PHYSICAL REVIEW D VOLUME 14, NUMBER 12 15 DECEMBER 1976

Capture of particles from plunge orbits by a black hole

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(Received 23 June 1976; revised manuscript received 30 September 1976)

Photon and "parabolic" particle orbits around a Kerr-Newman black hole are considered, both for uncharged particles moving along geodesics and for charged particles under the influence of the Lorentz force (neglecting radiative processes). Investigation of the radial equation of motion gives conditions under which a photon or particle is captured from a "plunge" orbit when incident from a large distance; the cross sections and accreted angular momenta are calculated for various fluxes of particles incident on the black hole. For example, the cross section of an extreme Kerr ($a = 1$) black hole to an isotropic flux of particles is 0.90 of that for a Schwarzschild hole, and the accreted angular momentum per unit mass of swallowed flux is -0.828 leading to rapid spin-down of the hole. The periastra of "escape" orbits are considered, and also the minimum periastron corresponding to the unstable spherical orbit that divides plunge and escape orbits. The evolution of the spin and charge of a black hole accreting particles and photons is discussed, including that of primordial black holes. It is shown that such black holes may be of the Schwarzschild type having spun-down due to consumption of radiation and particles at early times and subsequent neutralization in an ionized intergalactic medium. The statistics of proton and electron capture by the smallest surviving primordial black holes, $M = 10^{15}$ g, suggests that they are most likely to be found possessing a single quantum of positive charge.

I. INTRODUCTION

The existence of massive black holes ($M \simeq 10^8 M_\odot$) in the nuclei of galaxies has been postulated,[1] and the possibility that such a black hole may grow by tidally disrupting the stellar population of the nucleus has been considered.[2-6] Relativistic effects are important for a black hole of this size because the Roche limit for tidal disruption of a star is of the order of the size of the event horizon,[1] and also because a significant contribution to the growth may be made by the capture of stars in plunge orbits.[7]

The velocity dispersion of the stellar population, $\sigma_v \ll c$, and so the orbits of stars encountering the hole are, to a very good approximation, "parabolic" with energy at infinity $E = \mu$ (rest mass energy).[8] (In this case the small velocity at infinity combined with the impact parameter serves to define the angular momentum of the orbit.) This provides the motivation for a study of parabolic orbits in a Kerr metric, in particular the cross section for plunge orbits and the periastron[5] of escape orbits. Previous work has concentrated on bound orbits[9] and orbits in the equatorial plane.[8,10]

Photon orbits are also considered[11,12] and again the cross sections and accreted angular momenta for various types of flux are calculated, although sufficient density of radiation to affect the evolution of a black hole is only likely to have occurred in a "big bang" and to have affected primordial black holes.

The work is extended to include the possibility of charged black holes although their existence is, at the moment, questionable. It is, perhaps, unlikely that sufficient charge may exist on a black hole to perturb the metric [$Q \lesssim M$, i.e., $Q \lesssim 1.71 \times 10^{20}(M/M_\odot)$ coulomb], but more plausible is the fact that sufficient charge may exist on a black hole (from an electrodynamic process in an accretion disk, perhaps) for the Lorentz force on protons or electrons to be comparable to the gravitational force [$eQ/\mu M \simeq 1$, i.e., for electrons with $e/\mu = 2.04 \times 10^{21}$, $Q \simeq 4.90 \times 10^{-22} M$ or $Q \simeq 8.38 \times 10^{-2}(M/M_\odot)$ coulomb]. The vacuum polarization considerations[13] show that rapid discharge occurs if $Q \gtrsim M^2\mu^2/e$. Thus for a black hole with $Q \lesssim M$ to survive, $M \gtrsim 10^{39}\text{g} = 5 \times 10^5 M_\odot$ is required, and for a black hole with $eQ/\mu M \simeq 1$ to survive requires $M \gtrsim 5 \times 10^{17}$ g. The neutralization of charged primordial black holes in the intergalactic medium is discussed.

Section II introduces the notation and equations of motion, Sec. III discusses the radial motion and the question of escape versus plunge, and Sec. IV considers the evolution of the spin and charge of black holes under certain conditions.

II. NOTATION AND EQUATIONS OF MOTION

Units with $c = G = 1$ are used, the mass of the black hole $M = 1$, for particles the rest mass $\mu = 1$ (thus the energy, angular momentum, etc. are on a "per unit mass" basis), and for photons $\mu = 0$.

For a black hole of spin a and charge Q the metric in Boyer-Lindquist coordinates is

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - a dt]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \tag{2.1}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2r + a^2 + Q^2$.

The event horizon is located at $r_+ = 1 + (1 - a^2 - Q^2)^{1/2}$ with $Q^2 + a^2 \leq 1$. The vector potential of the electromagnetic field is

$$\underline{A} = -\frac{Qr}{\rho^2}(\underline{dt} - a\sin^2\theta\, \underline{d\phi}). \tag{2.2}$$

The conserved quantities in the motion of a particle or photon are

μ = rest mass ($\mu = 1$ for particles, $\mu = 0$ for photons),

e = charge per unit mass ($e = 0$ for photons),

E = energy at infinity

$= -(p_t + eA_t)$,

L_z = azimuthal angular momentum

$= (p_\phi + eA_\phi)$,

$\mathfrak{Q} = p_\theta{}^2 + \cos^2\theta[a^2(\mu^2 - E^2) + L_z{}^2/\sin^2\theta]$

or

$\mathcal{K} = \mathfrak{Q} + (L_z - aE)^2 \geq 0$.

Other quantities are

L = total angular momentum

$= (p^2 + L_z{}^2/\sin^2\theta)^{1/2}$,

b_z = azimuthal impact parameter

$= L_z/E$ (photons only),

b_θ = polar impact parameter

$= p_\theta/E$ (photons only),

b = total impact parameter

$= L/E$ (photons only).

