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Particle emission rates from a black hole. II. Massless particles from a rotating hole*

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The calculations of the first paper of this series (for nonrotating black holes) are extended to the emission rates of massless or nearly massless particles from a rotating hole and the consequent evolution of the hole. The power emitted increases as a function of the angular momentum of the hole, for a given mass, by factors of up to 13.35 for neutrinos, 107.5 for photons, and 26 380 for gravitons. Angular momentum is emitted several times faster than energy, so a rapidly rotating black hole spins down to a nearly nonrotating state before most of its mass has been given up. The third law of black-hole mechanics is proved for small perturbations of an uncharged hole, showing that it is impossible to spin up a hole to the extreme Kerr configuration. If a hole is rotating fast enough, its area and entropy initially increase with time (at an infinite rate for the extreme Kerr configuration) as heat flows into the hole from particle pairs created in the ergosphere. As the rotation decreases, the thermal emission becomes dominant, drawing heat out of the hole and decreasing its area. The lifetime of a black hole of a given mass varies with the initial rotation by a factor of only 2.0 to 2.7 (depending upon which particle species are emitted). If a nonrotating primordial black hole with initial mass 5×10^{14} g would have just decayed away within the present age of the universe, a hole created maximally rotating would have just died if its initial mass were about 7×10^{14} g. Primordial black holes created with larger masses would still exist today, but they would have a maximum rotation rate determined uniquely by the present mass. If they are small enough today to be emitting many hadrons, they are predicted to be very nearly nonrotating.

I. INTRODUCTION

Black holes, as Hawking and others have shown,[1-6] emit particles like thermal bodies. Paper I[7] reported numerical calculations of the emission rates from a nonrotating black hole. This paper gives the rates for the known particles of zero or negligible rest mass from a rotating (Kerr) black hole and shows how such a hole would evolve as it emitted these particles. These results are of interest in testing the validity of the simplifying assumption that most black holes which emit significantly today are not rotating (see, for example, Refs. 7–9).

Paper I noted that although a small black hole will quickly give up its electric charge,[10-12] it is much less certain whether the rotation will also become small. The main difference in the time scales of the two processes can be seen in the following way (using henceforth the dimensionless Planck units spelled out in paper I):

The parameters that determine the shape of a black hole are

$$a_* \equiv J/M^2 \quad \text{and} \quad Q_* \equiv Q/M\,, \tag{1}$$

where J is the angular momentum, Q is the charge, and M is the mass (which sets the scale of the size). These quantities have a domain limited by the constraint

$$a_*^2 + Q_*^2 \le 1\,. \tag{2}$$

Only black holes which emit quanta of much smaller energy than the hole mass will be considered, so that the adiabatic approximation used in the quantum calculations of the emission[1-6] will be valid. The quanta emitted have typical energies of the order of the black-hole temperature or of M^{-1} [with $(10^{15}\text{ g})^{-1} = 266$ MeV in conventional units], which we want much less than M, so we need

$$M \gg 1 \text{ (Planck mass)} = 2.18\times10^{-5}\text{ g}\,. \tag{3}$$

Then roughly M^2 quanta are needed to carry away the energy of the hole; i.e., the entropy in the radiation, which is roughly the number of quanta when thermally distributed, is of the same order as the initial entropy of the hole, which is one-fourth the area[13] or roughly M^2.

When a black hole is charged and/or rotating so that Q_* and/or a_* are significantly different from zero, and when it has temperature or electrostatic potential high enough to permit emission of electrons or positrons, it tends to emit most of its quanta with the same sign of the charge and/or angular momentum as the hole. A charged particle carries off charge

$$|\Delta Q| = e = 0.0854\,, \tag{4}$$

which is roughly of order unity, and a typical quantum also carries off an angular momentum

$$-\Delta J = m \tag{5}$$

of order unity. Since a_* and Q_* must have absolute values not greater than unity, the number of charged particles needed to neutralize the hole is

Q/e, which is only of order M, whereas the number of particles needed to carry off the angular momentum J is of order M^2. Thus the charge can be emitted fairly quickly, but the loss of angular momentum requires roughly the same number of particles as the loss of mass. Therefore, in this paper we will assume that the charge neutralization has already occurred but that the angular momentum may still be significant.

Though one expected a black hole to give up its angular momentum in the same order of time as it gives up its mass, it has not been known whether a_* tends to zero as the black hole evolves. Carter[12] argued that it would tend asymptotically toward a fixed value less than unity, but he gave no indication of what that value would be. Numerical calculations were needed to show whether $\ln J$ always decreases faster than $\ln M^2$, pushing a_* toward zero, or whether these two quantities decrease equally fast at some nonzero limiting value for a_*. There is some indirect evidence, to be given below, that if there were a large enough number of massless scalar fields (unknown at present and therefore not calculated in this paper) to dominate the emission, a_* might indeed get hung up at some nonzero value. However, this paper shows that emission of the known massless fields can only decrease a_* toward zero, and that in fact the decrease is rather rapid compared with the mass decrease.

Because black holes that died in recent epochs or that are emitting significantly today spend almost all their lives with temperatures of order 20 MeV, which is well above the mass of the electron but well below that of each known heavier particle, it is reasonable to do the calculations for the idealized case of emission of a fixed set of species with negligible rest mass. For example, the "canonical combination" used below is the set of known species with masses less than 20 MeV: gravitons, photons, electron and muon neutrinos with one helicity each, electrons, and the corresponding antileptons. However, the results will also be given for other sets of species, to include some of the possibilities (to be discussed below) of other near-massless particles in nature or of the emission from black holes too cold to emit electrons and positrons.

The quantities to be calculated in this paper are the rates at which energy and angular momentum are radiated, the evolution of the mass, rotation parameter, and area of the hole, the lifetimes of holes with different initial angular momenta, the masses of primordial black holes (PBH's) that would be just disappearing today, and the maximum rotation parameters that PBH's of various masses today could have. The remainder of the paper will derive the mathematical formulas for the quantities desired, describe the numerical methods used to calculate them, give the results, and discuss their properties.

II. MATHEMATICAL FORMULAS

Since the total number of particles emitted during the black-hole evolution, roughly M^2, is assumed to be very large, the emission may be approximated as a continuous process with negligible fluctuations due to particle discreteness. Then the rates are well-determined functions of M and a_* alone (assuming that $Q_*=0$, which was justified above). The rest masses of the particles emitted are assumed to be negligible, and the particle species emitted are assumed to be fixed (independent of M), so the only scale in the problem (other than the Planck units, which are here defined to be unity) is determined by M. All quantities to be calculated scale as some power of M and can therefore be put into a scale-invariant form (e.g., depending only on a_*) by dividing out this power of M—or, when one calculates the evolution of a hole [Eqs. (11) ff. below], by dividing out the value of M at some particular point on the evolutionary track.

First, let us consider the rates at which the mass and angular momentum of a black hole decrease, which are given in paper I by Eq. (I.12). Since the time t scales as M^3, we may define the scale-invariant quantities

$$f \equiv -M^3 d\ln M/dt = -M^2 dM/dt\,, \qquad (6)$$

$$g \equiv -M^3 d\ln J/dt = -M a_*^{-1} dJ/dt\,. \qquad (7)$$

These can be seen to be functions of a_* alone: If we define the scale-invariant energy of an emitted particle as

$$x \equiv M\omega\,, \qquad (8)$$

then Eq. (I.12) gives

$$\begin{pmatrix} f \\ g \end{pmatrix} = \sum_{j,l,m,p} \frac{1}{2\pi}\int_0^\infty dx\, \langle N_{jxlmp}\rangle \begin{pmatrix} x \\ m a_*^{-1} \end{pmatrix}, \qquad (9)$$

where the expected number of particles of the jth species of spin s emitted in the mode or state with energy $M^{-1}x$, spheroidal harmonic l, axial angular momentum m, and polarization p is

$$\langle N_{jxlmp}\rangle = \frac{\Gamma_{jlmp}(a_*, x)}{\exp\{4\pi[1+(1-a_*^2)^{-1/2}]x - 2\pi a_*(1-a_*^2)^{-1/2}m\} - (-1)^{2s}}\,. \qquad (10)$$

Here Eq. (I.4) has been used, with the values of the surface gravity, angular frequency, and electrostatic potential of the hole obtained from Eqs. (I.8), (I.9), and (I.10). Γ_{jlmp} is the absorption probability for an incoming wave of the mode considered and can be found by numerically solving the Teukolsky equation.[14,15] It can easily be seen to depend only on a_* and x in addition to the subscripts. The dependence on the species j and polarization p is only through the spin s (assumed positive) and the number of polarizations p that the species has; then l and m can take on any values such that $l-s$ and $l-|m|$ are non-negative integers.

