Absorption cross section of small black holes

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The absorption cross section of small nonrotating black holes is calculated for massive scalar and Dirac particles. The latter cross section is shown to be exactly 1/8 of the former for all energies in the limit where the wavelength is larger than the Schwarzschild radius.

Of interest recently has been the calculation of the spontaneous quantum evaporation of small black holes.[1] One parameter of particular interest in such a calculation is the absorption cross section of a black hole for particles of various spins and various energies. In my thesis,[2] these transmission coefficients for massive Dirac particles on small black holes were calculated, and shortly thereafter the absorption cross sections for small static black holes to massive Dirac particles and to massive scalar particles were calculated. These results are given in this paper. Some of these results have since been independently derived by Starobinski,[9] Ford,[7] and Page.[8]

The metric for a Schwarzschild black hole is given by

$$ds^2 = (1-2M/r)dt^2 - dr^2/(1-2M/r) - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

The scalar field equations and the Dirac equations are given by

$$g^{\mu\nu}\phi_{,\mu;\nu} + m^2\phi = 0, \tag{2a}$$

$$\gamma^\alpha \nabla_\alpha \psi + i\mu\psi = 0. \tag{2b}$$

In the following I will use units of mass and distance so that $2M = 1$ in the Schwarzschild metric.

The scalar and Dirac equations can be separated. For the scalar equation one obtains

$$\phi = e^{-i\omega t} f_{\omega l}(r) Y_{lm}(\theta,\phi), \tag{3a}$$

with

$$\left[\frac{\rho^2}{r^2}\frac{d}{dr} r^2\rho^2 \frac{d}{dr} + \omega^2 - \rho^2\left(m^2 + \frac{l(l+1)}{r^2}\right)\right] f_{\omega l}(r) = 0, \tag{3b}$$

$$\rho = (1-1/r)^{1/2}.$$

For the Dirac equation one obtains

$$\psi_{\epsilon k \tilde{m}} = \frac{e^{-i\epsilon t}}{r(1-1/r)^{1/4}} \begin{pmatrix} G_k(r)\eta_{k\tilde{m}}(\theta,\phi) \\ -iF_k(r)\vec{\mathrm{x}}\cdot\vec{\sigma}\eta_{k\tilde{m}}(\theta,\phi)\end{pmatrix}, \tag{4a}$$

$$\rho\frac{dG_k(r)}{dr} + \frac{k}{r}G_k(r) = \left(\frac{\epsilon}{\rho} + \mu\right)F_k(r),$$
$$\rho\frac{dF_k(r)}{dr} - \frac{k}{r}F_k(r) = \left(\frac{-\epsilon}{\rho} + \mu\right)G_k(r), \tag{4b}$$

where k is a positive or negative nonzero integer, $l = |k+\frac{1}{2}| - \frac{1}{2}$, and

$$\eta_{km} = \begin{cases} \begin{pmatrix} \left(\frac{l-\tilde{m}+\frac{1}{2}}{2l+1}\right)^{1/2} Y_{l,\tilde{m}-1/2}(\theta,\phi) \\ \left(\frac{l+\tilde{m}+\frac{1}{2}}{2l+1}\right)^{1/2} Y_{l,\tilde{m}+1/2}(\theta,\phi)\end{pmatrix}, & k>0 \\ (-1)^k i\begin{pmatrix} \left(\frac{l+\tilde{m}+\frac{1}{2}}{2l+1}\right)^{1/2} Y_{l,\tilde{m}-1/2}(\theta,\phi) \\ -\left(\frac{l-\tilde{m}+\frac{1}{2}}{2l+1}\right)^{1/2} Y_{l,\tilde{m}+1/2}(\theta,\phi)\end{pmatrix}, & k<0 \end{cases} \tag{5a}$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \tag{5b}$$

$$\vec{\mathrm{x}}\cdot\vec{\sigma} = \sin\theta\cos\phi\,\sigma_1 + \sin\theta\sin\phi\,\sigma_2 + \cos\theta\,\sigma_3.$$

Here $\tilde{m}$ is the angular momentum in the z ($\theta = 0$) direction, l is the total orbital angular momentum, and $j = |k| - \frac{1}{2}$ is the total angular momentum.