The quantities b_z, b_θ, b are defined for photons only; it is convenient because the orbit of a photon does not depend on its energy E but on its impact parameter.

The radial equation of motion is[14,15]

$$\rho^2\frac{dr}{d\tau} = R^{1/2}, \tag{2.3}$$

where

$$P = E(r^2 + a^2) - L_z a + \Lambda r, \tag{2.4}$$

$$R = P^2 - \Delta[\mu^2 r^2 + (L_z - aE)^2 + \mathfrak{Q}]. \tag{2.5}$$

The sign of the root may be taken to be positive or negative depending on the portion of the orbit in question. The quantity $\Lambda = -eQ$ in (2.4) is the ratio of the Newtonian gravitational attraction and the Coulomb electrostatic attraction, and $\Lambda = -1$ represents the case of "balanced forces" when the resultant motion is due to relativistic departures from the Newtonian description.

The Newtonian radial equation of motion (for particles) around a point mass is

$$r^2\frac{dr}{d\tau} = [2\epsilon r^4 + 2r^3(1+\Lambda) - L^2r^2]^{1/2}, \tag{2.6}$$

where

$$\epsilon \simeq \tfrac{1}{2}(E^2 - 1) = \text{kinetic energy at infinity}. \tag{2.7}$$

Electromagnetic and gravitational radiation processes are ignored, further justification being given in Sec. III. The latter process is negligible in considering capture of stars by a black hole in a galactic nucleus.[5]

III. RADIAL MOTION: ESCAPE AND PLUNGE ORBITS

(a) The radial equation of motion depends on θ only in the factor ρ^2 and is decoupled from ϕ and t. Thus the behavior of $R(r)$ determines the type of orbit and the question of escape versus plunge for given E, L_z, and $\mathfrak{Q}$. Since $R(r)$ is a quartic polynomial in r and motion is only possible when $R(r) \geq 0$, analysis of the position of its roots is a powerful method of investigation [see (14)]. Qualitatively $R(r)$ must behave as follows:

(i) $R(r)$ may have no roots in $r \geq r_+$ in which case the orbit is "plunge."

(ii) It may have 2 roots in $r \geq r_+$ when we have an escape orbit.

(iii) There may be a double root $R(r_*) = R'(r_*) = 0$ for $r_* \geq r$ [with $R''(r_*) > 0$] in which case the orbit is an unstable spherical orbit dividing the cases of escape and plunge (an incident particle with "critical" angular momentum will tend to this orbit).

(iv) The above cases give at most 2 roots of $R(r) = 0$ in $r \geq r_+$. If $r_+ = 1$ ($a^2 + Q^2 = 1$), $R(r)$ may have a double root at $r = r_+$ and an additional root in $r > r_+$ (3 roots in $r \geq r_+$). This is associated with the appearance of straight-line segments in the shape of the cross sections (see Fig. 2).

In all cases an unstable spherical orbit divides the plunge and escape orbits (see Ref. 11). A particle with nearly the critical angular momentum will tend to linger near $r = r_*$ in its orbit, winding many times around the black hole (see Ref. 9).

The two techniques that have been developed to analyze escape versus plunge are the following:

(i) a root search in $r \geq r_+$ for $R(r)$,[9,16] and

(ii) solution of $R(r_*) = R'(r_*) = 0$ to determine the

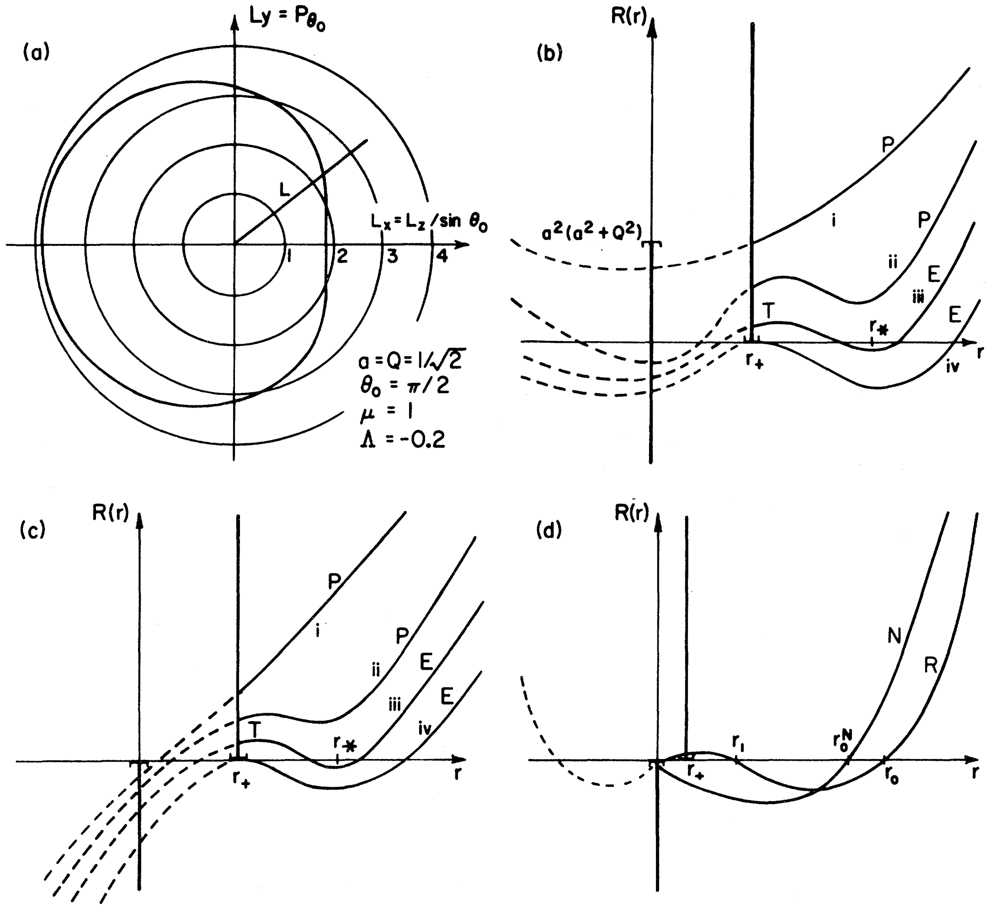

FIG. 1. Behavior of the function $R(r)$ governing the radial motion, with the various types of orbit indicated: P for plunge orbit, E for an escape orbit, and T for a trapped orbit. (a) Coordinate axes in angular momentum space, with a typical plunge region depicted. $L_y = p_{\theta_0}$ is the angular momentum along the spin axis of the black hole, $L_x = L_z/\sin\theta_0$ is the component perpendicular to the spin axis (positive for prograde orbits) and $L = (L_x{}^2 + L_y{}^2)^{1/2}$ is the total angular momentum. For photons the corresponding quantities are the impact parameters $b_y = b_{\theta_0}$, $b_x = b_z/\sin\theta_0$, and $b = (b_x{}^2 + b_y{}^2)^{1/2}$. (b) Behavior of $R(r)$ for photon orbits. Curves (i), (ii), and (iii) show the transition from plunge orbit to escape orbit, the marginal case being when the turning point touches the r axis at $r = r*$, giving an unstable spherical orbit. Curve (iv) is an example of the "exceptional case" which has 3 roots in $r \geq r_+$ (with a double root at $r = r_+$). (c) Behavior of $R(r)$ for particle orbits, $E = 1$, $\Lambda > -1$. Curves are identified as for photons. (d) Behavior of $R(r)$ for slightly hyperbolic particle orbits; $E = 1 + \delta$, $\Lambda < -1$. Both Newtonian (N) and relativistic (R) cases are plotted.

critical angular momentum dividing escape and plunge, other parameters being constant.[12]

Here we generalize previous work to include the case of charged particles in a Kerr-Newman metric as well as to perform computations for the uncharged particle case relevant to massive black holes in globular clusters or galactic nuclei. For completeness black holes with $Q = M$ are considered although the astrophysical applications are not evident at this time.

The orbits are specified by E, L_z, $\mathfrak{Q}$, μ, and Λ, and in addition the "incident" values (i.e., as $t \to -\infty$) of the polar angle θ_0 and polar angular momentum p_{θ_0} will also be used. The results are depicted in angular momentum space as in Fig. 1(a), with $L_x = L_z/\sin\theta_0$ and $L_y = p_{\theta_0}$, and showing a "plunge" region which is always symmetrical about the L_x axis since only $L_y^2 = p_{\theta_0}{}^2$ enters the problem via $\mathfrak{Q}$. The important consideration of the average angular momentum accreted through a cross section of this shape is obtained for a uniform flux by calculating the centroid of the region. These diagrams have been used previously in Refs. 11 and 12.

We shall consider orbits with $E = 1$ ("parabolic").

(b) Photons. The behavior of $R(r)$ is as described in Fig. 1(b), a sample of cross sections is shown in Fig. 2(a), and the numerical values of the cross sections and accreted angular momenta are shown in Figs. 3 and 4, having been computed with

