\ \tilde{\gamma}^1=\begin{pmatrix}0 & \sigma^3\\ -\sigma^3 & 0\end{pmatrix},$$
$$\tilde{\gamma}^2=\begin{pmatrix}0 & \sigma^1\\ -\sigma^1 & 0\end{pmatrix},\ \text{and}\ \tilde{\gamma}^3=\begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix}, \tag{55}$$

where σ^i are the usual Pauli matrices. We now have

$$\gamma^5=i\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}, \tag{56}$$

so a neutrino is represented by a spinor of the form

$$\Psi=\Psi_\nu=\begin{pmatrix}F\\ G\\ 0\\ 0\end{pmatrix}e^{i(m\phi-\omega t)}. \tag{57}$$

Let

$$F=f_1(\chi)f_2(\theta)\quad\text{and}\quad G=g_1(\chi)g_2(\theta). \tag{58}$$

Then Eq. (54) reduces to two pairs of first-order equations:

$$\frac{df_1}{d\chi}-\kappa_1\frac{g_1}{\sin\chi}+ia\omega f_1=0, \tag{59a}$$

$$\frac{dg_1}{d\chi}-\kappa_2\frac{f_1}{\sin\chi}-ia\omega g_1=0, \tag{59b}$$

$$\frac{df_2}{d\theta}-\frac{m}{\sin\theta}f_2-\kappa_2 g_2=0, \tag{60a}$$

and

$$\frac{dg_2}{d\theta}+\frac{m}{\sin\theta}g_2+\kappa_1 f_2=0, \tag{60b}$$

where κ_1 and κ_2 are separation constants. If one makes the substitution $f_1\to f_1$, $g_1\to(\kappa_2/\kappa_1)^{1/2}g_1$, $f_2\to f_2$, $g_2\to(\kappa_1/\kappa_2)^{1/2}g_2$, and κ_1 and $\kappa_2\to(\kappa_1\kappa_2)^{1/2}$, one finds that Eqs. (59) and (60) are transformed into a set of equations containing only one undetermined constant. This means that we may obtain all solutions of Eq. (54) by restricting $\kappa_1=\kappa_2=\kappa$ in Eqs. (59) and (60), which we will do henceforth. The substitution $\kappa\to-\kappa$, $f_1\to f_1$, $g_1\to-g_1$, $f_2\to f_2$, and $g_2\to-g_2$ leaves these equations and the function F and G unchanged, so we may also restrict κ to be nonnegative without excluding any solutions.

The solutions of Eqs. (59) and (60) may be expressed in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}$. Let

$$f_1=\tfrac{1}{2}(K_1+iK_2) \tag{61a}$$

and

$$g_1 = \tfrac{1}{2}(K_1 - iK_2). \tag{61b}$$

Then

$$K_1(z) = (1+z)^{1/2}(1-z^2)^{\kappa/2} P_n^{(\kappa-1/2,\kappa+1/2)}(z) \tag{62a}$$

and

$$K_2(z) = (-1)^n (1-z)^{1/2}(1-z^2)^{\kappa/2} P_n^{(\kappa+1/2,\kappa-1/2)}(z), \tag{62b}$$

where $z = \cos\chi$ and $n = a\omega - \kappa - \frac{1}{2} = 1, 2, \ldots$. Likewise, if $m > 0$,

$$f_2(x) = (1+x)^{1/2}(1-x^2)^{m/2} P_j^{(m-1/2,m+1/2)}(x) \tag{63a}$$

and

$$g_2(x) = (-1)^j (1-x)^{1/2}(1-x^2)^{m/2} P_j^{(m+1/2,m-1/2)}(x), \tag{63b}$$

where $x = \cos\theta$ and $j = k - m - \frac{1}{2} = 0, 1, \ldots$. If $m < 0$,

$$f_2(x) = (1-x)^{1/2}(1-x^2)^{-m/2} P_j^{(-m+1/2,-m-1/2)}(x) \tag{64a}$$

and

$$g_2(x) = (-1)^j (1+x)^{1/2}(1-x^2)^{-m/2} P_j^{(-m-1/2,-m+1/2)}(x), \tag{64b}$$

where $j = \kappa + m - \frac{1}{2} = 0, 1, \ldots$.