For future use, I will also define v, λ by

$$v = \begin{cases} (1-m^2/\omega^2)^{1/2} & \text{for scalar equation,} \\ (1-\mu^2/\epsilon^2)^{1/2} & \text{for Dirac equation,}\end{cases}$$
$$\lambda = (1-v^2)^{1/2}. \tag{6}$$

I will be interested in the solutions to the above equations (3) and (4) when ω and ϵ are much less than 1 (i.e., when the wavelength of the particle is larger than the Schwarzschild radius of the black hole).

For this case, the above equations may be solved by an asymptotic expansion technique in which one solves a simplified set of equations in which certain small terms are neglected in various regions for the parameter r. Since it is difficult to estimate the relative magnitude of various parameters in the first-order coupled neutrino equations (as it is difficult *a priori* to estimate the relative magnitudes of F and G), I will convert these to a second-order set of equations.

For $\epsilon > 0$, define the new variable x such that

$$\frac{dx}{dr} = \frac{1+\lambda\rho}{\rho^2}. \tag{7}$$

Then F may be elimated from the equations (4b) to give a second-order equation for G.

$$\frac{d^2G}{dx^2} + \left[\epsilon^2\left(\frac{1-\lambda\rho}{1+\lambda\rho}\right) - \frac{k^2\rho^2}{(1+\lambda\rho)^2r^2} + \frac{d}{dx}\left(\frac{k\rho}{(1+\lambda\rho)r}\right)\right] G = 0, \tag{8}$$

where r, ρ are now implicit functions of x. A similar equation for F can be obtained by taking $\lambda \to -\lambda$ and $k \to -k$, but is singular in general.

The various regions and the approximations to the field equations applicable in each region will now be defined.

Region I: Near the horizon

Scalar equation. Region defined by $(r-1) < \omega^2/l(l+1) \ll 1$:

$$\left(\rho^2\frac{d}{dr}\rho^2\frac{d}{dr} + \omega^2\right)f = 0. \tag{9}$$

If we define a new variable r^* by $r^* = r + \ln(r-1)$, this equation becomes

$$\left(\frac{d^2}{dr^{*2}} + \omega^2\right)f = 0. \tag{10}$$

All neglected terms go to 0 with $r^* \to -\infty$ $(r \to 1)$ at least as fast as $e^{r^*}f$, and are thus much smaller than the term $\omega^2 f$ in the region of interest.

Dirac equation. Region defined by $\rho \approx r-1 < \epsilon^2/k^2 \ll 1$:

$$\frac{d^2}{dx^2}G + \frac{\epsilon^2}{(1+\lambda^2)}G = 0. \tag{11}$$

Again, the terms which have been neglected are of order $e^{x/(1+\lambda)}G$, which go to zero as $x \to -\infty$ $(r \to 1)$.

Region II: Intermediate region

Scalar equation. The terms in ω^2, m^2 are much smaller than all other terms:

$$\left[\frac{d}{dr}r^2\rho^2\frac{d}{dr} - l(l+1)\right]f = 0. \tag{12}$$

Dirac equation. The terms in ϵ^2, μ^2 are much smaller than other terms:

$$\frac{d^2}{dx^2}G + \left[\frac{-k^2\rho^2}{(1+\lambda\rho)^2r^2} + \frac{d}{dx}\left(\frac{k\rho}{(1+\lambda\rho)r}\right)\right]G = 0. \tag{13}$$

This equation is most easily solved by breaking it into a pair of coupled first order equations by defining a new function

$$H = \frac{dG}{dx} + \frac{k\rho G}{(1+\lambda\rho)r}. \tag{14}$$

Then the above equation becomes

$$\frac{dH}{dr} - \frac{k}{\rho r}H = 0. \tag{15}$$

Note that this pair of coupled equations is not the same as those obtained by simply neglecting ϵ, μ in the coupled equations for F, G [Eq. (4b)].

Region III: Far from the black hole

All terms of order higher than $(1/r)^2$ are neglected.

Scalar equation. The scalar equation is

$$\left[\frac{d^2}{dr^2} + \left((\omega^2-m^2) + \frac{2\omega^2-m^2}{r} - \frac{l(l+1)}{r^2}\right)\right]\left(\frac{rf}{\rho}\right) = 0. \tag{16}$$

An additional term of order ω^2/r^2 has also been neglected.

