Density perturbations in cosmological models

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An explicitly coordinate-independent method, based on previous work by Hawking, is used to study linearized density perturbations of $k = 0$ Friedmann cosmological models.

I. INTRODUCTION

Linear perturbations of spatially homogeneous and isotropic universes have been investigated since the pioneering work of Lifshitz,[1] who studied the growth law of density perturbations in relation to the problem of galaxy formation. More recent applications include the anisotropy of the 3 °K background radiation (Sachs and Wolfe,[2] Silk,[3] Peebles and Yu,[4] Sunyaev and Zeldovich[5]), inhomogeneity in primordial element production (Silk and Shapiro,[6] Ipavich,[7] Gisler, Harrison, and Rees[8]), and the possibility of primordial black-hole formation (Carr and Hawking,[9] Carr,[10] Mészaros[11]).

Density perturbations in cosmological models have been studied by Bonnor,[12] Peebles,[13] and Savedoff and Vila[14] in Newtonian theory, and by Irvine[15] and Peebles[16] in the Newtonian limit of general relativity. Relativistic treatments include those of Arons and Silk,[17] Field and Shepley,[18] Hawking,[19] Lifshitz,[1] Lifshitz and Khalatnikov,[20] Nariai,[21] Nariai *et al.*,[22] Peebles and Yu,[4] Sachs and Wolfe,[2] Sakai,[23] Silk,[24] and Weinberg,[25] with review articles by Harrison,[26] Rees and Sciama,[27] and Rees.[28]

However, a difficulty arises in the interpretation of density perturbation results. While the density ρ is, of course, a scalar invariant under coordinate transformations, the density perturbation $\delta\rho$ is a gauge-dependent quantity, i.e., $\delta\rho$ is *not* invariant under infinitesimal coordinate transformations. Since previous authors have used a variety of coordinate conditions, they have obtained differing results for $\delta\rho$. For example, different gauge choices can give different exponents in the power-law growth (or decay) typical for long-wavelength density perturbations; for details, see Sakai.[23]

In fact, for arbitrary perturbations of an expanding Friedmann universe, there always exists an appropriate choice of the time coordinate such that one can set $\delta\rho = 0$ identically in that coordinate system. Conversely, in an unperturbed Friedmann background model, apparent "density inhomogeneities" can be generated by the choice of a time coordinate other than comoving proper time, since the $t =$ constant hypersurfaces for an arbitrary time coordinate t need not coincide with the homogeneous $\rho =$ constant hypersurfaces. In general, with only local information on the density ρ at coordinate time t, an observer cannot distinguish *a priori* between physically significant density fluctuations and nonphysical coordinate effects.

The primary purpose of this paper is to show that the evolution of density perturbations of $k = 0$ Friedmann universes can be calculated by an explicitly coordinate-independent technique, which follows a prescription originally devised by Hawking[19] and does not require the introduction of metric tensor perturbations. This method, based on the continuity equation and the Raychaudhuri equation (Sec. II), gives the evolution of the perturbations as seen by observers comoving with the fluid and as measured with respect to τ, the proper time along the fluid world lines. The density perturbation is defined relative to comoving proper time, which has physical significance, rather than via an arbitrary time coordinate. Therefore, apart from one remaining degree of freedom to be discussed in Sec. IV, the gauge-dependent arbitrariness of $\delta\rho$ is eliminated.

Although the basic ideas for the coordinate-independent methods are taken from the work of Hawking,[19] the present results differ from those of Hawking, as an error in one of his linearized evolution equations is corrected here; see Appendix B. A secondary purpose of this paper is to reconcile coordinate-independent techniques and results with those of other previous authors. In particular, Field and Shepley,[18] Ellis,[29] and Bardeen[30] have independently noted certain conflicts with Hawking's coordinate-independent analysis; for the present results (Sec. IV) derived from the corrected equations, these disagreements no longer exist.

Although Hawking's analysis is also applicable for the $k = \pm 1$ background models, the present work is restricted for simplicity to perturbations of $k = 0$ Friedmann universes.

II. BASIC EQUATIONS

As given by Hawking[19] and Ellis,[29] the continuity equation and the Raychaudhuri equation for a perfect fluid are

$$\dot{\rho} + (\rho + p)\theta = 0 \tag{1}$$

and

$$\dot{\theta}=2(\omega^2-\sigma^2)-\tfrac{1}{3}\theta^2-\tfrac{1}{2}(\rho+3p)+\dot{u}^a{}_{;a}\,, \tag{2}$$

where ρ is the density, p is the pressure, the expansion rate $\theta=u^a{}_{;a}$ is the divergence of the four-velocity u^a, and ω and σ are the rotation and shear scalars. Covariant differentiation along a fluid world line is denoted by an overdot, i.e., $\dot{\rho}\equiv D\rho/D\tau\equiv\rho_{;a}u^a$. The four-acceleration caused by pressure gradients is

$$\dot{u}^a=u^a{}_{;b}u^b=-\frac{h^{ab}p_{;b}}{\rho+p}\,, \tag{3}$$

where $h^{ab}=g^{ab}+u^au^b$ is the projection tensor orthogonal to the four-velocity u^a (assuming normalization $u^au_a=-1$). Units are chosen such that $8\pi G=c=1$.