Next, let us consider the evolution of the black hole. Equations (6) and (7) give the rates of change of M and J with respect to time once f and g have been calculated. However, since f and g are functions of a_*, it is easier to solve the equations if a_* is considered as the independent variable. Furthermore, dividing Eq. (7) by Eq. (6) shows us that

$$\frac{d\ln a_*}{d\ln M}=\frac{d\ln J}{d\ln M}-2=\frac{g}{f}-2\equiv h(a_*)\,, \tag{11}$$

which approaches a constant value as a_* approaches zero (assuming that the value is positive so indeed $a_*\to 0$ as $M\to 0$). Because of the logarithms in Eq. (11), it is convenient to define the independent variable to be

$$y\equiv -\ln a_*\,. \tag{12}$$

To cover the greatest range of possibilities, the evolution will first be calculated from $a_*=1$ or $y=0$ to $a_*=0$ or $y=\infty$; a black hole starting at a different value of a_* will simply follow the evolutionary track from that point onward.

Now the object is to find how the mass and time vary with y. Let the starting mass at $a_*=1$ be

$$M_1\equiv M(y=0)\,; \tag{13}$$

this will be the mass that sets the scale. With an eye back on Eq. (11), set

$$z\equiv -\ln(M/M_1)\,, \tag{14}$$

which has the initial value

$$z(0)=0 \tag{15}$$

and evolves according to the reciprocal of Eq. (11) as

$$dz/dy=1/h=f/(g-2f)\,. \tag{16}$$

It has been noted that the time scales as the mass cubed, so define the scale-invariant time parameter as

$$\tau\equiv M_1^{-3}t \tag{17}$$

with initial value

$$\tau(0)=0\,. \tag{18}$$

Then Eq. (6) combined with Eq. (16) gives

$$\frac{d\tau}{dy}=\frac{e^{-3z}}{fh}=\frac{e^{-3z}}{g-2f}\,. \tag{19}$$

From the solutions $z(y)$ and $\tau(y)$ of the coupled differential equations (16) and (19), one can get $y(\tau)$ and $z(\tau)$, and hence a_* and M/M_1, as a function of time. From these, one can find how other quantities evolve, such as the area

$$A=8\pi M^2[1+(1-a_*^2)^{1/2}]\,. \tag{20}$$

Once one has the evolution of a black hole from $a_*=1$, one can consider holes with other initial values a_{*i} of the rotation parameter. They will follow the same solution $z(y)$ and $\tau(y)$ but with different initial values:

$$y_i\equiv -\ln a_{*i}\,, \tag{21}$$

$$z_i\equiv z(y_i)=-\ln(M_i/M_1)\,, \tag{22}$$

$$\tau_i\equiv\tau(y_i)=M_1^{-3}t_i\,. \tag{23}$$

These equations determine M_1 and t_i such that the hole would have mass M_i and rotation a_{*i} at time t_i if it had started with $M=M_1$ and $a_*=1$ at time $t=0$. In terms of M_i and a_{*i}, the evolution follows

$$M=M_1e^{-z}=M_ie^{z_i-z}\,, \tag{24}$$

$$t-t_i=M_1^3(\tau-\tau_i)=M_i^3e^{3z_i}(\tau-\tau_i)\,. \tag{25}$$

Equation (25) and the "standard evolution law" $z(y)$ and $\tau(y)$ can be inverted to get τ and hence y and a_* as functions of time, and then Eq. (24) gives the mass.

A particular quantity desired is the lifetime $T(M_i, a_{*i})$ of a black hole with initial mass M_i and rotation parameter a_{*i}. It can be seen from Eq. (25), assuming that the black hole does evolve to $a_*\to 0$ or $y\to\infty$ as $M\to 0$, that this is

$$T(M_i,a_{*i})\equiv t(y=\infty)-t_i=M_i^3e^{3z_i}(\tau_f-\tau_i)\,, \tag{26}$$

where

$$\tau_f\equiv\tau(y=\infty) \tag{27}$$

is the lifetime in units of M_1^3 of a hole that started with $a_*=1$. The mass dependence of the lifetime can be divided out to get the scale-invariant quantity

$$\theta_i\equiv M_i^{-3}T(M_i,a_{*i})=e^{3z_i}(\tau_f-\tau_i)\,, \tag{28}$$

thus written in terms of quantities previously calculated. Once the lifetime of any black hole is known, one can calculate the initial mass of a primordial black hole that has just disappeared within the present age t_0 of the universe:

$$M_i(a_{*i}, t_0) = t_0^{1/3}\theta_i^{-1/3} = t_0^{1/3} e^{-z_i}(\tau_f - \tau_i)^{-1/3}. \quad (29)$$

Since PBH's would have been spinning down since their creation at time t_0 ago, their present values of a_* should have an upper limit $a_{*\max}(M, t_0)$ less than unity, depending upon the present mass M. It is simpler to solve for the inverse function $M_{\min}(a_*, t_0)$, the minimum mass of a PBH with a_* today. By combining Eqs. (24) and (25) with $t - t_i = t_0$, one finds that

$$M = t_0^{1/3}(\tau - \tau_i)^{-1/3} e^{-z}, \quad (30)$$

where τ and z are evaluated at the present value of y or a_*. Clearly, the minimum occurs at the smallest value of τ_i, which is zero if PBH's can be created with a_{*i} up to unity, so in that case

$$M_{\min}(a_*, t_0) = t_0^{1/3}[\tau(-\ln a_*)]^{-1/3} e^{-z(-\ln a_*)}, \quad (31)$$

where $-\ln a_*$ is shown explicitly as the argument of $\tau(y)$ and $z(y)$. If a_{*i} has a smaller maximum value, the corresponding minimum for τ_i is to be used in Eq. (30) to give $M_{\min}(a_*, t_0)$. One can see that for fixed τ_i, M in Eq. (30) is a monotonically increasing function of a_*, assuming that $g - 2f$ is always positive so that τ is a decreasing function of a_* by Eq. (19). Then the inverse $a_{*\max}(M, t_0)$ is uniquely defined and is a monotonically increasing function of M.

III. NUMERICAL METHODS

The major part of the numerical calculations consisted of computing the functions $f(a_*)$ and $g(a_*)$ by Eqs. (9) and (10), which was done at 14 values of a_* from 0.01 to 0.999 99 to an accuracy of one part in roughly 10^4 or better at low a_* and 10^3 at high a_*. The basic method is briefly summarized in Sec. III of paper I. In order to cover different possibilities for the set of particle species, the contributions to f and g from each species were calculated separately. Thus $f_{1/2}$ and $g_{1/2}$, f_1 and g_1, and f_2 and g_2 were calculated as the contributions from one species with two polarizations of spin $\frac{1}{2}$, 1, and 2, respectively:

$$\begin{pmatrix} f_s(a_*) \\ g_s(a_*) \end{pmatrix} = \sum_{l,m} \frac{1}{\pi} \int_0^\infty dx \langle N_{sxlm}(a_*)\rangle \begin{pmatrix} x \\ m a_*^{-1} \end{pmatrix}. \quad (32)$$

Here the dependence on the species is only through its spin s, and the sum over the two polarizations has already been taken, since the expected number emitted in a mode labeled by x, l, and m is independent of the polarization. Then

$$\begin{pmatrix} f \\ g \end{pmatrix} = n_{1/2}\begin{pmatrix} f_{1/2} \\ g_{1/2} \end{pmatrix} + n_1\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} + n_2\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}, \quad (33)$$

where $n_{1/2}$, n_1, and n_2 are the number of species with spin $\frac{1}{2}$, 1, and 2, respectively, assuming that there are no massless particles of other spins.

A total of 463 angular modes (a combination of s, l, m, and a_*) were calculated and integrated over frequency: 170 modes for $s = \frac{1}{2}$, 155 for $s = 1$, and 138 for $s = 2$. For example, at low a_* all the modes up through $l = \frac{5}{2}$ for $s = \frac{1}{2}$ and through $l = 3$ for $s = 1$ and $s = 2$ were calculated. At high a_* the $l = m$ modes were calculated up to $l = \frac{25}{2}$ for $s = \frac{1}{2}$, $l = 11$ for $s = 1$, and $l = 9$ for $s = 2$, and several $l = m + 1$ modes were calculated (with considerably smaller results), but no modes with $l - m > 1$. At intermediate values of a_*, some combination between these two extremes was taken. The modes calculated appeared to include nearly all of the radiation, though estimates for the small contributions of all the other modes were added in, assuming that the sum over m dropped off exponentially in l roughly as the calculated modes did.