Dirac equation. The Dirac equation is

$$\left[\frac{d^2}{dr^2} + \left((\epsilon^2-\mu^2) + \frac{2\epsilon^2-\mu^2}{r} - \frac{k(k+1)}{r^2}\right)\right] \times \left[\frac{(1+\lambda)}{1+\lambda\rho}\right]^{1/2}\rho G = 0. \tag{17}$$

Again a term of order ϵ^2/r^2 has also been neglected. Note that the effective gravitational mass is $(2\omega^2 - m^2)$, not either ω^2 or m^2.

These sets of equations in the three regions can be solved by means of well-known functions. By matching these solutions in the areas of overlap between the regions, one can find an approximate solution for the wave equation, and in particular determine the fraction of the incident wave which is transmitted down the black hole.

Region I

Only those solutions are retained which represent no particles coming out of the past horizon of the black hole.

Scalar solution. This is

$$f_{\text{I}} = A_{\text{I}}e^{i(\omega r^* + \delta)}, \tag{18}$$

where δ is a phase which will be conveniently chosen later.

Dirac solution. This is

$$G_{\text{I}} = \alpha_{\text{I}}e^{i(\epsilon r^* + \delta)}. \tag{19}$$

Region II

Scalar solution. The scalar solution is

$$f_{II}(r) = A_{II} P_l(2r-1) + B_{II} Q_l(2r-1), \tag{20}$$

where P_l, Q_l are the Legendre functions.

Dirac solution. We can solve Eq. (14) readily for the function H:

$$H = \beta_{II}\left(\frac{1-\rho}{1+\rho}\right)^{-k}. \tag{21}$$

Then the solution for G may be written as

$$G_{II} = \alpha_{II}\left(\frac{1-\rho}{1+\rho}\right)^{k} + \beta_{II}\mathcal{G}, \tag{22}$$

where $\mathcal{G}$ is a particular solution to

$$\frac{d\mathcal{G}}{dr} + \frac{k}{\rho r}\mathcal{G} = \frac{1+\lambda\rho}{\rho^2}\left(\frac{1-\rho}{1+\rho}\right)^{-k}. \tag{23}$$

For $k<0$ we obtain

$$\mathcal{G} = \left(\frac{1-\rho}{1+\rho}\right)^{k}\int_1^{\rho}\frac{2(1+\lambda\rho')}{\rho'}\frac{(1-\rho')^{2|k|-2}}{(1+\rho')^{2|k|+2}}\,d\rho', \tag{24}$$

whereas for $k>0$ $\mathcal{G}$ may be written as

$$\mathcal{G} = \left(\frac{1-\rho}{1+\rho}\right)^{k}\left\{\ln\rho^2 + \int_0^{\rho}\frac{2}{\rho'}\left[\frac{(1+\lambda\rho')}{(1-\rho'^2)^2}\left(\frac{1+\rho'}{1-\rho'}\right)^{2k} - 1\right]d\rho'\right\}. \tag{25}$$

Region III

Both equations have solutions in terms of Coulomb wave functions F_l^C and G_l^C (see Ref. 3).

Scalar solution. The scalar solution is

$$f_{III} = A_{III}\frac{F_l^C(-\omega(1+v^2)/2v, \omega v r)}{r} + B_{III}\frac{G_l^C(-\omega(1+v^2)/2v, \omega v r)}{r}. \tag{26}$$

Dirac solution. Define $l=k$ for $k>0$ and $l=-k-1$ for $k<0$ to write the solution as

$$G_{III}(r) = \frac{(1+\lambda\rho)^{1/2}}{(1+\lambda)^{1/2}\rho}\left[\alpha_{III}F_l^C\left(-\left(\frac{\epsilon(1+v^2)}{2v}\right), \epsilon v r\right) + \beta_{III}G_l^C\left(-\left(\frac{\epsilon(1+v^2)}{2v}\right), \epsilon v r\right)\right]. \tag{27}$$

We can obtain one restriction on A_{III}, B_{III} and α_{III}, β_{III} by demanding that f and G have unit incoming amplitude. From the asymptotic form for F^C and G^C (see Ref. 3) one finds that this can be satisfied by

$$A_{III} + iB_{III} = 2,$$
$$\alpha_{III} + i\beta_{III} = 2. \tag{28}$$

The relation between the various coefficients is now found by joining the functions in the overlap between the various regions.