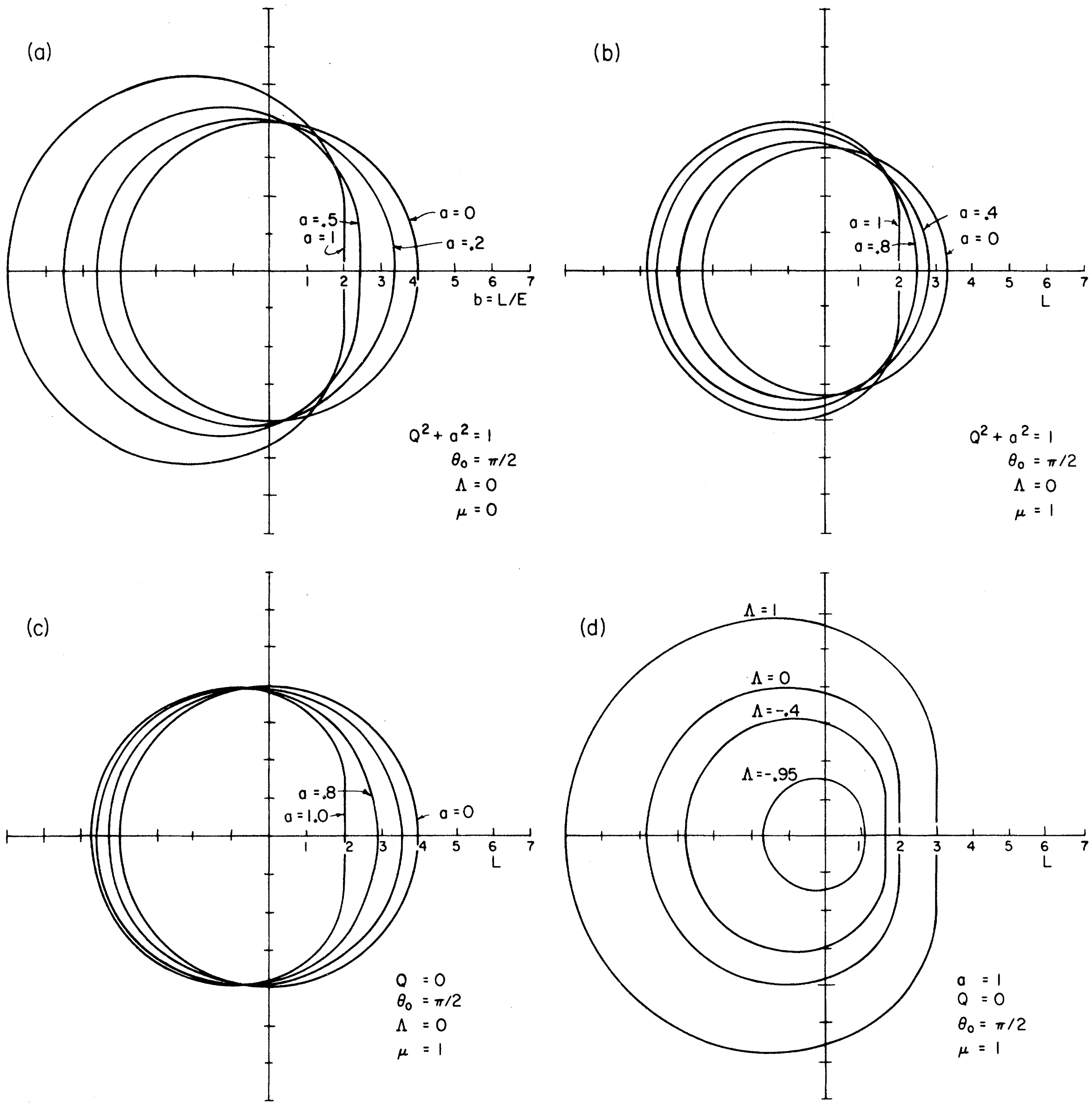

FIG. 2. Region in angular momentum space occupied by plunge orbits for particles in parabolic orbits, or photons, incident upon a black hole from infinity. (a) Photons ($\mu=0$, $\Lambda=0$) equatorially incident ($\theta_0=\frac{1}{2}\pi$) open an extreme Kerr-Newman black hole ($a^2+Q^2=1$). $b=L/E$ is the impact parameter. (b) Uncharged particles ($\mu=1$, $\Lambda=0$) incident equatorially ($\theta_0=\frac{1}{2}\pi$) upon a Kerr-Newman black hole ($a^2+Q^2=1$). L is the angular momentum per unit mass. (c) Uncharged particles equatorially incident upon a Kerr hole ($Q=0$). (d) Charged particles incident equatorially upon a Kerr hole. This represents the limit $Q\to 0$; $|e|\to\infty$ with $\Lambda=-eQ$ fixed.

technique (i). Some special cases are the following:

(i) $Q=0$, $\theta_0=\frac{1}{2}\pi$, $b_\theta=0$ (orbits in the equatorial plane of the Kerr metric[8]). Then

$$r_* = 2\{1+\cos[\tfrac{2}{3}\cos^{-1}(\mp a)]\} \tag{3.1}$$

(upper sign prograde, lower sign retrograde orbits). The critical impact parameter is given by

$$r_*^{3/2} = \pm(b_{z,\mathrm{crit}} - a) \tag{3.2}$$

and the periastron is found from

$$b_z = \{2a + [4a^2 + (r_q - 2)(r_q^3 + a^2 r_q + 2a^2)]^{1/2}\}/(2 - r_q). \tag{3.3}$$

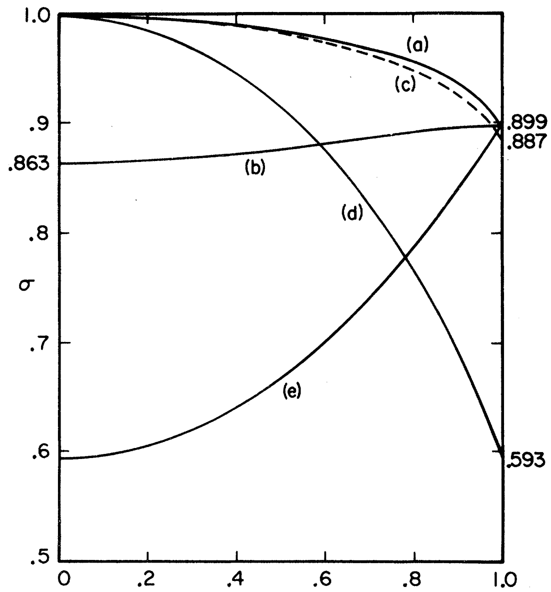

FIG. 3. Cross section σ of a Kerr-Newman black hole to photons incident from infinity, referred to $\sigma = 27\pi$ (Schwarzschild hole). (a) x-axis units are hole spin a, $Q=0$, $\theta_0 = \frac{1}{2}\pi$. (b) is in units of θ_0 ($0 \le \theta_0 \le \frac{1}{2}\pi$), $Q=0$, $a=1$. (c) is in units of a, $Q=0$; isotropic photon flux, uniform from all θ_0. (d) is in units of Q, $a=0$. (e) is in units of a, $a^2+Q^2=1$, $\theta_0=\frac{1}{2}\pi$.

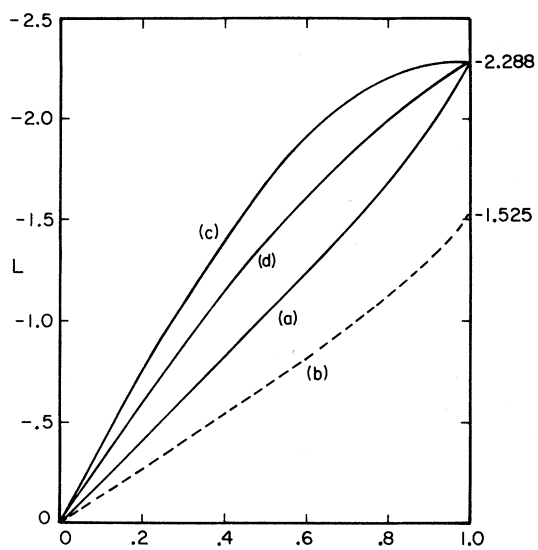

FIG. 4. Average angular momentum accreted by a Kerr-Newman black hole capturing photons incident from infinity. (a) Units of x axis are hole spin a, $Q=0$, $\theta_0=\frac{1}{2}\pi$. (b) is in units of a, $Q=0$; isotropic photon flux. (c) is in units of θ_0 ($0 \le \theta_0 \le \frac{1}{2}\pi$); $Q=0$, $a=1$. (d) is in units of a; $a^2+Q^2=1$, $\theta_0=\frac{1}{2}\pi$.