Because the spinors change spin under the rotation $\phi \to \phi + 2\pi$, m must be half-integral and κ must be a positive integer. For fixed κ, m takes on the values

$$m = -\kappa + \tfrac{1}{2}, \ldots, -\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \kappa - \tfrac{1}{2}, \tag{65}$$

and for fixed ω, κ takes on the values

$$\kappa = 1, 2, \ldots, a\omega - \tfrac{1}{2}. \tag{66}$$

This specifies a complete set of neutrino solutions of Eq. (43) of the form

$$u_\lambda = \frac{A_\lambda}{\sin\chi(\sin\theta)^{1/2}} \begin{pmatrix} F_\lambda \\ G_\lambda \\ 0 \\ 0 \end{pmatrix} e^{i(m\phi - \omega t)}, \tag{67}$$

where $\lambda = (\omega, \kappa, m)$, with eigenfrequencies

$$\omega = \omega_n = a^{-1}(n + \tfrac{1}{2}), \quad n = 1, 2, \ldots \tag{68}$$

and a degeneracy of $n(n+1)$. Likewise, a set of antineutrino solutions is given by

$$v_\lambda = \frac{A_\lambda}{\sin\chi(\sin\theta)^{1/2}} \begin{pmatrix} 0 \\ 0 \\ F_\lambda \\ G_\lambda \end{pmatrix} e^{i(m\phi + \omega t)}. \tag{69}$$

If the normalization constant A_λ is appropriately choosen, we have the orthonormality relations

$$\int u_\lambda^\dagger u_{\lambda'} \sqrt{-g}\, d^3x = \int v_\lambda^\dagger v_{\lambda'} \sqrt{-g}\, d^3x = \delta_{\lambda\lambda'}. \tag{70}$$

B. Quantization

The Lagrangian from which Eq. (43) may be derived is

$$\mathcal{L} = i\bar{\psi}\gamma^\alpha \nabla_\alpha \psi, \tag{71}$$

where $\bar{\psi} = \psi^\dagger \tilde{\gamma}^0$. The canonical momentum is defined by

$$\pi = \frac{\delta \mathcal{L}}{\delta(\nabla^0 \psi)} = i\bar{\psi}\gamma_0. \tag{72}$$

The quantization is carried out by imposition of the canonical anticommutation relations

$$\{\psi_\alpha(x^i, t), \pi_\beta(y^i, t)\} = i\delta_{\alpha\beta}\delta(x^i, y^i), \tag{73}$$

where α and β are Dirac indices and $\delta(x^i, y^i)$ satisfies

$$\int \delta(x^i, y^i)\sqrt{-g}\, d^3x = 1. \tag{74}$$

The field operator ψ may be expanded as

$$\psi = \sum_\lambda (a_\lambda u_\lambda + b_\lambda^\dagger v_\lambda), \tag{75}$$

where

$$\{a_\lambda, a_{\lambda'}^\dagger\} = \{b_\lambda, b_{\lambda'}^\dagger\} = \delta_{\lambda\lambda'}. \tag{76}$$

The creation operator for neutrinos is $a_\lambda^\dagger$ and that for antineutrinos is $b_\lambda^\dagger$. The vacuum state $|0\rangle$ is defined by

$$a_\lambda |0\rangle = b_\lambda |0\rangle = 0 \quad \text{for all } \lambda. \tag{77}$$

The energy-momentum tensor for the neutrino field is

$$T_{\mu\nu} = \tfrac{1}{4} i [\bar{\psi}\gamma_\mu \nabla_\nu \psi + \bar{\psi}\gamma_\nu \nabla_\mu \psi - (\nabla_\mu \bar{\psi})\gamma_\nu \psi - (\nabla_\nu \bar{\psi})\gamma_\mu \psi]. \tag{78}$$

Its trace vanishes, $T^\mu_\mu = 0$, as expected for a conformally invariant field. The vacuum expectation value of $T_{\mu\nu}$ is given by

$$\rho_0 = 3P_0 = -(2\pi^2 a^3)^{-1} \sum_\lambda \omega. \tag{79}$$

The formal, divergent expression for the zero-point energy is negative, as usual in the case of fermions. It may be thought of either as corresponding to $-\omega$ for each antineutrino mode, or to $-\frac{1}{2}\omega$ for each neutrino and antineutrino mode. If we adopt the latter point of view, the degeneracy of each eigenfrequency becomes $2n(n+1)$. Of course, ρ_0 and P_0 are the same in either case.