Once the functions f_s and g_s were found at 14 values of a_*, an interpolation algorithm was needed to evaluate them at other values of a_* or y. These functions varied by factors of up to 25 000 from $a_* = 0.01$ to $a_* = 0.999\,99$, and the variation with a_* was particularly rapid at the upper end. To find smooth relationships, various functions of the f's and g's were plotted against various functions of a_*. Of the combinations tried, a small fractional power of the f's and g's versus the surface gravity κ of the hole was the most linear. Therefore, cubic spline fits,[16] minimizing the sum of the squares of the third derivative discontinuities at the 14 values of a_*, were made of $f_s^{0.4}$ and $g_s^{0.4}$ versus

$$4M\kappa = 2[1 + (1 - a_*^2)^{-1/2}]^{-1}, \quad (34)$$

which varies from 0 at $a_* = 1$ or $y = 0$ to 1 at $a_* = 0$ or $y = \infty$. The fits of these variables indeed were quite smooth, with the slopes never changing by a factor of more than 3.6 (even though the values of the fractional powers themselves changed by factors exceeding 50) and with only four of the 84 values of the second derivatives of the splines at the knots exceeding unity in magnitude.

The functions f_s and g_s were evaluated at 363 values of a_* from 1 down to 0.0005 by the cubic spline interpolation algorithm and then were combined by Eq. (33) for some combination of n's to get f and g at each point. A fourth-order Runge-Kutta method was used to integrate Eqs. (16) and (19) simultaneously over the corresponding range of y with the initial values set by Eqs. (15) and (18). At every other point (since the integration requires two points per step), the values of M/M_1, θ, $M_i(a_{*i}, t_0)$, $M_{\min}(a_*, t_0)$, and A/A_1 (where $A_1 = 8\pi M_1^2$ was the area at $a_* = 1$) were calculated by Eqs. (24), (28), (29), (31), and (20). As a check on the accuracy of the numerical integration, the step size was halved, which resulted in agreement to four or five decimal places.

For a_* smaller than 0.0005, the values of f and g at $a_*=0$ or $y=\infty$ were used:

$$\alpha \equiv f(a_*=0)\,, \tag{35}$$

$$\beta \equiv g(a_*=0)\,. \tag{36}$$

Then Eqs. (16) and (19) become

$$dz/dy \underset{y\to\infty}{\sim} \alpha/(\beta-2\alpha) \equiv \gamma\,, \tag{37}$$

$$d\tau/dy \underset{y\to\infty}{\sim} e^{-3z}/(\beta-\alpha)\,, \tag{38}$$

so the solution is

$$z \sim \gamma y + \delta\,, \tag{39}$$

$$\tau \sim \tau_f - \tfrac{1}{3}\alpha^{-1}e^{-3z}\,, \tag{40}$$

where

$$\delta \equiv \int_0^\infty \left(\frac{f}{g-2f} - \gamma\right) dy \tag{41}$$

is a constant that was simply estimated as $z-\gamma y$ at $a_*=0.0005$. The solution for large y or small a_* gives the asymptotic forms

$$M/M_1 \sim e^{-\gamma y - \delta} = e^{-\delta} a_*^{\gamma} \sim [3\alpha(\tau_f-\tau)]^{1/3}\,, \tag{42}$$

$$\theta \sim \tfrac{1}{3}\alpha^{-1} \tag{43}$$

[cf. Eq. (I.26), where $M_0 = M_i$ and $\tau = T(M_i, a_{*i}=0)$],

$$M_i(a_{*i}\to 0, t_0) \sim (3\alpha t_0)^{1/3}\,, \tag{44}$$

$$M_{\min}(a_*, t_0) \sim (3\alpha t_0)^{1/3}(3\alpha\tau_f e^{3\delta} a_*^{-3\gamma} - 1)^{-1/3} \sim (t_0/\tau_f)^{1/3} e^{-\delta} a_*^{\gamma} = M_i(a_{*i}=1, t_0) e^{-\delta} a_*^{\gamma}\,, \tag{45}$$

$$A/A_1 \sim 2M^2/M_1^2 \sim 2[3\alpha(\tau_f-\tau)]^{2/3}\,. \tag{46}$$

IV. RESULTS

The values of f_s, the scale-invariant power in a two-helicity particle species of spin s, and of g_s, the scale-invariant torque per angular momentum of the hole, are listed at the 14 values of a_* in Table I, along with the extrapolated values for $a_*=1$. The cubic spline interpolations are graphed in Figs. 1 and 2, which show that below roughly $a_* \approx 0.6$ the neutrino ($s=\frac{1}{2}$) power dominates, followed by photons ($s=1$) and finally gravitons ($s=2$). However, at greater values of a_* the order is reversed, with gravitons dominating the emission and photons and neutrinos coming second and third, respectively.

This behavior can be explained qualitatively in the following way: For a slowly rotating hole, the coupling depends most strongly on the spheroidal harmonic index l (which reduces to the total, not the orbital, angular momentum when $a_*=0$) rather than on the axial angular momentum m or the spin s. The coupling is greater at lower l values (e.g., paper I showed that the emission rate at low frequencies goes as ω^{2l+1}), but $l \geq s$, so the emission is greater at lower values of s, which allow lower values of l. On the other hand, a rapidly rotating hole couples strongly with the axial angular momentum and also with the spin,[17] so the $s=l=m$

TABLE I. Power and torque emitted by a black hole. For each spin s, $f_s(a_*)$ and $g_s(a_*)$ are the contributions of one species with two polarizations to $f=-M^3 d\ln M/dt$ and $g=-M^3 d\ln J/dt$ at that value of the rotation parameter a_*. The first 14 rows were calculated by Eq. (32); the last row came from a cubic spline extrapolation. The values for $a_*=0$ are nearly the same as for $a_*=0.01$; see Table II for more precise values.

$a_*=J/M^2$	$f_{1/2}(a_*)$	$f_1(a_*)$	$f_2(a_*)$	$g_{1/2}(a_*)$	$g_1(a_*)$	$g_2(a_*)$
0.010 00	8.185×10^{-5}	3.366×10^{-5}	3.845×10^{-6}	6.161×10^{-4}	4.795×10^{-4}	1.064×10^{-4}
0.100 00	8.343×10^{-5}	3.580×10^{-5}	4.684×10^{-6}	6.174×10^{-4}	4.895×10^{-4}	1.167×10^{-4}
0.200 00	8.830×10^{-5}	4.265×10^{-5}	7.732×10^{-6}	6.218×10^{-4}	5.207×10^{-4}	1.514×10^{-4}
0.300 00	9.669×10^{-5}	5.525×10^{-5}	1.494×10^{-5}	6.299×10^{-4}	5.759×10^{-4}	2.233×10^{-4}
0.400 00	1.089×10^{-4}	7.570×10^{-5}	3.116×10^{-5}	6.430×10^{-4}	6.599×10^{-4}	3.603×10^{-4}
0.500 00	1.258×10^{-4}	1.080×10^{-4}	6.822×10^{-5}	6.631×10^{-4}	7.845×10^{-4}	6.236×10^{-4}
0.600 00	1.487×10^{-4}	1.594×10^{-4}	1.574×10^{-4}	6.946×10^{-4}	9.668×10^{-4}	1.155×10^{-3}
0.700 00	1.804×10^{-4}	2.450×10^{-4}	3.909×10^{-3}	7.457×10^{-4}	1.245×10^{-3}	2.322×10^{-3}
0.800 00	2.284×10^{-4}	4.014×10^{-4}	1.104×10^{-3}	8.366×10^{-4}	1.706×10^{-3}	5.286×10^{-3}
0.900 00	3.195×10^{-4}	7.520×10^{-4}	4.107×10^{-3}	1.034×10^{-3}	2.632×10^{-3}	1.544×10^{-2}
0.960 00	4.567×10^{-4}	1.313×10^{-3}	1.305×10^{-2}	1.343×10^{-3}	3.976×10^{-3}	4.057×10^{-2}
0.990 00	6.708×10^{-4}	2.151×10^{-3}	3.578×10^{-2}	1.810×10^{-3}	5.829×10^{-3}	9.555×10^{-2}
0.999 00	9.253×10^{-4}	3.057×10^{-3}	7.251×10^{-2}	2.340×10^{-3}	7.723×10^{-3}	1.753×10^{-1}
0.999 99	1.074×10^{-3}	3.555×10^{-3}	9.785×10^{-2}	2.641×10^{-3}	8.730×10^{-3}	2.271×10^{-1}
1.000 00	1.093×10^{-3}	3.616×10^{-3}	1.012×10^{-1}	2.678×10^{-3}	8.851×10^{-3}	2.338×10^{-1}

rate.[1-6]