Assuming an equation of state of the form $p=p(\rho)$, the speed of sound is given by $(v_s)^2=dp/d\rho$.

In the unperturbed $k=0$ Friedmann model with expansion factor $R(\tau)$, $\rho=3(\dot{R}/R)^2$ and $\theta=3(\dot{R}/R)$, while $\omega=\sigma=\dot{u}^a=0$.

III. LINEARIZED EQUATIONS

Define the density contrast $\delta\rho/\rho$ by

$$\rho\equiv\rho_0(\tau)\left(1+\frac{\delta\rho}{\rho}\right)\,, \tag{4}$$

where $\rho_0(\tau)$ is that function of comoving proper time τ which corresponds to the density observed in the $k=0$ background universe at the proper time τ after the big bang. A related small quantity S can be defined by

$$\rho\equiv\tfrac{1}{3}\theta^2(1+S)\,. \tag{5}$$

Roughly speaking, S can be thought of in linearized theory as a "spatial curvature perturbation." This identification becomes precise for irrotational models ($\omega=0$, Ellis[29]), where the Ricci scalar curvature of the hypersurfaces orthogonal to the four-velocity u^a is ${}^{(3)}R=2(\rho-\frac{1}{3}\theta^2+\sigma^2)$, implying to linear order that $S={}^{(3)}R/2\rho$.

Hereafter, the dynamical equations will be linearized in ω, σ, S, $\delta\rho/\rho$, and their derivatives. Combining Eqs. (1), (2), and (5) gives the linearized result

$$\theta\dot{S}=(\rho+3p)S-2\dot{u}^a{}_{;a}\,. \tag{6}$$

If the spatial gradient of the density is defined as $X^a\equiv h^{ab}\rho_{;b}$, then the equation of state and Eq. (3), when linearized, imply

$$\dot{u}^a{}_{;a}=-\frac{(v_s)^2}{\rho+p}\operatorname{div}X\,, \tag{7}$$

where $\operatorname{div}X\equiv X^a{}_{;a}$. Equation (6) can therefore be reexpressed as

$$\theta\dot{S}=(\rho+3p)S+2\frac{(v_s)^2}{\rho+p}\operatorname{div}X\,. \tag{8}$$

The evolution equation for div X is obtained by taking the first and second spatial gradients of Eq. (1); the linearized result, derived in Appendix A, is

$$\theta(\operatorname{div}X)^{\boldsymbol{\cdot}}=[-5\rho-\tfrac{3}{2}(\rho+p)](\operatorname{div}X)-\tfrac{3}{2}\rho(\rho+p)\frac{n^2}{R^2}S\,, \tag{9}$$

where $n^2=$constant, and a plane-wave expansion of S has been made.

Equations (8) and (9) are sufficient to determine the evolution of S, since div$X$ can be eliminated. The quantity $\delta\rho/\rho$ can then be related to S by the continuity equation (1) in a manner that depends on the equation of state. For example, if $p=\alpha\rho$, as in the three examples given below, then

$$S=\left(\tau\frac{\delta\rho}{\rho}\right)^{\boldsymbol{\cdot}}\,, \tag{10}$$

where the relations $R(\tau)\sim\tau^{2/(3+3\alpha)}$ and $\rho_0(\tau)=4\tau^{-2}(1+\alpha)^{-2}/3$ have been used for the $k=0$ Friedmann background. S must be integrated once to obtain

$$\frac{\delta\rho}{\rho}=\frac{1}{\tau}\left(C+\int S\,d\tau\right)\,, \tag{11}$$

where C satisfies $\dot{C}=0$, and where the integral is taken along a fluid world line.

IV. EXAMPLES

A. $p=0$

In the background model, $R\propto\tau^{2/3}$ and $\rho_0(\tau)=4\tau^{-2}/3$. Equation (8) becomes

$$\tau\dot{S}-\tfrac{2}{3}S=0\,,$$

with solutions

$$S\propto\tau^{2/3}\ \text{and}\ \frac{\delta\rho}{\rho}\propto\{\tau^{2/3},\tau^{-1}\}\,,$$

where the curly brackets are taken here to mean that the general solution is a linear combination of the modes within the brackets.