Regions I and II

The definitions of regions I and II do not really lead to an overlap region. However, near the point $r_0 \approx 1 + \omega^2/(l+1)^2$ or $r_0 \approx 1+\epsilon^2/k^2$, all of the nonderivative terms in the differential equations for f and G are small. One can approximate the solutions by linear combinations of constant terms and terms proportional to $\ln(r-1)$. Furthermore, since $\ln(r_0-1)$ is equal to $\ln[\omega^2/(l+1)^2]$ or $\ln(\epsilon^2/k^2)$, I will neglect terms of the form $\omega\ln(r_0-1)$ or $\epsilon\ln(r_0-1)$ when compared to unity.

Scalar equation. Near $r=r_0$ we have from Eq. (18)

$$f_I \approx A_I(1 - i\omega(r^* - r_0^*))$$
$$\approx A_I([1 + i\omega\ln(r_0-1)] - i\omega\ln(r-1)), \tag{29}$$

where δ has been chosen equal to $(-\omega r_0^*)$.

f_{II} can be determined from the behavior[3] of P_l, Q_l for $2r-1$ near 1:

$$P_l(2r-1) = 1 + O(r-1),$$
$$Q_l(2r-1) = -\tfrac{1}{2}\ln(r-1) + \tfrac{1}{2}\ln 2 + \sum_1^l \frac{1}{m} + O(r-1)$$
$$\approx -\tfrac{1}{2}\ln(r-1) + a, \tag{30}$$

where a is a constant.

Equating f_I and f_{II} near r_0 gives the following relation among A_{II}, B_{II}, and A_I:

$$A_{II} + B_{II}(-\tfrac{1}{2}\ln(r-1) + a) \approx A_I(1 - i\omega\ln(r-1)). \tag{31}$$

Keeping terms only to lowest order in ω we obtain

$$A_{II} \approx A_I,$$
$$B_{II} \approx 2i\omega A_I. \tag{32}$$

Dirac equation. For the Dirac solutions, a similar analysis applies. Taking

$$r_0 \approx (1+\epsilon^2/k^2) \tag{33}$$

we have from Eq. (19) with $\delta = -\epsilon r_0^*$

$$\begin{aligned} G_{\rm I} &\approx \alpha_{\rm I}(1 - i\epsilon(r^* - r_0^*)) \\ &\approx \alpha_{\rm I}(1 - i\epsilon \ln(r-1)), \end{aligned} \tag{34}$$

where we have neglected terms of the form $\epsilon \ln(\epsilon^2/k^2)$ with respect to 1 as well as terms of order $(r-1)$.

$G_{\rm II}$ must be evaluated for the two cases of $k>0$ and $k<0$. First, as $r \to 1$, we have that $\rho \to 0$. Then for $k<0$ we obtain from Eqs. (22) and (24)

$$G_{\rm II} \approx \alpha_{\rm II} + \beta_{\rm II}[\ln(r-1)+b] + O(r-1), \tag{35}$$

where

$$b = \int_1^0 \frac{2}{\rho'}\left[\frac{(1+\lambda\rho')}{(1-\rho'^2)^2}\left(\frac{1-\rho'}{1+\rho'}\right)^{2|k|} - 1\right] d\rho' \tag{36}$$

is a constant.

Equating $G_{\rm I}$ and $G_{\rm II}$, we obtain

$$\alpha_{\rm II} + \beta_{\rm II}(\ln(r-1)+b) \approx \alpha_{\rm I}(1 - i\epsilon \ln(r-1)). \tag{37}$$

Keeping terms only to lowest order in ϵ we obtain

$$\begin{aligned} \alpha_{\rm II} &\approx \alpha_{\rm I}, \\ \beta_{\rm II} &\approx -i\epsilon\alpha_{\rm I}. \end{aligned} \tag{38}$$

For $k>0$, a similar analysis applies and one again obtains

$$\alpha_{\rm II} = \alpha_{\rm I}, \quad \beta_{\rm II} = -i\epsilon\alpha_{\rm I}. \tag{39}$$

Regions II and III

The overlap between these two regions will occur when $1/r \ll 1$ but when $1/r \gg \omega v$ or ϵv. We can therefore neglect $1/r$ with respect to unity and terms in ωv or ϵv with respect to both unity and $1/r$ in Eqs. (20), (26), (22), and (27).