Figures 1 and 2 also show the power and relative torque for various combinations $(n_{1/2}, n_1, n_2)$ of the numbers of species emitted with spin $\frac{1}{2}$, 1, and 2, respectively. Since photons and gravitons are the only massless bosons known or commonly theorized to exist as free particles (thus excluding gluons, which are conjectured to exist only in color-singlet configurations[31]), only combinations with $n_1 = n_2 = 1$ have been given. There is a greater uncertainty about $n_{1/2}$, the number of 2-helicity spin-$\frac{1}{2}$ species. The simplest picture consistent with experiment is that $n_{1/2} = 2$, corresponding to the $(\nu_e, \bar{\nu}_e)$ species with a left-handed electron neutrino and its right-handed antiparticle, and the $(\nu_\mu, \bar{\nu}_\mu)$ species with its muon neutrino and antineutrino. However, both of these species may also have the opposite helicity states, which would couple to gravity even if the $V-A$ weak interaction[32] did not couple them to other leptons, thus making $n_{1/2} = 4$. Indeed, vectorlike gauge theories of elementary particles have been made[33-35] in which there are additional neutrino states. Furthermore, black holes small enough to evaporate within the present age of the universe are hot enough to emit ultrarelativistic electrons and positrons,[7] each with two spin states, so $n_{1/2}$ must be augmented by 2 over the number for neutrinos if we consider all rest masses less than 20 MeV as negligible. Therefore, curves are given for $n_{1/2}$ from 2 to 10. In Figs. 3, 4, 5, 6, 7, and 8, the "canonical combination" $(n_{1/2}, n_1, n_2) = (4, 1, 1)$ is labeled as "everything emitted," meaning all of the presently known species with rest mass below 20 MeV, as listed explicitly in Fig. 9.

Figure 3 graphs the lifetime of a black hole—in units of its initial mass cubed—[i.e., θ_i of Eq. (28)], versus the initial rotation parameter a_{*i}, for various combinations of $(n_{1/2}, n_1, n_2)$. [The conversion factor in cgs units is $(10^{15}\text{ g})^3 = 5.23 \times 10^{15}$ sec $= 1.66 \times 10^8$ yr.] Then Fig. 4 gives the initial mass of a PBH that just evaporates today

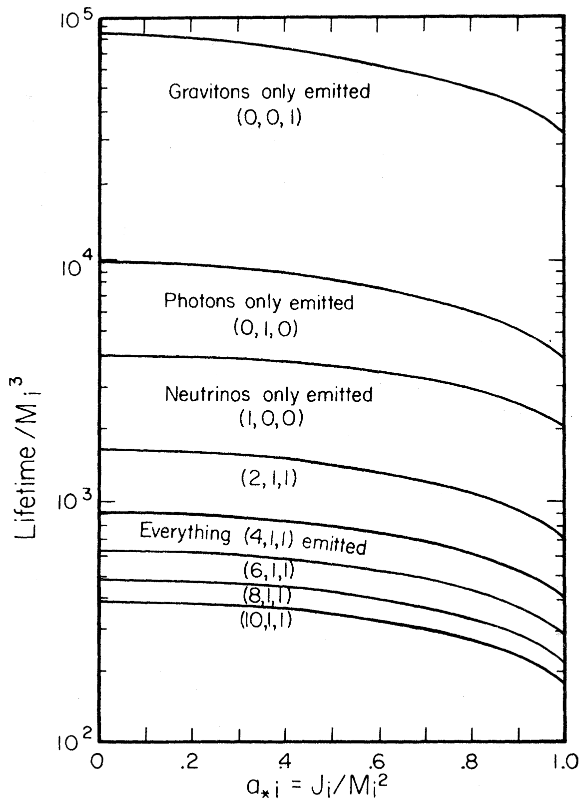

FIG. 3. Lifetime of a black hole, scaled by the initial mass cubed to give θ_i, versus the initial rotation parameter a_{*i}. "Everything emitted" means the canonical combination (4, 1, 1) of all known particles with masses less than 20 MeV.

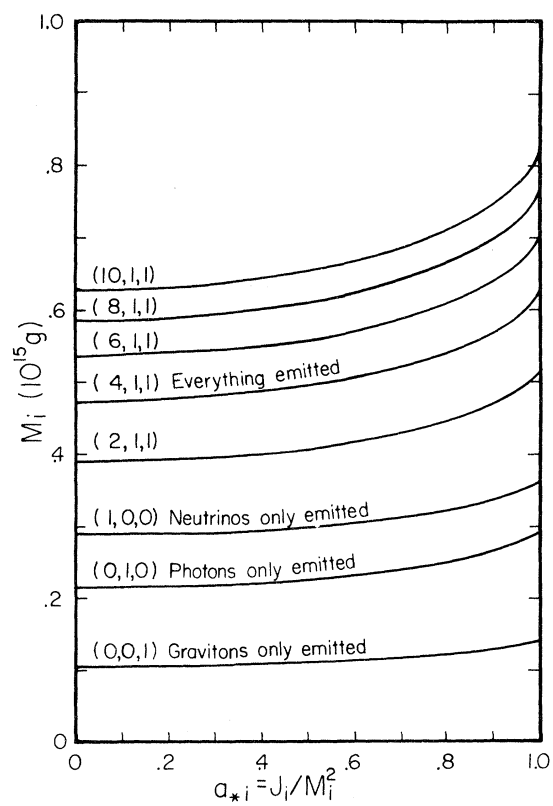

FIG. 4. Initial mass of a primordial black hole created with rotation parameter a_{*i} 16×10^9 yr ago that just goes away today, assuming that it emits the combination $(n_{1/2}, n_1, n_2)$ of species with negligible rest mass. The emission of all known particles, including those with masses greater than 20 MeV, would give a curve slightly above the (4, 1, 1) curve labeled "Everything emitted." However, if there are additional neutrino states, the true curve would be slightly above one of the higher curves shown.

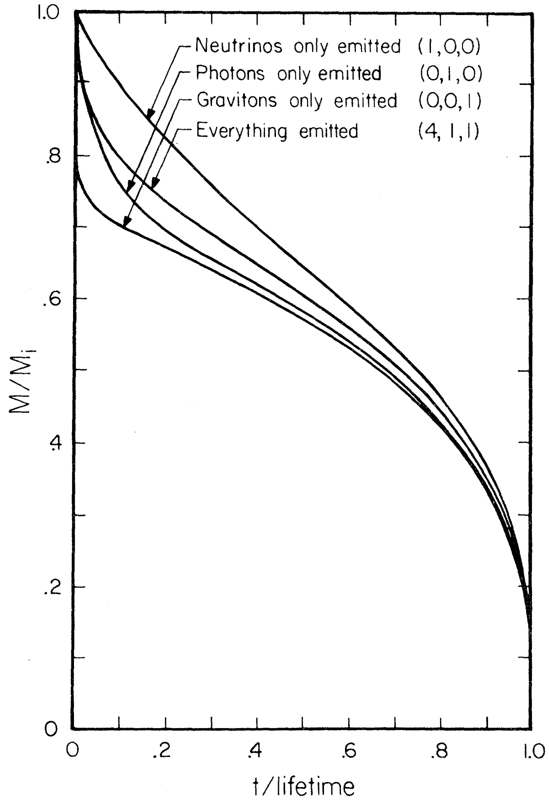

FIG. 5. Time evolution of the mass of a black hole which started out maximally rotating. The vertical and horizontal axes have been scaled by the initial mass $M_i = M_1$ and lifetime $M_i{}^3\theta_i = M_1{}^3\tau_f$. For a black hole that starts with $a_{*i} < 1$, one can use one of the same curves but shrink the axes so that the upper left corner of the graph is on the curve at a later point (to be determined from the value of t/lifetime at $a_* = a_{*i}$ in Fig. 6) and the lower right corner stays fixed, at the end point of the curve.