B. $p=\rho/3$

In the background model, $R\propto\tau^{1/2}$ and $\rho_0(\tau)=3\tau^{-2}/4$. Equations (8) and (9) combine to give

$$\tau^2\ddot{S}-\tfrac{1}{2}\tau\dot{S}+(\tfrac{1}{2}+\tfrac{1}{4}\kappa^2\tau)S=0\,,$$

where $\kappa=$constant. The solutions are

$$S\propto\begin{Bmatrix}x\sin x\\ x\cos x\end{Bmatrix}$$

and

$$\frac{\delta\rho}{\rho} \propto \left\{ \begin{matrix} \tau^{-1} \\ \left[\left(1-\frac{2}{x^2}\right)\cos x - \frac{2}{x}\sin x + \frac{2}{x^2}\right] \\ \left[\left(1-\frac{2}{x^2}\right)\sin x + \frac{2}{x}\cos x\right] \end{matrix} \right\},$$

where $x \equiv \kappa\sqrt{\tau}$. In the low-frequency limit ($x \ll 1$) the solution becomes

$$\frac{\delta\rho}{\rho} \to \{\tau^{-1}, \tau, \tau^{1/2}\}.$$

The high-frequency limit ($x \gg 1$) gives

$$\frac{\delta\rho}{\rho} \to \left\{ \begin{matrix} \tau^{-1} \\ \cos\kappa\sqrt{\tau} \\ \sin\kappa\sqrt{\tau} \end{matrix} \right\},$$

and in this limit the oscillatory modes represent acoustic waves with constant amplitude and with acoustic frequency red-shifting as R^{-1}.

C. $p = \rho$

In the background model, $R \propto \tau^{1/3}$ and $\rho_0(\tau) = \tau^{-2}/3$. Equations (8) and (9) combine to give

$$\tau^2\ddot{S} - \tfrac{5}{3}\tau\dot{S} + (\tfrac{16}{9} + \tfrac{4}{9}\beta^2\tau^{4/3})S = 0,$$

where β = constant. The solutions are Bessel functions

$$S \propto \left\{ \begin{matrix} y^2 J_0(y) \\ y^2 N_0(y) \end{matrix} \right\},$$

where $y \equiv \beta\tau^{2/3}$. The expression obtained for $\delta\rho/\rho$ by integrating has simple limiting forms. The low-frequency limit ($y \ll 1$) gives

$$\frac{\delta\rho}{\rho} \to \{\tau^{-1}, \tau^{4/3}, \tau^{4/3}\ln\tau\}.$$

The high-frequency limit ($y \gg 1$) gives

$$\frac{\delta\rho}{\rho} \to \left\{ \begin{matrix} \tau^{-1} \\ \tau^{1/3}\sin(\beta\tau^{2/3}) \\ \tau^{1/3}\cos(\beta\tau^{2/3}) \end{matrix} \right\},$$

i.e., the acoustic wave modes oscillate with amplitude growing as R, and with acoustic frequency red-shifting as R^{-1}.

D. Discussion

Present in each of these three examples is a nonoscillatory decaying mode for which $(\delta\rho/\rho) \sim \tau^{-1}$. For this mode, and only this mode, the quantity S vanishes identically, as can be seen from Eq. (10). Furthermore, the interpretation of the decaying mode is not straightforward because of the one remaining degree of freedom left in the definition of the density perturbation in terms of comoving proper time.

If τ represents comoving proper time, then so does τ^* defined by $\tau \to \tau^* = \tau + {}_0\tau$, where ${}_0\tau$ is a small quantity satisfying ${}_0\tau_{;a}u^a = 0$. The freedom inherent in ${}_0\tau$ corresponds to each comoving observer choosing the event along his world line at which he initializes his local clock, with later evolution determined by $\tau_{;a}u^a = \tau^*{}_{;a}u^a = 1$. For the equation of state $p = \alpha\rho$, definitions (4) and (5) imply to linear order that

$$\frac{\delta\rho}{\rho} \to \frac{\delta\rho^*}{\rho} = \frac{\delta\rho}{\rho} + 2\frac{{}_0\tau}{\tau},$$

while S remains unchanged. Any τ^{-1} mode, but no other mode, can be eliminated from the linear density perturbation by appropriate choice of ${}_0\tau$. Although the physical significance of the decaying mode is therefore not clear from the work given so far, this problem in interpretation can be resolved by considering other quantities in addition to $\delta\rho/\rho$ and S. Either by using coordinate-invariant techniques or by adapting the work of the previous authors cited in Sec. I, it can be shown that in the $p = 0$ case (but not in the $p = \rho/3$ and $p = \rho$ cases) the nonoscillatory decaying mode, for which $(\delta\rho/\rho) \sim \tau^{-1}$, also appears in such gauge-invariant quantities as the spatial gradient of the density (X_a), its divergence (div$X$), and the fluid shear tensor (σ_{ab}), all of which are unaffected by the above transformation involving ${}_0\tau$. In this sense the decaying mode is "real" for $p = 0$ (i.e., removable from $\delta\rho/\rho$, but not from other quantities, by resetting the origin of proper time), but it is "fictitious" (i.e., entirely removable) for the cases with $p = \rho/3$ and $p = \rho$. This difference occurs because the coupling between the various perturbed quantities can depend on the equation of state, as can be seen, for example, for S and div$X$ in Eq. (8).