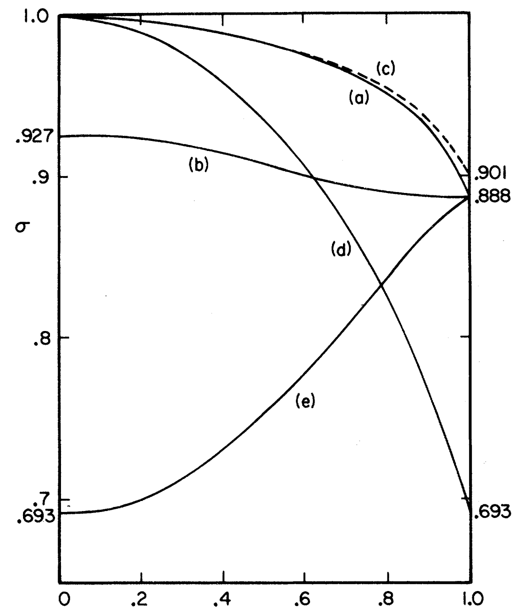

FIG. 5. Cross section σ of a Kerr-Newman black hole to uncharged particles in parabolic orbits incident from infinity referred to $\sigma = 16\pi$ (Schwarzschild hole). (a) Units of x axis are a, $Q=0$, $\theta_0=\frac{1}{2}\pi$. (b) is in units of θ_0 ($0 \le \theta_0 \le \frac{1}{2}\pi$), $Q=0$, $a=1$. (c) is in units of a, $Q=0$; isotropic particle flux. (d) is in units of Q, $a=0$. (e) is in units of a, $Q^2+a^2=1$, $\theta_0=\frac{1}{2}\pi$.

The reference cross section is that of the Schwarzschild hole,

$$\sigma_{\rm Sch} = 27\pi \,. \tag{3.4}$$

(ii) $a=0$, then only b need by specified since the metric is spherically symmetric:

$$r_* = \tfrac{1}{2}[3 + (9 - 8Q^2)^{1/2}]\,, \tag{3.5}$$

$$b_{\rm crit} = [2r^3/(r-1)]^{1/2}\,, \tag{3.6}$$

$$b = r_q^{\,2}/(r_q^{\,2} - 2r_q + Q^2)^{1/2}\,. \tag{3.7}$$

Qualitatively the effects of charge and spin on the hole are the following:

(i) They decrease the cross section, charge having a greater effect reducing σ to 16π when $Q=M$.

(ii) The polar cross section of a spinning hole ($a=1$) is less than the equatorial cross section by a small (3%) amount.

(iii) The average accreted angular momentum for a uniform flux of captured particles is always negative and reaches $\langle L\rangle = -2.288$ for equatorial incidence onto an extreme Kerr black hole.

(iv) Spin $a=1$ ($Q=0$) will allow a particle to graze the event horizon at $r_+=1$ and subsequently escape; but $Q=1$ ($a=0$) will only permit $r_q \ge 2$ with escape.

(c) Particles. For $\Lambda > -1$ (i.e., net attraction)

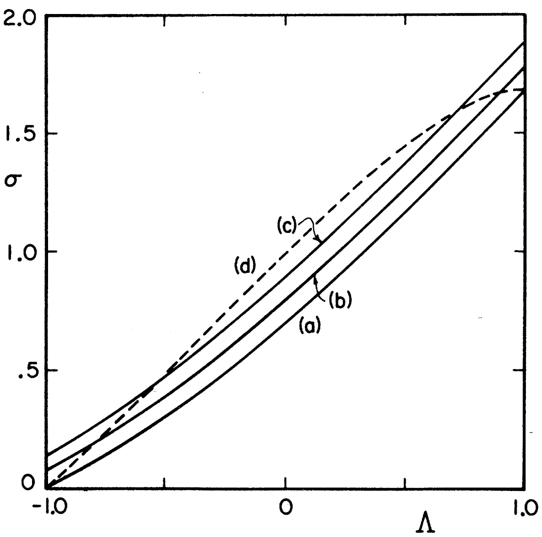

FIG. 6. Cross sections σ for particles of charge $\Lambda > -1$ referred to the Schwarzschild case, $\sigma = 16\pi$. (a) $Q = 1$, $a = 0$. (b) $Q = a = 2^{-1/2}$, $\theta_0 = \frac{1}{2}\pi$. (c) $Q = 0$, $a = 1$, the limit $Q \to 0$, $|e| \to \infty$, with $\Lambda = -eQ$ fixed. (d) $a = 0$, charge $Q = \Lambda$. Thus the dashed curve is a particle of constant charge in the metric of varying Q.

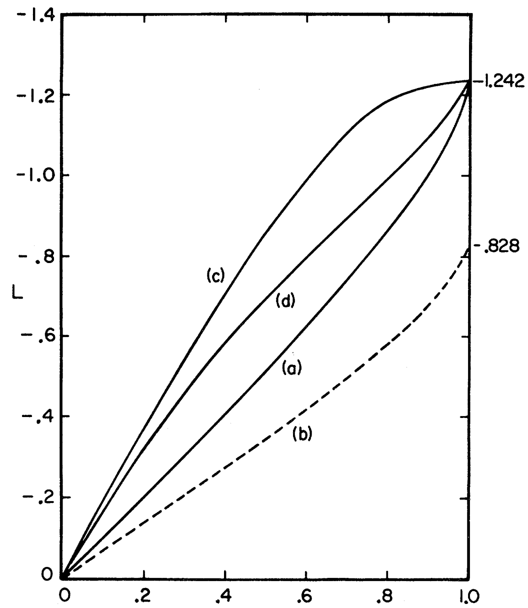

FIG. 7. Average angular momentum L accreted from capture of uncharged particles in parabolic orbits. (a) Units of x axis are a, $Q = 0$, $\theta_0 = \frac{1}{2}\pi$. (b) is in units of a, $Q = 0$; isotropic particle flux. (c) is in units of θ_0 $(0 \le \theta_0 \le \frac{1}{2}\pi)$, $Q = 0$, $a = 1$. (d) is in units of a, $Q^2 + a^2 = 1$, $\theta_0 = \frac{1}{2}\pi$.