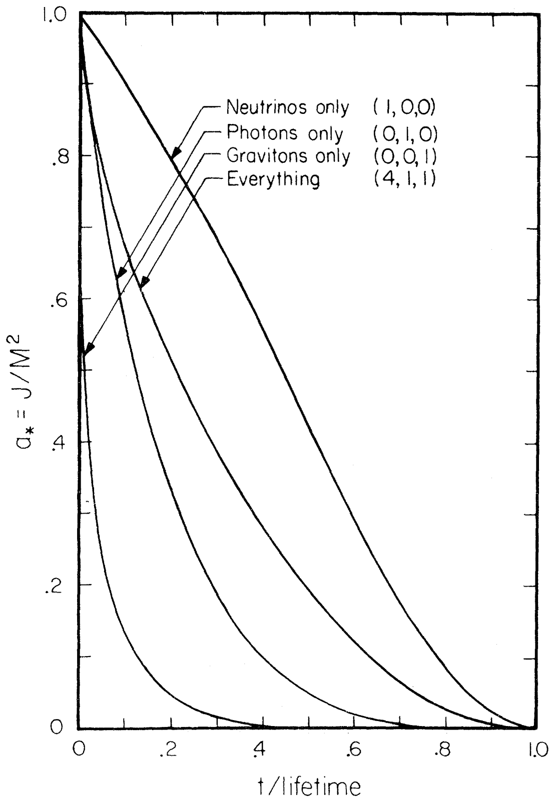

FIG. 6. Time evolution of the rotation parameter a_* of a black hole that started with $a_{*i} = 1$. For any given curve representing the emission of an assumed combination of species, the evolution from $a_{*i} < 1$ can be gotten by moving the left vertical axis to the right until it intersects the curve at $a_* = a_{*i}$, meanwhile shrinking the horizontal axis appropriately to leave its right end fixed.

[Eq. (29)], assuming that the present age of the universe is

$$t_0 = 16 \times 10^9 \text{ yr} = 9.37 \times 10^{60}, \tag{48}$$

so that

$$t_0{}^{1/3} = 2.11 \times 10^{20} = 4.59 \times 10^{15} \text{ g}. \tag{49}$$

For example, a PBH emitting the canonical combination of all known species (except for the small amount of muons and heavier particles emitted) would have just given up all its mass by now if its initial mass had been 4.73×10^{14} g if nonrotating or 6.26×10^{14} g if initially maximally rotating. The curves marked "neutrinos only emitted" in Figs. 3 and 4, as in Figs. 1 and 2, give the results if only one species of neutrinos is emitted; for successive graphs it does not matter how many neutrino species there are for the curves labeled "neutrinos only," since those graphs have the rates scaled out and depend only on the ratios of f's and of g's at different values of a_*.

The time evolution of the mass and rotation parameter are shown in Figs. 5 and 6. The curves for neutrinos, photons, or gravitons only cover the purely hypothetical cases in which the black hole emits only particles of one spin; they are included to illustrate the different behavior that would result. For example, gravitons cause the mass and particularly a_* to decrease more rapidly at large a_*, as compared with the behavior at small a_*, than photons or neutrinos do. Only one combination with species of all three spins being emitted is included, $(n_{1/2}, n_1, n_2) = (4, 1, 1)$, since other combinations gave curves only slightly different. One can see that for this canonical combination, a black hole which started at $a_* = 1$ will lose half its initial mass in 71% of its lifetime but half its initial a_* in only 21% of its lifetime. (Half the angular

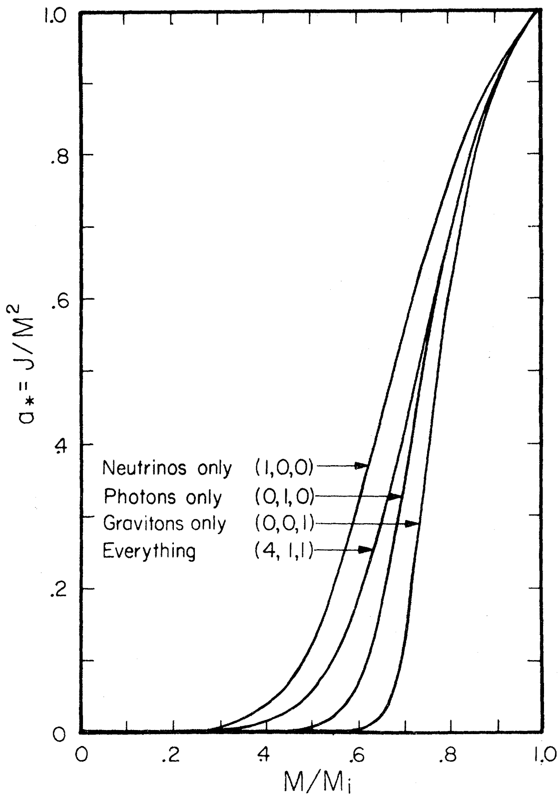

FIG. 7. Variation of the rotation parameter with the mass during the evolution of a black hole, which proceeds from the upper right to the lower left corner. The evolution from $a_{*i} < 1$ can be gotten by keeping the left end of the horizontal axis fixed and shrinking the scale so that $M/M_i = 1$ falls at $a_* = a_{*i}$ on the curve considered.

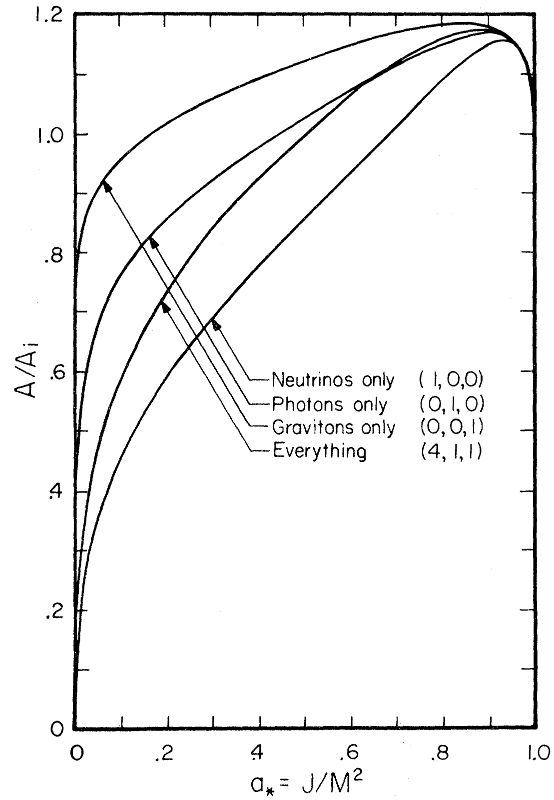

FIG. 8. Evolution of the area A of a black hole, scaled by the initial area A_i in the case $a_{*i} = 1$. For general a_{*i}, the evolution starts at that value of a_* with the vertical axis rescaled to give $A/A_i = 1$ there, and proceeds to the left along the appropriate curve as a_* decreases with time. The evolution of A is plotted versus a_* rather than time to spread out the very rapid changes near $a_* = 1$, where A actually increases with time. Since the area is four times the entropy of the hole, these curves can also be viewed as giving the evolution of the entropy.

momentum $J = M^2 a_*$ is lost in only 6.7% of the lifetime.)

Figure 7 shows how a_* varies with the mass as the black hole gives up its angular momentum and energy. The emission of gravitons causes a_* to decrease at the fastest rate compared with M, essentially because gravitons have the greatest spin and thus carry off the most angular momentum per quantum. For the canonical combination of species, Fig. 6 showed that a_* is reduced to 0.19 after half of the lifetime from $a_* = 1$, but since it takes 71% of the lifetime to reduce M to half its original value, a_* is further reduced to 0.06 by then, as Fig. 7 illustrates directly. A check of the values represented by Fig. 1 reveals that f is then only 1% greater than its value at $a_* = 0$. Therefore, a black hole decaying by the emission of gravitons, photons, the presently known neutrinos, and ultrarelativistic electrons and positrons will emit more than 50% of its energy when it is so slowly rotating that its power is within 1% of the Schwarzschild value given in paper I. This result gives a fairly strong justification for the usual simplifying assumption, mentioned in the Introduction, that emitting black holes are not rotating.[7-9]

One might note that this result was not apparent *a priori*, since $h(a_*)$ in Eq. (11) might have gone to zero at a nonzero value of a_*, in which case the curves in Fig. 7 would have leveled out at that value of a_* as M decreased. In fact, although the calculations have not been made for hypothetical massless spin-0 particles, there are two reasons for suspecting that h might indeed go to zero somewhere if the emission were predominantly in scalar radiation:

(1) If one defines $h_s(a_*)$ by Eq. (11) with f and g replaced by f_s and g_s, one has the logarithmic slope of a_* vs M in the curves for only one spin emitted in Fig. 7. These curves thus have a_* go-

