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Why is a black hole hot?

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The distribution function as well as the fluctuation spectrum of the radiation emitted from a spherically symmetric black hole is derived within the basic framework of statistical mechanics from the fact that it is an entity endowed with a densely spaced quantum level spectrum. The fluctuation spectrum of the emitted radiation is thus found to exhibit deviations from that of pure thermal radiation.

The purpose of this note is to identify in what way the densely spaced quantum energy-level structure of an incipient black hole gives rise to (a) emission of radiation which is blackbody in its essence, i.e., both in regard to its energy and its fluctuation spectrum, and (b) deviations from blackbody radiation, deviations that serve as a signature for the quantum level structure of the black hole. The statistical-mechanical treatment of thermal emission from a many-level compound nucleus is well known.[1] That a black hole should have "internal (quantum) configurations" was first stated by Bekenstein[2] and was also used by Bekenstein[3] and Hawking[4] to discuss its statistical-thermodynamical aspects.

Consider an isolated black hole. Denote by Ω^{BH} the number of microscopic states accessible[5,6] to that black hole. If the black hole can indeed be characterized by such states then it follows that its entropy is

$$S^{BH} = k \ln \Omega^{BH} . \tag{1}$$

The entropy and, hence, the number of accessible states is determined by certain macroscopic parameters only. For a spherically symmetric black hole there is only one parameter, its mass M. This fact is expressed by the equation

$$\Omega^{BH} = \Omega^{BH}(M) .$$

The entropy of a black hole is

$$S^{BH} = \alpha k M^2 / L_P^2 . \tag{2}$$

The constants k and $L_P^2 = \hbar/Gc$ are Boltzmann's constant and the squared Planck length, respectively. On dimensional-theoretic grounds Wheeler[7] estimates the dimensionless constant α to "of order unity." On statistical-thermodynamic[2,8] as well as information-theoretic[9] grounds Bekenstein estimates $\alpha \leqslant 96$. On quantum-field-theoretic grounds, Hawking[10] and others[11,12] give $\alpha = 4\pi$. On the basis of classical field theory applied to zero-point fluctuations one obtains[13] the same value.

Consider a gas of n photons each of frequency $\hbar\omega$. The total energy of this gas is

$$\eta = n\hbar\omega .$$

Let these photons be distributed over ΔG_ω phase-space cells ("Planck oscillators," outward-traveling modes each satisfying the wave equation). The number of accessible quantum states of this radiated gas of n photons (i.e., of this ensemble of ΔG_ω oscillators) is[14]

$$\Omega_\omega^{rad}(\eta) = (\Delta G_\omega + n - 1)!/(\Delta G_\omega - 1)!\, n! .$$

In terms of the entropy of the ("monochromatic") photon gas under consideration this number is

$$\Omega_\omega^{rad}(\eta) = \exp[S_\omega^{rad}(\eta)/k] . \tag{3}$$

For large n and ΔG_ω the entropy, we recall, has the form first determined by Planck,

$$S_\omega^{rad}(\eta) = k\Delta G_\omega [(f+1)\ln(f+1) - f \ln f] , \tag{4}$$

where

$$f = \eta/\Delta G_\omega \hbar\omega = n/\Delta G_\omega \tag{5}$$

is the mean number of photons occupying any one of the ΔG_ω phase-space cells.

Consider an incipient black hole (a star during late collapse) at that instant of time when its mass energy is $M^{(1)}$. It proceeds to make a transition to a state of lower energy $M^{(2)}$ and thereby emits a gas of photons of energy $\eta^{(2)}$ occupying ΔG_ω phase-space cells. Next, the black hole makes another transition to a state of mass-energy $M^{(3)}$ and emits thereby a photon gas of energy $\eta^{(3)}$ also occupying ΔG_ω phase-space cells, etc. We have the following family of processes.

$$\begin{aligned} &M^{(1)} \to M^{(2)} + \eta^{(2)} \\ &M^{(2)} \to M^{(3)} + \eta^{(3)} \\ &\cdots \\ &\qquad \to M + \eta \\ &\cdots \end{aligned} \tag{6}$$

These processes are stimulated at the star's surface by the zero-point fluctuations of the vacuum and are ordered in a time-sequential way.[13] A typical process is one in which the black hole makes a transition from a state of mass-energy $M+\eta$ to a state of lower mass M.