Fig. 1(c) shows typical radial functions $R(r)$ and plunge regions are shown in Fig. 2(b)–2(d). Figures 5, 6, and 7 graph the numerical results. Again some special cases:

(i) $Q = 0$, $\theta_0 = \frac{1}{2}\pi$, $\mathfrak{Q} = 0$ (see Ref. 7):

$$r_* = 2 \mp a + 2(2 \mp a)^{1/2}, \tag{3.8a}$$

$$L_{z,\,\mathrm{crit}} = 2r_*^{\,1/2} = r_* + a. \tag{3.8b}$$

The reference (Schwarzschild) cross section is now $\sigma_{\mathrm{Sch}} = 16$ and the periastron is

$$r_q = \tfrac{1}{4} L_z^2 \{1 + [1 - 16(L_z - a)^2/L_z^4]^{1/2}\}. \tag{3.8c}$$

(ii) $a = 0$, $\Lambda = 0$:

$$Q = r_*^{\,1/2}(2 - r_*^{\,1/2})^{1/2}, \tag{3.9}$$

$$L_{\mathrm{crit}} = (r_*^{\,1/2} - 1)^{-1/2}, \tag{3.10}$$

$$L^2 = r_q^{\,2}(2r_q - Q^2)/(r_q^{\,2} - 2r_q + Q^2) \tag{3.11}$$

(iii) $Q = 0$, $a = 0$, $\Lambda > -1$ (the limit $Q \to 0$, $|e| \to \infty$, $\Lambda = -eQ$ fixed):

$$r_* = 2 + [4 + \Lambda^2/(1 + \Lambda)]^{1/2}, \tag{3.12}$$

$$L_{\mathrm{crit}} = r_*(1 + \Lambda)^{1/2}, \tag{3.13}$$

and as $\Lambda \to \infty$, $r_* \sim \Lambda^{1/2}$, $L_{\mathrm{crit}} \sim \Lambda$,

$$L^2 = r_q^{\,2}[2r_q(1 + \Lambda) - (Q^2 - \Lambda^2)]/(r_q^{\,2} - 2r_q + Q^2). \tag{3.14}$$

The qualitative effects are similar to those for photons:

(i) The minimum cross section is again found for $Q = 1$, $a = 0$ (when $\Lambda = 0$), and is $\sigma = \pi\,(3.330)^2$.

(ii) The polar cross section of a spinning hole ($a = 1$) is now some 4% larger than the equatorial cross section (when $\Lambda = 0$).

(iii) When $\Lambda = 0$, $\langle L \rangle = -1.242$ is maximum for equatorial incidence onto an extreme Kerr hole ($a = 1$) but may be increased if charged particles are accreted and $Q \ne 0$.

(iv) The relativistic effects always decrease r_q below the Newtonian value $r_q = L^2/2(1 + \Lambda)$; the greatest effect being for prograde orbits in the equatorial plane of a Kerr hole. Since tidal forces vary as r^3/ρ^6 this increases greatly the cross section of a spinning black hole to tidal disruption (for detailed calculations involving stars in parabolic orbits see Ref. 5).

For charged particles with $\Lambda < -1$ [see Fig. 1(d) for the behavior of $R(r)$] we note that $E = 1 + \delta > 1$ for motion to be possible. Then we see a central "pit in the potential," absent in the Newtonian case, that can trap a particle near $r = r_+$ and haul it into the hole despite the savage repulsion. It would, however, take a contrived situation to set this up since the energy of a stationary particle near a black hole is

$$E = -(p_t + eA_t) = \Lambda r/\rho^2\,. \tag{3.15}$$

When $\Lambda \gg 1$, noting that $r_* \sim \Lambda^{1/2}$, $L_z \sim \Lambda$ (as $\Lambda \to \infty$) we find

$$R(r) \sim 2r^3\Lambda + r^2(\Lambda^2 - L^2) + 2r(L^2 - a\Lambda L_z), \quad (3.16)$$

$$r_q \sim \left(\frac{L^2-\Lambda^2}{4\Lambda}\right)\left\{1 + \left[1 - \frac{16\Lambda(L^2 - a\Lambda L_z)}{(L^2-\Lambda^2)}\right]^{1/2}\right\}, \quad (3.17)$$

$$L_{\text{crit}} \sim \Lambda + 2\Lambda^{1/2}(1 - aL_z/L)^{1/2}, \quad (3.18)$$

$$r_* \sim \Lambda^{1/2}(1 - aL_z/L)^{1/2}. \quad (3.19)$$

Despite the strong attraction, we find as $L_z/L \to 1$ and $a \to 1$ the particle may swoop down to graze the horizon at $r_+ = 1$ and then escape, when incident upon a "straight-line segment" in the angular momentum space located at

$$L_z = \Lambda + 2, \quad \mathfrak{L} \le 2(1+\Lambda). \quad (3.20)$$

(d) Particles. With $\Lambda = -1$ in a "balanced force" situation we find a curious set of neutrally stable circular orbits when $R(r) \equiv 0$. This requires that all the coefficients of the quartic $R(r)$ should vanish, which happens if and only if

$$L_z = L = a, \quad \mathfrak{L} = 0, \quad \theta = \tfrac{1}{2}\pi, \quad \Lambda = -1, \quad a^2 + Q^2 = 1. \quad (3.21)$$

Then the particle executes circular orbits in the equatorial plane of an extreme Kerr-Newman hole with the same angular momentum per unit mass as the black hole. If $a = 0$ ($Q = 1$), the particle may be stationary in neutral equilibrium (as in the Newtonian case), but for $a \neq 0$ the frame-dragging of the hole demands that the particle should orbit. Small perturbations cause a particle to coast away from equilibrium, eventually succumbing to an instability. As regards capture of "balanced" particles we note that

$$L_{\text{crit}} = (1 - Q^2)^{1/2} \quad (3.22)$$

is independent of a and θ_0. The dividing case can now be a neutrally stable orbit instead of an unstable circular orbit.