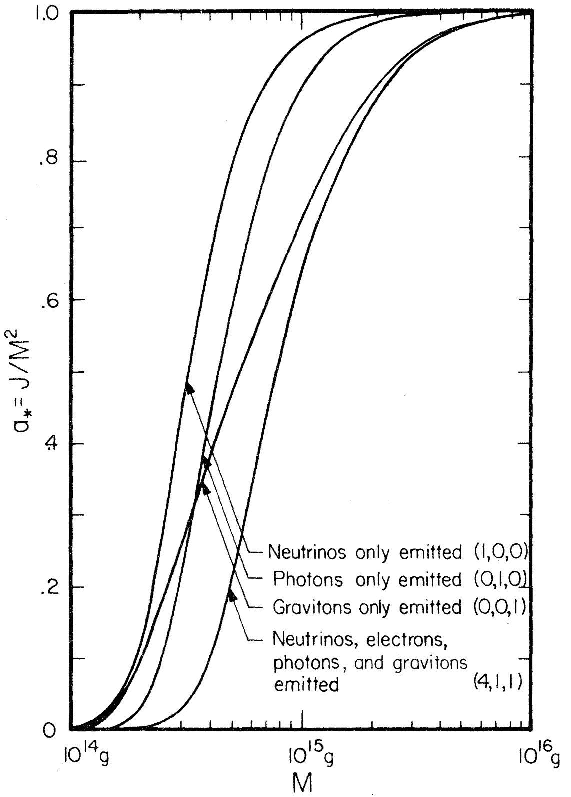

FIG. 9. Maximum present rotation parameter a_* of a primordial black hole with mass M today, assuming it was created 16 billion years ago with unity as the upper limit on the rotation parameter then. Under these assumptions, the actual maximum is probably near (particularly for $M > 10^{15}$ g) or somewhat below (particularly for $M < 10^{15}$ g) the bottom curve given, depending upon the additional emitted species not covered in the canonical combination.

ing as some power of M for small a_*, where the power is $h_s(a_*=0)$. The numerical calculations indicate that there is a remarkably linear relationship between $h_s(a_*=0)$ and the spin s for $s=\frac{1}{2}$, 1, and 2:

$$h_s(a_*=0) \simeq 13.4464s - 1.1948 \tag{50}$$

is accurate to one part in 10^4 for all three values, roughly the accuracy of the numerical calculations. Although there is no apparent theoretical reason to suspect such a highly linear relationship, which comes only after one evaluates integrals over frequency and sums over angular modes in Eq. (32) and therefore seems to be accidental, it is tempting to extrapolate it to $s=0$ to get a negative value for $h_0(a_*=0)$. One can easily see that the emission of any species makes $h(a_*=1)>0$, since Eq. (47) says that the emission from a maximally rotating hole is entirely in the superradiant regime where each quantum contributes

$$\frac{\Delta \ln a_*}{\Delta \ln M} = \frac{\Delta \ln J}{\Delta \ln M} - 2 \simeq \frac{M \Delta J}{J \Delta M} - 2 = \frac{1}{M a_*}\frac{m}{\omega} - 2 = \frac{1}{a_*}\left(\frac{m}{x} - 2a_*\right) > 0 . \tag{51}$$

Therefore, if $h_0(a_*)$ is continuous and is negative at $a_*=0$, it must become zero at some intermediate a_*.

(2) The dominant angular mode at small a_* is presumably the $l=s$ mode, as it is for $s=\frac{1}{2}$, 1, and 2. For $s=0$ that mode carries off energy but no angular momentum, so unless higher angular modes contribute significantly, one would expect $g_0(a_*=0)$ to be roughly zero and hence $h_0(a_*=0)$ to be roughly -2. The higher angular modes would raise $h_0(a_*=0)$ above -2 [conceivably to the value -1.1948 predicted by Eq. (50)] but would probably leave it negative, so again one deduces that $h_0(a_*)$ may be zero for some a_* between zero and one.

If either (1) or (2) is valid and if scalar radiation dominates sufficiently at low a_* for the total radiation to give $h(a_*=0)<0$, then the black hole will spin down only to the nonzero value of a_* at which $h(a_*)=0$. This does not occur for emission of the canonical combination of species, which causes the hole to spin down rapidly toward $a_*=0$, as shown in the curve marked "everything" in Fig. 7. Once a_* is reduced to a small value, it decreases as a power law of M, with the exponent being

$$h(a_*=0) = 6.3611 \tag{52}$$

in the canonical case.

Another interesting result is the evolution of the black-hole area A, which is illustrated in Fig. 8. The area first *increases* with time at large a_* and then decreases to zero along with a_* and the mass. This can be seen formally by using Eq. (11) to differentiate Eq. (20):

$$\frac{d\ln A}{d\ln a_*} = \frac{g}{g-2f} - (1-a_*^2)^{-1/2} . \tag{53}$$

One may further use Eq. (6) to express the time derivative as

$$\frac{dA}{dt} = AM^{-3}[(1-a_*^2)^{-1/2}(g-2f) - g] . \tag{54}$$

For small a_*, the right-hand side of Eq. (54) becomes $-2AM^{-3}f$. This means that the area decreases logarithmically at twice the rate the mass does from Eq. (6), which is obvious since at small a_* the area is simply proportional to M^2. At large a_*, it was shown above that $h>0$, and hence $g-2f>0$ since $f>0$. But $(1-a_*^2)^{-1/2}$ diverges as $a_* \to 1$, so dA/dt becomes positive and even goes infinite as $a_* \to 1$ (cf. the vertical behavior of the curves at the right edge of Fig. 8). The area is at a maximum where

$$2f = [1 - (1 - a_*^2)^{1/2}]g. \quad (55)$$

For the canonical combination of species, this occurs at $a_* = 0.8868$, where the area is 17.3% greater than the original value, after a time of only $6.729M_i^3$ or 1.7% of the total lifetime $394.5M_i^3$ of a hole with $a_{*i} = 1$.

Physically, the behavior of the area can be understood by thermodynamic arguments, since the area is proportional to the entropy of the black hole (as was first suggested by Bekenstein,[36] though there were problems with this interpretation for a black hole immersed in a background of very low temperature until Hawking discovered that black holes not only absorb but also emit thermal radiation[13]). At high values of a_*, the emission is primarily the spontaneous emission discovered by Zel'dovich[18] that corresponds to the stimulated emission of superradiant scattering. In this process, pairs are created in the ergosphere with particles (say) being emitted to infinity with positive energies and their antiparticles going down the hole with negative energies as measured at infinity but positive energies as measured locally. In fact, the antiparticles can even be on classical trajectories at the horizon. Thus heat flows down the hole as well as out to infinity, increasing the entropy of both. On the other hand, at lower values of a_* the emission is primarily thermal, drawing entropy out of the hole. The process may still be regarded as the creation of pairs, with antiparticles going down the hole having negative energies with respect to infinity, but outside the superradiant regime (which becomes negligible at small a_*), the antiparticles also have negative energy locally at the horizon and therefore cannot be on classical trajectories. Instead, they are tunneling through a classically forbidden region in virtual states that actually bring heat out of the hole.

There is still some entropy produced by the partial scattering off the gravitational potential barrier surrounding the hole, but outside the superradiant regime this can only partially cancel the entropy flow out of the hole and serves in effect to increase the entropy emitted to the surrounding region for a given entropy loss by the hole. For example, numerical calculations for a nonrotating hole show that the emission of $s = \frac{1}{2}$ particles into empty space increases the external entropy by 1.6391 times the entropy drawn out of the hole, $s = 1$ particles increase it by a factor of 1.5003, $s = 2$ particles by 1.3481, and the canonical combination of species gives 1.6233 times as much entropy in radiation as the entropy decrease of the hole.