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APPENDIX A

The purpose of this appendix is to derive the linearized evolution equation for the quantity div$X$ $= X^a{}_{;a}$, where $X^a \equiv h^{ab}\rho_{;b}$ is the spatial gradient of the density. If $Y^a \equiv h^{ab}\theta_{;b}$ is defined as the spatial gradient of the expansion rate, then taking the spatial gradient of Eq. (1) gives

$$\dot{X}^a = -\tfrac{4}{3}\,\theta X^a - (\rho + p) Y^a, \tag{A1}$$

where this equation, like the other numbered equations in this appendix, has been linearized. Taking the divergence of Eq. (A1) yields

$$(\mathrm{div}X)^{\dot{}} = -\tfrac{5}{3}\,\theta(\mathrm{div}X) - (\rho + p)(\mathrm{div}Y), \tag{A2}$$

where the only nontrivial step, involving an interchange in the order of differentiation, is

$$\begin{aligned} X^a{}_{;b;a} u^b &= X^a{}_{;a;b} u^b + X_e R^{ae}{}_{ab} u^b \\ &= (\mathrm{div}X)^{\dot{}}, \end{aligned}$$

and where the curvature tensor term vanishes via

$$X_e R^{ae}{}_{ab} u^b = X_e R^e{}_b u^b = X_e T^e{}_b u^b = 0,$$

using the Einstein field equations, $R_{ab} - \frac{1}{2} g_{ab} R = T_{ab}$, and the stress tensor for a perfect fluid, $T_{ab} = p g_{ab} + (\rho + p) u_a u_b$.

Taking the divergence of the spatial gradient of definition (5) gives

$$\mathrm{div}X = \tfrac{2}{3}\theta\, \mathrm{div}Y - \tfrac{1}{3}\theta^2 \left(\frac{n^2 S}{R^2}\right), \tag{A3}$$

assuming an expansion in plane waves, as in Hawking,[19] to set

$$h^a{}_c (h^{dc} S_{;d})_{;a} = -\frac{n^2 S}{R^2},$$

where $n^2 = $ constant.

Finally, combining Eqs. (A2) and (A3) gives the result which is Eq. (9) in the text:

$$\theta(\mathrm{div}X)^{\dot{}} = [-5\rho - \tfrac{3}{2}(\rho + p)]\,(\mathrm{div}X) - \tfrac{3}{2}\rho(\rho + p)\frac{n^2 S}{R^2}. \tag{A4}$$

APPENDIX B

The purpose of this appendix is to indicate how the present work differs from that of Hawking.[19] The difference lies in the treatment of the $\dot{u}^a{}_{;a}$ term (the four-acceleration term) in the Raychaudhuri equation. In those cases where the four-acceleration of the fluid is important (as in the case of acoustic waves, for example), Hawking's analysis is not valid, as has been noted by Ellis[29] and Bardeen.[30]

Using the equation of state, definition (4), and the identity $[\rho_0(\tau)]_{;b} = \dot{\rho}_0 \tau_{;b}$, the linearized form of Eq. (7) can be written as

$$\dot{u}^a{}_{;a} = -\frac{\rho (v_s)^2}{\rho + p}\left[h_a{}^c \left\{h^{ab}\left(\frac{\delta\rho}{\rho}\right)_{;b}\right\}_{;c}\right] - \frac{\dot{p}}{\rho + p}[(h^{ab}\tau_{;b})_{;a}]. \tag{B1}$$

Though this expansion is not explicitly required in the calculations of Sec. III and Appendix A, the treatment there is equivalent to keeping both of the terms on the right-hand side of Eq. (B1).

Hawking keeps the first term from the right-hand side of Eq. (B1), rewriting the factor in brackets as $[-(n^2/R^2)(\delta\rho/\rho)]$, after expansion in plane waves. The second term does not appear in Hawking's analysis; however, in general, this second term does not vanish and should be retained. For those cases in which the fluid four-velocity u^b is orthogonal to the $\tau = $ constant hypersurfaces, the second term does vanish immediately, since in those cases $\tau_{;b} = -u_b$, and the relation $h^{ab} u_b = 0$ is an identity. However, u_b and $\tau_{;b}$ need not be parallel in general; indeed, they *cannot* be parallel in the presence of nonzero four-acceleration (since $u_b = \psi \tau_{;b}$ immediately implies both $\psi = -1$ and $\dot{u}_b = u_{b;a} u^a = 0$).

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