(e) Neglect of radiation damping. The effects of radiation damping are to radiate off energy and angular momentum according to

$$dE/dt = -\tfrac{2}{3}e^2\ddot{\vec{v}}^2, \quad (3.23a)$$

$$d\vec{L}/dt = -\tfrac{2}{3}e^2\vec{v}\times\ddot{\vec{v}}. \quad (3.23b)$$

In performing a simple Newtonian calculation, let us consider the effects of radiation in a circular orbit at distance αM from the black hole, to simulate one "wind" of a near-plunge orbit around the hole. The energy and angular momentum radiated in one revolution are

$$E_{\text{rad}} = \frac{2e^2}{3}\frac{2\pi(1+\Lambda)^{3/2}}{M\alpha^{5/2}}, \quad L_{\text{rad}} = \frac{2e^2}{3}\frac{2\pi(1+\Lambda)^{1/2}}{\alpha^{1/2}}v, \quad (3.24)$$

and we require that

$$E_{\text{rad}} \ll \mu E, \quad L_{\text{rad}} \ll \mu L = \mu(\alpha M)v. \quad (3.25)$$

Thus we obtain

$$\frac{Q}{e} \gg \Lambda(\Lambda+1)^{1/2}\frac{8\pi}{3\alpha^{3/2}}, \quad \frac{Q}{e} \gg \Lambda(\Lambda+1)^{1/2}\frac{4\pi}{3\alpha^{3/2}}, \quad (3.26)$$

and upon setting $\alpha = 4$ for a Schwarzschild black hole, we require that

$$\frac{Q}{e} \gg \Lambda(\Lambda+1)^{1/2}. \quad (3.27)$$

For accretion of elementary particles by a massive black hole with $Q \ll M$ it is clear that (3.27) is easily satisfied because the length scales involved are too large to permit savage enough accelerations to cause radiation damping. It is for small black holes near the Hawking limit (10^{15} g) with $Q = e$ that we should concern ourselves with the possible breakdown of (3.27). Then for protons $|\Lambda_p| = 2\times10^{-3}$, which is quite safe, but for electrons $|\Lambda_e| = 3.7$, which may cause trouble.

We shall estimate the change in the critical angular momentum by taking it to be such that $L_{\text{rad}} \simeq \mu L$, then we require that

$$\left(\frac{\alpha}{4}\right)^{3/2} \simeq \frac{e}{Q}\Lambda(\Lambda+1)^{1/2}$$

and if L_{crit} becomes βL_{crit}, $\frac{1}{4}\alpha \simeq \beta^2[1+(1-\beta^2)^{1/2}]$ (in the Schwarzschild case). As an example of the small black hole with $|\Lambda_e| = 3.7$, $\frac{1}{4}\alpha \simeq 4$, $\beta \simeq 1.5$ which changes the capture cross section and thus (4.15) only slightly.

We shall not consider here more complicated processes whereby radiation of energy results in the particle being trapped in orbit around the black hole.

IV. EVOLUTION OF A BLACK HOLE

(a) If a black hole is bathed in mass-energy flux F, then the evolution in time dt is described by

$$\begin{aligned} dm &= F\,dt\,(\sigma M^2),\\ dM &= (1-E)\,dm,\\ d(aM^2) &= (LM)\,dm,\\ d(QM) &= e\,dm = -(\Lambda/Q)\,dm, \end{aligned} \quad (4.1)$$

where we have reintroduced the black-hole mass M explicitly; a, Q are as before; σM^2 is the cross section of the black hole; LM is the average accreted angular momentum; E is the energy radiated during the plunge; $\Lambda = -eQ$ is the charge parameter of the captured particles; and dm is the captured mass-energy. Then

$$\begin{aligned} dM/dt &= (1-E)F(\sigma M^2), \\ da/d(\ln M) &= -2a + L/(1-E), \\ dQ/d(\ln M) &= -Q + e/(1-E) \\ &= -Q - (\Lambda/Q)/(1-E). \end{aligned} \tag{4.2}$$

If a charged black hole accretes highly charged particles ($|e| >> 1$, as for protons and electrons), then the evolution in mass is negligible, and is approximately

$$\begin{aligned} M\,dQ/dt &= (eF)(\sigma M^2) \\ &= -(\Lambda F/Q)(\sigma M^2). \end{aligned} \tag{4.3}$$

The orbits are not appreciably affected by quadrupole gravitational radiation if $M/\mu >> 1$, and similarly are unperturbed by dipole electromagnetic radiation if $Q/e >> \Lambda(1+\Lambda)^{1/2}$, in which cases $E \simeq 0$.

(b) The spin-down of a Kerr hole due to accretion of an isotropic flux of uncharged particles or photons may be calculated by integration of the second equation of (4.2) with $E \simeq 0$ and L taken from Figs. 7 and 4, respectively. The resulting evolution with mass proceeds as

$$\begin{aligned} &a \propto M^{-10/3} \quad (L \simeq -4a/3) \quad \text{(photons)}, \\ &a \propto M^{-8/3} \quad (L \simeq -2a/3) \quad \text{(particles)} \end{aligned} \tag{4.4}$$

and the hole may be effectively spun-down from $a=1$ as the mass increases by a factor of 2. This process may include the following cases:

(i) *Black holes in galactic nuclei*, where for $M \gtrsim 10^7 M_\odot$ the consumption of stars in plunge orbits is appreciable. This has been discussed with the added effect of an accretion disk[6] opposing the spin-down.