Within the context of several transitions as exhibited by Eq. (6) introduce the mean energy η_0 of the concomitant (monochromatic) photon gases so that the energy of a typical gas is

$$\eta=\eta_0+\epsilon\,. \tag{7}$$

Then ask: What is the probability that a black hole, whose initial mass is $M_0+\eta_0$, emits a gas of photons of energy $\eta=\eta_0+\epsilon$ and thereby makes a transition to a state whose final mass is

$$M=M_0-\epsilon\ ? \tag{8}$$

The answer is

$$(\text{probability})_\omega=\text{const}\times\Omega^{\text{BH}}(M_0-\epsilon)\Omega_\omega^{\text{rad}}(\eta_0+\epsilon)\,. \tag{9}$$

The first factor is the number of final states accessible to the black hole making such a transition. For each of these final states accessible to a black hole the emitted photon gas is in any one of *its* accessible states whose number is given by the second factor. It follows that the transition probability for the process

$$M_0+\eta_0\rightarrow(M_0-\epsilon)+(\eta_0+\epsilon)\,,$$

which is proportional to the total number of final states accessible to the black-hole–photon-gas system, is given by the product of the two factors of Eq. (9).

[One may note that the probability for a black hole to make a transition from an initial mass $M_0+\eta_{01}+\eta_{02}+\cdots$ to a final mass $M_0-\epsilon_1-\epsilon_2-\cdots$ by emitting photon gases characterized by frequencies $\omega_1,\omega_2,\ldots$ and having respective energies $\eta_{01}+\epsilon_1$, $\eta_{02}+\epsilon_2,\ldots$ is

$$\text{probability}=\text{const}\times\Omega^{\text{BH}}(M_0-(\epsilon_1+\epsilon_2+\cdots))\times\Omega_{\omega_1}(\eta_{01}+\epsilon_1)\Omega_{\omega_2}(\eta_{02}+\epsilon_2)\cdots\,.$$

Such a more general and complete expression has to be understood, but does not have to be mentioned explicitly every time we mention in this paper the total number of states accessible to the black-hole–photon-gas quantum system.]

The connecting link between the quantum-mechanical microstructure and the thermodynamic macrostructure of the total system is the inverted Planck-Boltzmann relation[15]

$$\Omega^{\text{total}}=\exp(S^{\text{total}}/k)=\Omega^{\text{BH}}(M_0-\epsilon)\Omega_\omega^{\text{rad}}(\eta_0+\epsilon)\,.$$

Equations (2) and (4) show that the first factor is a rapidly decreasing function of ϵ, while the second factor is a rapidly increasing one. The product, one expects, has a maximum corresponding to that macroscopic configuration of the total system which has the largest number of quantum-mechanical energy levels. This maximum is reflected in the maximum of the entropy of the total system. A Taylor-series expansion around $\epsilon=0$ yields

$$\begin{aligned}S^{\text{total}}&=S^{\text{BH}}+S_\omega^{\text{rad}}\\&=S^{\text{BH}}(M_0)+S_\omega^{\text{rad}}(\eta_0)\\&\quad+\left\{\frac{d(S^{\text{BH}}+S_\omega^{\text{rad}})}{d\epsilon}\right\}_0\epsilon+\frac{1}{2}\left\{\frac{d^2(S^{\text{BH}}+S_\omega^{\text{rad}})}{d\epsilon^2}\right\}_0\epsilon^2+\cdots\,.\end{aligned} \tag{10}$$

Let η_0 be that photon-gas energy which extremizes the number of accessible final states and, hence, the entropy of the total system. Thus the linear term vanishes. Substituting Eqs. (2) and (4) into this extremum condition, using Eq. (7) and then finally Eq. (5), one obtains

$$\begin{aligned}\eta_0&=\Delta G_\omega\hbar\omega f_0\\&=\Delta G_\omega\hbar\omega[\exp(8\pi M\omega)-1]^{-1}\,,\end{aligned} \tag{11}$$

the most probable energy of a set of photon gases, each occupying ΔG_ω phase-space cells.