The fact that $g - 2f > 0$ at $a_* = 1$ allows one to prove the third law of black-hole mechanics[37] for small perturbations of an uncharged black hole. (Similar reasoning can presumably be made also for an electrically charged hole.) The third law states that it is impossible to reduce the surface gravity κ to zero by a finite sequence of operations: Using Eqs. (6) and (7) to differentiate the expression for κ in Eq. (I.8) [cf. Eq. (34)], one finds that the emission of particles makes

$$\frac{d\kappa}{dt} = \frac{[1 - (1 - a_*^2)^{1/2}](g - 2f) + (1 - a_*^2)f}{2M^4(1 - a_*^2)^{1/2}[1 + (1 - a_*^2)^{1/2}]} \underset{\kappa \to 0}{\sim} \frac{g - 2f}{4M^5\kappa}, \quad (56)$$

which diverges as $a_* \to 1$ or $\kappa \to 0$. One cannot balance this divergence with a finite accretion rate, as shown by the following argument: Bosons incident in the superradiant regime are not absorbed but amplified, increasing κ. Fermions in this regime are absorbed and do decrease κ, but the exclusion principle prevents more than one incident particle per mode. Even if each such mode with $\omega < m\Omega$ is filled, the excess of absorption over emission goes as $\Gamma\{1 + \exp[-2\pi\kappa^{-1}(\omega - m\Omega)]\}^{-1}$. This dies sufficiently rapidly as $\kappa \to 0$ that these fermion modes cannot balance the effect on κ of the boson superradiant modes. Outside the superradiant regime, each accreting particle of energy $dM \ll M$ can be shown to contribute $d\kappa > -M^{-2}dM$, which can only decrease κ at a finite rate with a finite accretion rate dM/dt. As one tries to reduce κ by accretion (at least if the accretion is only a small perturbation at any one time), eventually the emission dominates and keeps κ away from zero. Thus it is impossible to spin up a black hole adiabatically to the extreme Kerr configuration.

Figure 9 gives the maximum present value of the rotation parameter a_* for a PBH with present mass M that was created 16 billion years ago, assuming no spin up from incident particles. The curves resulting from the emission of neutrinos, photons, or gravitons only are purely illustrative; the true maximum is probably near or somewhat below the curve for the canonical combination of particle species, since those species and possibly a few others are the one predominantly emitted for the mass range shown. For example, electrons, positrons, and all lighter particles will be emitted with negligible effects from their rest masses over the whole range shown, and muons and heavier particles will also be emitted at a significant rate for $M < 5 \times 10^{14}$ g, as paper I pointed out. The graph shows that a PBH with $M < 10^{15}$g should have $a_* < 0.64$ today.

The asymptotic behavior of the graphs in Figs.

1–9 at small a_* was given in functional form by Eqs. (35)–(46), and the parameters α, β, γ, δ, τ_f, and $M_i(a_{*i}=0, t_0)$ are given in Table II for the various combinations $(n_{1/2}, n_1, n_2)$ of species of spin $\frac{1}{2}$, 1, and 2. Note that

$$\left(\frac{d\ln M}{d\ln a_*}\right)_{a_*=0} \equiv \gamma \equiv \frac{\alpha}{\beta - 2\alpha} \equiv \frac{f(a_*=0)}{g(a_*=0) - 2f(a_*=0)} = \frac{1}{h(a_*=0)} \tag{57}$$

is the reciprocal of the exponent of the power-law behavior of a_* versus M at the lower left edge of Fig. 7. The ratio of the lifetime of a black hole with $a_{*i}=1$ to one of the same initial mass with $a_{*i}=0$ is $3\alpha\tau_f$, so Eq. (29) gives the initial mass of a PBH with $a_{*i}=1$ that would just go away today as

$$M_i(a_{*i}=1, t_0) = (t_0/\tau_f)^{1/3} = (3\alpha\tau_f)^{-1/3} M_i(a_{*i}=0, t_0), \tag{58}$$

here written in terms of the parameters in Table II. Then $M_{\min}(a_*, t_0)$ can be directly evaluated from the last quantity in Eq. (45) at small a_*. One can invert this asymptotic formula to obtain

$$a_{*\max}(M, t_0) \sim [e^{\delta} M/M_i(a_{*i}=1, t_0)]^{1/\gamma} \tag{59}$$

for $M \ll M_i(a_{*i}=1, t_0)$. For example, the canonical combination of species gives

$$a_{*\max}(M, t_0) \sim (M/4.870\times 10^{14}\text{ g})^{6.361} = 4.234\times 10^{-5}(M/10^{14}\text{ g})^{6.361}. \tag{60}$$

The actual maximum is almost certainly somewhat lower than this, since muons and other particles omitted in the calculation will have decreased the spin even more, and the upper limit on a_{*i} may be lower than unity; but unless small black holes were formed significantly more recently than 16 billion years ago, one may predict that any black hole found today with $M < 10^{14}$ g will have $a_* < 0.000\,042\,3$.

One can also get asymptotic forms near $a_*=1$. The lifetime has already been given by Eq. (26) with $z_i=0$, $\tau_i=0$, and τ_f listed in Table II, and $M_i(a_{*i}=1, t_0)$ was given by Eq. (58). If we set

$$\alpha_1 \equiv f(a_*=1), \tag{61}$$

$$\beta_1 \equiv g(a_*=1), \tag{62}$$

which can be evaluated by combining the numbers of the last row of Table I according to Eq. (33), then integrating Eqs. (6) and (7) for a small time $t \ll M_1^3$ from $t=0$ at $a_*=1$ and $M=M_1$ gives

$$M \sim M_1(1 - \alpha_1 M_1^{-3} t), \tag{63}$$

$$J \sim M_1^2(1 - \beta_1 M_1^{-3} t), \tag{64}$$

$$a_* \equiv J/M^2 \sim 1 - (\beta_1 - 2\alpha_1) M_1^{-3} t. \tag{65}$$

Since the mass decreases only infinitesimally within the age of the universe if $M_1^3 \gg \alpha_1 t_0$, one can use Eq. (65) with $M=M_1$ and $t=t_0$ as an asymptotic approximation to $a_{*\max}(M, t_0)$ for large M. For example, the canonical combination of species gives

$$a_{*\max}(M, t_0 = 16\times 10^9\text{ yr}) \sim 1 - (M/1.500\times 10^{15}\text{ g})^{-3} = 1 - 0.003\,378(M/10^{16}\text{ g})^{-3}. \tag{66}$$

This formula depends only weakly on the number of spin-$\frac{1}{2}$ species, since gravitons dominate the emission. However, since f and g change so rap-

TABLE II. Parameters in the asymptotic behavior of a slowly rotating black hole: $\alpha \equiv f(a_*=0) \sim -M^2 dM/dt$, $\beta \equiv g(a_*=0) \sim -M a_*^{-1} dJ/dt$, $\gamma \equiv \alpha/(\beta - 2a) \sim d\ln M/d\ln a_*$, $\delta \equiv \int_0^1 (h^{-1} - \gamma) a_*^{-1} da_* \sim \gamma \ln a_* - \ln(M/M_1)$, $\tau_f \equiv \tau(a_*=0) =$ (lifetime from $a_*=1$)/(initial mass $M_1)^3$, and $M_i(a_{*i}=0, t_0) \equiv$ (initial mass of a Schwarzschild hole with lifetime $t_0) = (3\alpha t_0)^{1/3}$. These are given for various combinations of $n_{1/2}$ spin-$\frac{1}{2}$, n_1 spin-1, and n_2 spin-2 species.

$(n_{1/2}, n_1, n_2)$	$10^4\alpha$	$10^4\beta$	γ	δ	τ_f	$M_i(a_{*i}=0, t_0)$
(1, 0, 0)	0.818 30	6.161 08	0.180 86	0.323 38	2011.52	2.8728×10^{14} g
(0, 1, 0)	0.336 38	4.793 64	0.081 63	0.295 39	3856.63	2.1360×10^{14} g
(0, 0, 1)	0.038 36	1.062 65	0.038 91	0.280 73	32 560.0	1.0359×10^{14} g
(0, 1, 1)	0.374 75	5.856 29	0.073 38	0.264 05	3449.2	2.2143×10^{14} g
(1, 1, 1)	1.193 04	12.017 36	0.123 87	0.282 14	1159.5	3.2575×10^{14} g
(2, 1, 1)	2.011 33	18.178 43	0.142 09	0.240 84	695.08	3.8770×10^{14} g
(3, 1, 1)	2.829 63	24.339 52	0.151 48	0.245 51	502.44	4.3442×10^{14} g
(4, 1, 1)	3.647 93	30.500 59	0.157 21	0.250 47	394.50	4.7280×10^{14} g
(5, 1, 1)	4.466 23	36.661 67	0.161 07	0.255 06	325.27	5.0580×10^{14} g
(6, 1, 1)	5.284 52	42.822 75	0.163 84	0.259 17	277.01	5.3497×10^{14} g
(8, 1, 1)	6.921 11	55.144 90	0.167 57	0.266 10	214.02	5.8531×10^{14} g
(10, 1, 1)	8.557 71	67.467 06	0.169 96	0.271 64	174.65	6.2823×10^{14} g

idly with a_* near one (e.g., decreasing roughly 10% between $a_* = 1$ and $a_* = 0.9999$), these asymptotic formulas are only accurate very near $a_* = 1$.