(ii) *Primordial black holes*. The growth of such holes by accretion after their formation may be discussed as in Ref. 17. If a hole of mass $M_1 = \eta t_1$ is formed at time t_1 (where $\eta < 1$ such that the hole if formed within the particle horizon), then at time t

$$M \simeq t\Big/\left[1 + \frac{t}{t_1}\left(\frac{t_1}{M} - 1\right)\right] \to M_1\big/(1-\eta) \quad (t \to \infty). \tag{4.5}$$

If the hole manages to grow by a factor of 2 (i.e., $\eta > \frac{1}{2}$), most of the growth occurring at early times, and the captured flux is isotropic, then it will be spun down.

For $t_1 \gtrsim 1$ sec ($M_1 \gtrsim 2\times10^5 M_\odot$) the universe is radiation-dominated and photon flux causes the spin damping; for $t_1 \lesssim 1$ sec pair production is important and it is a photon-particle fluid that is accreted.

(c) The neutralization of a charged black hole in the intergalactic medium. Taking the medium as a fully ionized plasma with $n_e = n_p = 10^{-6}\ \text{cm}^{-3}$ and $T = 10^6\ ^\circ\text{K}$ for the present epoch (n, T higher in the past), then the mean free paths and rms velocities of protons and electrons are

$$\begin{aligned} &\lambda_p = 2\times10^{21}\ \text{cm}, \quad w_p = 2\times10^6\ \text{cm sec}^{-1}, \\ &\lambda_e = 2\times10^{21}\ \text{cm}, \quad w_e = 7\times10^7\ \text{cm sec}^{-1}. \end{aligned} \tag{4.6}$$

The radius of influence of the black hole, r_h, may be estimated by

$$M/r_h + eQ/r_h \simeq \mu w^2 \quad \text{or} \quad r_h/M \simeq (\Lambda+1)/w^2. \tag{4.7}$$

Thus even for $M \simeq 10^5 M_\odot$ ($\simeq 1.5\times10^{10}$ cm) we may have $|\Lambda_p| \simeq 10^3$ or $|\Lambda_e| \simeq 10^6$ for $r_h << \lambda_p, \lambda_e$. Under these conditions we may treat the accretion of electrons or protons as a noninteracting particle process. The time required to pull a particle into the hole, the orbit travel time, will be neglected. The mean free paths of the particles may be significantly reduced when an intergalactic magnetic field exists and is stronger than the field of the black hole at $r = r_h$.

(i) $\Lambda >> 1$. Significant accretion occurs only for particles of opposite charge to that of the hole. For particles with velocity at infinity v_∞, the cross section for plunge orbits is,[18]

$$\Sigma = \sigma v_\infty^{-2}, \quad \sigma_{\text{Sch}} = 16M^2. \tag{4.8}$$

From (3.18),

$$\Sigma = \pi\Lambda^2 M^2 v_\infty^{-2}. \tag{4.9}$$

Averaging over a Maxwell distribution of rms (space) velocity w, we find

$$\langle \Sigma v_\infty \rangle = \pi\Lambda^2 M^2 \left(\frac{6}{\pi}\right)^{1/2} w^{-1}, \tag{4.10}$$

and the accreted flux is

$$F = \Lambda^2 M^2 (6\pi)^{1/2} q w^{-1}, \tag{4.11}$$

where q is the charge density of the medium from particles of the type to be accreted. From (4.3),

$$\begin{aligned} M\frac{dQ}{dt} &= -F \\ &= -\Lambda^2 M^2 (6\pi)^{1/2} q w^{-1} \\ &= -e^2 Q^2 M^2 (6\pi)^{1/2} q w^{-1}. \end{aligned} \tag{4.12}$$

This integrates to give a "half-life" for the charge of

$$t_{1/2}^{-1} = 2(6\pi)^{1/2} e^2 q w^{-1} (MQ), \tag{4.13}$$

which depends only on the total charge (MQ) on the black hole. For proton and electron neutralization, with (QM) in coulombs,

$$\begin{aligned} t_{1/2}(p) &= (1.8\times10^3)/(QM)\ \text{sec}, \\ t_{1/2}(e) &= (1.8\times10^{-2})/(QM)\ \text{sec}. \end{aligned} \tag{4.14}$$

The difference is due to the greater charge per unit mass, e, of an electron, slightly countered by its larger rms velocity.

It is clear that significant charges will decay rapidly and that charges of only a few quanta may survive.

(ii) For a small black hole, $M \simeq 10^{15}$ g, $QM = e = 1.6 \times 10^{-19}$ coulombs, which avoids both the Hawking radiation and the vacuum polarization limits, $|\Lambda_e| = 3.7$, $|\Lambda_p| = 2\times 10^{-3}$ so that (4.14) are not applicable. Ignoring effects of quantum gravity and electromagnetic radiation, use of the cross sections for the above Λ values gives

$$t(p)/t(e) = 0.17 \tag{4.15}$$

for a black hole accreting the particle of opposite sign.

When in a neutral state, the times required to accrete a single proton or electron are

$$t^*(p)/t^*(e) = 0.029, \quad t^*(p) = t(p) \tag{4.16}$$

With the above values for the density of the intergalactic medium,

$$t(p) = 5.7 \times 10^{15} \text{ sec.}$$

This time scale is likely to have been much shorter in the past when the intergalactic medium was much denser, and in a flat universe the total flux accreted by a fixed cross section in the lifetime of that universe is enhanced by a factor of 3. Even allowing for errors in the adopted values of the density and temperature of the intergalactic medium, it is likely that $t(p) \ll H_0^{-1} = 5\times 10^{17}$ sec. Then the above time scales suggest that the statistical fluctuations of the accreted protons and electrons by small black holes would result in the majority being positively charged by a single excess proton.

ACKNOWLEDGMENTS

I thank Paul Chrzanowski and William Jeffreys for discussion and critical reading of this manuscript.

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