Such a statistical quantum-level formulation of the emitted-radiation–black-hole system yields not only its extremal thermodynamic properties but also the nature of the deviations from the extremum, namely the properties of the fluctuations of the emitted radiation. Indeed, in terms of only the first nontrivial term of the Taylor expansion, Eq. (10), the expression for the spectral-emission probability, Eq. (9), becomes

$$(\text{probability})_\omega=\text{const}\times\exp\left(\frac{1}{k}\left\{\frac{d^2S^{\text{BH}}}{d\epsilon^2}+\frac{d^2S^{\text{rad}}}{d\epsilon^2}\right\}_0\frac{\epsilon^2}{2}\right)\,.$$

This expression gives a Gaussian probability distribution for the emission of photon gases, each occupying ΔG_ω phase-space cells. The probability is a function of ϵ, the deviation from the mean energy η_0 of the gases. The content of the curly bracket gives the inverse squared width of the distribution, the mean squared value $\overline{\epsilon^2}$ of the energy fluctuation of (monochromatic) radiation to be found in ΔG_ω phase-space cells. Using Eqs. (2) and (4) with the help of Eqs. (5), (7), (8), and (11) one obtains for a single ($\Delta G_\omega=1$) phase-space cell

$$\overline{\epsilon^2}=\left[\frac{1}{(\hbar\omega)^2f_0(f_0+1)}-\frac{8\pi}{E_P{}^2}\right]^{-1} \tag{12a}$$

$$=(\hbar\omega)^2f_0(f_0+1)\times\left[1+\frac{1}{8\pi}\left(\frac{L_P}{M}\right)^2(8\pi M\omega)^2f_0(f_0+1)+\cdots\right]\,. \tag{12b}$$