V. CONCLUSIONS

The power emitted from a black hole in particles of negligible mass and of spin $\frac{1}{2}$, 1, and 2 are strongly increasing functions of the rotation parameter $a_* = J/M^2$, varying in the range $a_* = 0$ to $a_* = 1$ by factors of 13.35 for spin $\frac{1}{2}$, 107.5 for spin 1, and 26 380 for spin 2. The power increases 299.3 times for the "canonical combination" of 4 spin-$\frac{1}{2}$, 1 spin-1, and 1 spin-2 species that represent all of the presently known particles with rest masses less than 20 MeV. The power is greatest in spin-$\frac{1}{2}$ particles for $a_* \lesssim 0.6$, followed by spin 1 and then spin 2; but for $a_* \gtrsim 0.6$ the order is reversed.

The emission of angular momentum also increases greatly with a_*, even after the linear dependence expected at small a_* is factored out to get the relative torque or logarithmic rate of decrease in the angular momentum of the hole. The relative torque g behaves similar to the relative power f with respect to spin and a_*, but it is always sufficiently greater than $2f$, for the three spins calculated, that a black hole spins down toward a Schwarzschild configuration much faster than it loses energy. More than half of the energy is emitted after a_* is reduced below a small value, less than 0.06 for the canonical combination of species. At this point the power is within 1% of its Schwarzschild value, so the assumption that decaying black holes have negligible rotation is generally valid.

Even though the power emitted is such a strong function of a_*, the fact that a black hole loses a_* so rapidly means that the total lifetime for a given mass varies only by a factor between 2.02 (for the emission of spin $\frac{1}{2}$ only) and 2.67 (for spin 2 only) over all a_{*i}. A black hole emitting the canonical species has a lifetime 2.32 times as long if initially nonrotating as one the same mass maximally rotating initially. The initial mass of a PBH created 16 billion years ago that just disappears today varies from 4.73×10^{14} g for a Schwarzschild hole to 6.26×10^{14} g for an extreme Kerr hole initially. (This is for the emission of the canonical species; the emission of muons and heavier particles will make these masses somewhat greater, say 5×10^{15} g and 6.6×10^{14} g, respectively.)

A black hole evolving from $a_{*i} = 1$ initially has its area and entropy increase as heat flows into the hole from particle pairs created in the ergosphere. Then as a_* falls low enough (below 0.89 for the canonical species), the nonsuperradiant thermal emission begins to dominate, taking heat out of the hole and thus causing the entropy and area to decrease. The maximum increase in the area is about 17.3% for the canonical emission. For a Schwarzschild hole that emits its energy into the canonical species in empty space, the emission process increases the entropy of the universe ($\frac{1}{4}A$ + entropy outside) by 62.3% of the black hole's initial entropy.

Finally, it was shown that a black hole cannot be spun up to $a_* = 1$. A PBH today is predicted to have a maximum rotation parameter as a function of mass that is given by Fig. 9 for $10^{14}\text{ g} < M < 10^{16}$ g and by Eqs. (59) and (66) for larger and smaller values of the mass. Black holes that are small enough to emit many muons and heavier particles today are seen to be very nearly nonrotating.

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angular mode dominates greatly and now has an effect increasing with s.

It is of interest to note that as $a_* \to 1$, the surface gravity and hence temperature of the black hole go to zero, but the emission does not. In fact, Eq. (10) becomes

$$\langle N_{jxlmp}\rangle \underset{a_*\to 1}{\sim} (-1)^{2s+1}\Gamma_{jlmp}(a_*, x)H(m-2x)\,, \qquad (47)$$

where $H(m-2x)$ is the Heaviside step function (0 if $m-2x<0$, 1 if $m-2x>0$), so one gets simply the spontaneous emission (first discovered by Zel'dovich[18]) in the superradiant regime where the angular velocity ω/m of the wave is lower than the angular velocity

$$\Omega \underset{a_*\to 1}{\sim} \tfrac{1}{2}M^{-1}$$

of the hole. For bosons ($2s$ even), Γ is negative in the superradiant regime, as predicted by Zel'dovich[18,19] and confirmed by Misner,[20] Starobinsky,[21] and Press and Teukolsky[22] for scalar waves, and by Teukolsky[15] and Starobinsky and Churilov[23] for electromagnetic and gravitational waves. That is, the waves gain amplitude on reflection and extract rotational energy from the hole in the wave analog of the Penrose process.[24] Bekenstein[25] has shown that this result follows from Hawking's area theorem[26] for waves with positive-definite energy density. For fermions ($2s$ odd), Γ is always positive, as Unruh[27] has shown for the classical neutrino field, which has a negative energy density near the hole in the superradiant regime. In the quantum analysis, the amplification of a boson wave corresponds to stimulated emission, whereas the Pauli exclusion principle prevents fermions from being amplified. The fact that this behavior shows up in the solutions of the classical wave equations is a manifestation of the connection between spin and statistics.[28] Field-theoretic derivations of the spontaneous emission from a rotating black hole with the appropriate initial state for no thermal emission have been given by Unruh[29] and Ford,[30] but one must remember that a black hole formed by collapse has a nonzero temperature (except when $a_* = 1$) and thus emits at a greater

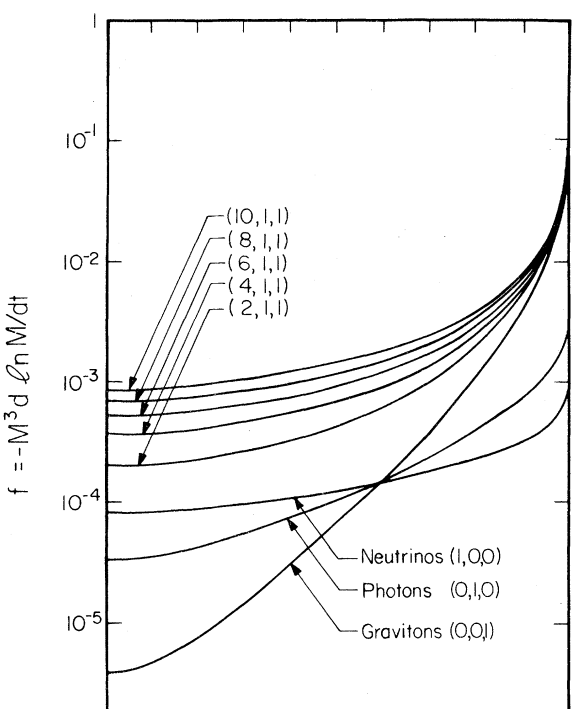

FIG. 1. Power emitted in various combinations of species by a rotating black hole, expressed in a scale-invariant way by f. The symbol $(n_{1/2}, n_1, n_2)$ denotes a combination of $n_{1/2}$ spin-$\frac{1}{2}$, n_1 spin-1, and n_2 spin-2 species, where each species is assumed to have two polarizations (e.g., left-handed neutrino plus right-handed antineutrino).

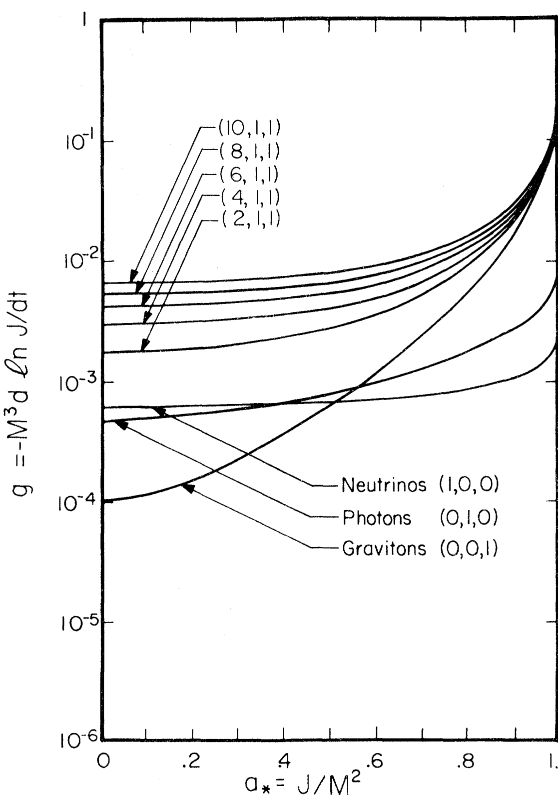

FIG. 2. Relative torque emitted by a black hole (i.e., the rate of emission of angular momentum, divided by the angular momentum of the hole), expressed in a scale-invariant form by g.

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