We may now proceed to calculate the regularized vacuum energy and pressure as in the electromagnetic case. That is, the cutoff-dependent energy density and pressure are defined by

$$\tilde{\rho} = 3\tilde{P} = -(2\pi^2 a^4)^{-1} \sum_{n=1}^{\infty} n(n+1)(n+\tfrac{1}{2})\, f(\alpha, \Omega), \tag{80}$$

where in this case the cutoff function is taken to be a function of

$$\Omega = \omega^4 - \tfrac{1}{2}\omega^2 a^2. \tag{81}$$

The vacuum energy density and pressure are found to be

$$\rho = \frac{17}{1920\pi^2 a^4} \tag{82a}$$

and

$$P = \tfrac{1}{3}\rho. \tag{82b}$$

Aside from the numerical coefficient, this answer is of the same form as in the case of the electromagnetic and massless conformal fields. Of particular interest is that ρ and P are positive in the case of neutrinos in spite of the fact that the divergent quantities are negative.

VI. DISCUSSION

In the previous sections, the vacuum energy density and pressure were calculated for the massive scalar field, the electromagnetic field, and the neutrino field in the Einstein universe. In all of these cases there is a natural definition of the vacuum state, which is obtained by using the global timelike Killing vector to define positive-frequency solutions of the wave equations. In the case of a closed, expanding Robertson-Walker universe this is not generally the case. In the case of the massive scalar field there will be particle production, and it will not in general be possible to give an unambiguous definition of particle number. The vacuum state is then not well defined, except in the sense of in or out vacuums if the universe is asymptotically static in the past or future. This does not prevent one from defining a regularized energy-momentum tensor for a given state vector, as might be done using adiabatic regularization or other regularization prescriptions.

In the case of the conformally invariant fields there is no particle production, so a vacuum state can be defined which will coincide with both the in and out vacuums. The question then arises as to what the vacuum energy density and pressure for such a state should be. In I it was proposed to obtain the regularized energy-momentum tensor for such a case by conformally transforming the result obtained in the Einstein universe to the general closed Robertson-Walker universe. For the massless conformal scalar field this yields

$$\rho = 3P = \frac{1}{480\pi^2 A^4(t)}. \tag{83}$$

Similarly, for the electromagnetic and neutrino fields one simply replaces a by $A(t)$ in Eqs. (40) and (82), respectively.

Dowker and Critchley[13] have pointed out, however, that in the case of de Sitter space $[A(t) = b\cosh b^{-1}t]$ this leads to a result which is not invariant under the isometry group of the space. It also disagrees with the result which they obtain by dimensional regularization. There seem to be at least two possible explanations for this discrepancy. One is that it is not permissible to conformally transform regularized energy-momentum tensors because the process of regularization breaks the conformal invariance that classical field theory possesses. The other, perhaps related, possibility is that there are different definitions of the vacuum state, some of which are de Sitter invariant and others of which are not. Davies and Fulling[3] have pointed out that this is the case in two-dimensional de Sitter space. In either case more work needs to be done before a complete understanding of the case of a general closed universe is attained. One would also like to treat the case of the Einstein universe by other methods such as point separation or dimensional regularization, to compare with the results obtained here.

Note added in proof. S. G. Mamayev, V. M. Mostepanenko, and A. A. Starobinsky [Zh. Eksp. Teor. Fiz. 70, 1577 (1976)] have recently also analyzed the vacuum energy and pressure of the conformally coupled scalar field in a closed isotropic universe and reached conclusions similar to those obtained in Sec. III.

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