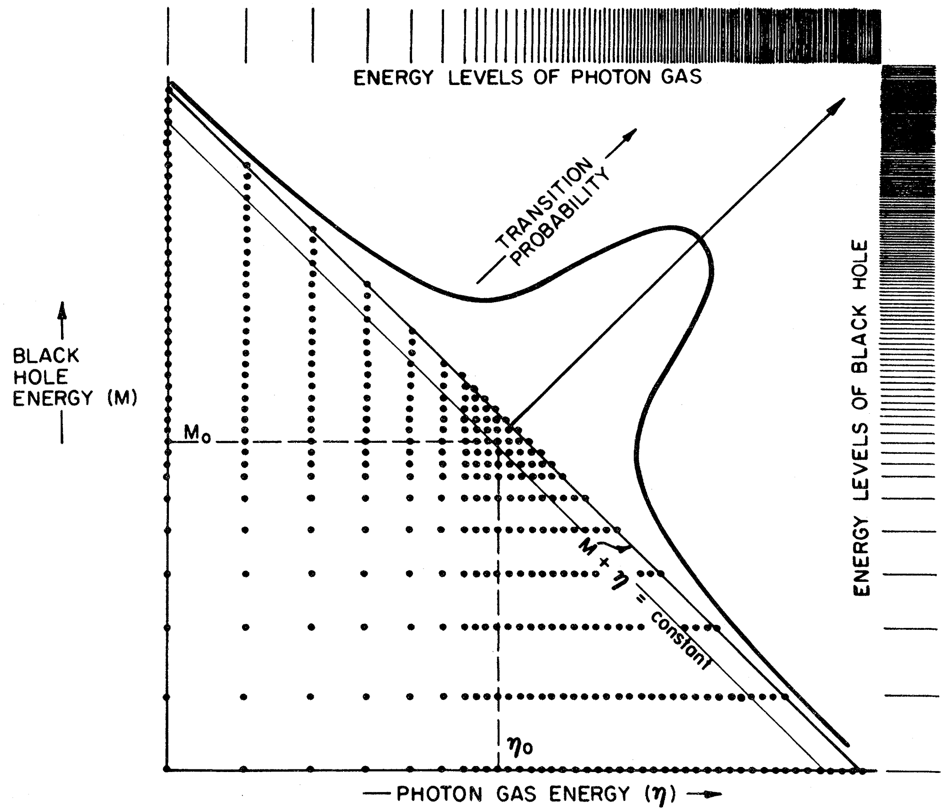

FIG. 1. Probability for making transition to final states accessible to a photon-gas–black-hole system. Each dot represents a state of this combined system. A given horizontal row of dots represents a fixed black-hole state with the photon gas assuming any one of the energy values of its quantum states. Similarly a vertical column of dots represents a photon-gas state with a black hole assuming any one of the energy values of its quantum states. As indicated by the spacing of the columns and the spacing of the rows, the accessible states of a black hole and those of an emitted photon gas are rapidly increasing functions of energy. Energy is conserved when a black hole makes a transition from some initial state of, say, energy $M_0+\eta_0$ to a final photon-gas–black-hole state. Consequently, the number of accessible final states is limited to a subset within the indicated diagonal narrow strip within which energy is constant (to a prescribed degree of accuracy). The Gaussian probability curve plots the number of accessible final states as a function of the deviation ϵ away from the mean photon gas energy η_0. There are no laws of physics (e.g., selection rules) that favor one of these states over any other. Consequently, all are equally probable. The photon-gas–black-hole system will therefore share the energy $M_0+\eta_0=M+\eta$ in a probable fashion determined by the density of accessible states along the narrow region of constant energy. It is therefore clear that the conservation of energy gives rise to the property "temperature" (T). The inverse temperature, T^{-1}, is merely the magnitude of the fractional change in the black hole (or photon gas) states per unit energy in that neighborhood where the density of accessible states of the combined system has an extremum. In the picture this happens at $M=M_0$ and $\eta=\eta_0$. The fluctuations in the energy of the emitted photon gas as well as in the mass-energy of the black hole are reflected in the finite width of the Gaussian density of accessible-final-states curve. Upon making another transition the combined system makes a jump to another state in a second constant-energy strip parallel, adjacent, and to the left of the one indicated in the picture. In other words, as the black hole evaporates a representative point executes a kind of random walk towards the origin of this picture.

Here the physical constants G and c have been reintroduced by replacing Eq. (8) by $M=M_0-\epsilon G/c^4$. The constant $E_P{}^2=(\hbar c^5/G)$ is the squared Planck energy. The first term is important for macroscopic black holes, $M^2 \gg L_P{}^2$. This term is precisely the mean squared value of the fluctuations in the radiative energy emitted by a black hole.[13] This agreement between the detailed computation for an incipient black hole and the general results of statistical mechanics not only shows that such a black hole is a blackbody in the precise sense of the term but also accounts for the emission of the radiation in terms of the black hole making quantum transitions among its densely spaced set of energy levels.

These fluctuations correspond, we recall, to a mixture of statistically independent Boltzmann gases.[13,14,16] Thus, the fluctuations are primarily due to the particle nature of the emitted radiation.

The second term becomes important to the extent that the black hole is no longer macroscopic, $M^2 \sim L_P^2$. Under such a circumstance, the fluctuations are no longer due to only the quantized nature of the emitted radiation, but also due to the yet-to-be-determined detailed structure of the set of accessible energy levels of the black hole itself. Therefore, our considerations do not approximate the black hole as an "infinite thermodynamic" reservoir. However, this common approximation is made implicitly or explicitly in the computations of Refs. 10–13.

The determination of transition probabilities, Eq. (9), via the method of entropy extremization, Eq. (10), constitutes, we recall, the foundation[2,3] of statistical thermodynamics. It is applicable not only to a photon-gas–black-hole system, but also to the system consisting of a black hole together with any particle gas; one merely has to know the entropy.

The fluctuations, Eqs. (12), evidently give an indeterminacy in the observed magnitude of the mass-energy of a black hole. If its size is M (cm^{-1}), then the mean squared value of the energy fluctuations is $\sim(\hbar/8\pi M)^2 =$ (Compton energy)^2 for such an object,

$$\overline{\epsilon^2} = \left(\frac{\hbar}{8\pi M}\right)^2 x^2 f_0(f_0+1) \times \left[1+\frac{1}{16\pi}\left(\frac{L_P}{M}\right)^2 x^2 f_0(f_0+1)+\cdots\right].$$

Here $x=8\pi M\omega$ is the dimensionless "Boltzmann exponent."

It is appropriate to remind oneself of the logical status of the accessible-state formulation of the energy and the fluctuation-spectral properties of radiation emitted from a black hole. Is the premise of the existence of quantum states accessible to a black hole necessary for the understanding of these properties? The answer is no only if one does not care to identify the underlying idea(s) responsible for the energy spectrum and the fluctuation spectrum associated with a black hole.

Presently there is at least one direct benefit from the accessible-state formulation of the statistical many-level structure of a black hole: As a black hole radiates away its mass energy, and thereby approaches a microscopic black hole ($M \sim L_P$), its finite size and, hence, its quantum structure plays an increasingly important role in its evolution. This role manifest itself, as is evident from Eqs. (12), in a fluctuation spectrum which is characterized by a mean squared deviation which is different from that associated with radiation coming from a body with an "infinite" heat capacity. If a black hole could be isolated sufficiently from other noise-producing sources then there would exist the possibility of actually observing the difference and thus verifying the basic quantum nature of a black hole.

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