Scalar equation. To evaluate $f_{\rm II}$ in this overlap region we need an asymptotic expansion[3] for P_l and Q_l:

$$\begin{aligned} P_l(2r-1) &= \frac{(2l)!}{2^l(l!)^2}(2r-1)^l\left[1+O\left(\frac{1}{r}\right)\right] \\ &\approx \frac{(2l)!}{(l!)^2} r^l, \\ Q_l(2r-1) &= \frac{(l!)^2}{2(2l+1)!\,r^{l+1}}\left[1+O\left(\frac{1}{r}\right)\right]. \end{aligned} \tag{40}$$

Therefore we have

$$f_{\rm II} \approx A_{\rm II}\frac{(2l)!}{(l!)^2} r^l + B_{\rm II}\frac{(l!)^2}{2(2l+1)!\,r^{l+1}}. \tag{41}$$

Similarly we use the asymptotic expansions[3] for the Coulomb functions near $\omega v r \approx 0$:

$$\begin{aligned} f_{\rm III} = {} & \frac{A_{\rm III}}{r}\left[C_l\left(\frac{-\omega(1+v^2)}{2v}\right)(\omega v r)^{l+1}[1+O(\omega v r)]\right] \\ & + \frac{B_{\rm III}}{r}\left[\frac{(\omega v r)^{-l}}{(2l+1)C_l(-\omega(1+v^2)/2v)}[1+O(\omega v r)]\right], \end{aligned} \tag{42}$$

where

$$C_l(\eta) = 2^l e^{-\pi\eta/2}\frac{|\Gamma(l+1+i\eta)|}{(2l+1)!}. \tag{43}$$

Therefore we obtain by equating $f_{\rm III}$ and $f_{\rm II}$

$$\begin{aligned} A_{\rm II}\frac{(2l)!}{(l!)^2} &\approx A_{\rm III}\,C_l\left(\frac{-\omega(1+v^2)}{2v}\right)(\omega v)^{l+1}, \\ \frac{B_{\rm II}(l!)^2}{2(2l+1)!} &\approx B_{\rm III}\left(\frac{(\omega v)^{-l}}{(2l+1)C_l(-\omega(1+v^2)/2v)}\right). \end{aligned} \tag{44}$$

Dirac equation. For $k<0$ we have

$$\begin{aligned} G_{\rm II} = {} & \alpha_{\rm II}\left(\frac{1-\rho^2}{(1+\rho)^2}\right)^k \\ & + \beta_{\rm II}\left(\left(\frac{1-\rho^2}{(1+\rho)^2}\right)^k \int_\infty^r \frac{(1+\lambda\rho)}{\rho^2}\left(\frac{1-\rho^2}{(1+\rho)^2}\right)^{2|k|} dr\right). \end{aligned} \tag{45}$$

Recalling that $1-\rho^2 = 1/r$, we have that $\rho \approx 1$ and

$$\begin{aligned} G_{\rm II} &\approx \alpha_{\rm II}\left(\frac{1}{4r}\right)^k + \beta_{\rm II}\left(\frac{1}{4r}\right)^k \int_\infty^r \frac{(1+\lambda)}{(4r)^{2|k|}}\, dr \\ &\approx \alpha_{\rm II}(4r)^{|k|} + \frac{\beta_{\rm II}(1+\lambda)}{4(2|k|-1)(4r)^{|k|-1}}. \end{aligned} \tag{46}$$

For $k>0$ we have

$$\begin{aligned} G_{\rm II} = {} & \alpha_{\rm II}\left(\frac{1-\rho^2}{(1+\rho)^2}\right)^k \\ & + \beta_{\rm II}\left(\left(\frac{1-\rho^2}{(1+\rho)^2}\right)^k\left[E + \int^r \frac{(1+\lambda)}{\rho^2}\left(\frac{(1+\rho)^2}{(1-\rho^2)}\right)^{2k} dr\right]\right) \end{aligned} \tag{47}$$

$$\begin{aligned} &\approx \frac{\alpha_{\rm II}}{(4r)^k} + \beta_{\rm II}\left(\frac{1}{(4r)^k}\left[E + \int (1+\lambda)(4r)^{2k} dr\right]\right) \\ &\approx \frac{\alpha_{\rm II}}{(4r)^k} + \frac{\beta_{\rm II}(1+\lambda)}{4(2k+1)}(4r)^{k+1}, \end{aligned} \tag{48}$$

where E was an integration constant assumed small with respect to r^{k+1}. Again, terms of order $1/r$ smaller than those kept have been neglected.

We use an approximation for the Coulomb functions to determine $G_{\rm III}$:

$$G_{\rm III} \approx \alpha_{\rm III} C_l\left(\frac{-\epsilon(1+v^2)}{2v}\right)(\epsilon v r)^{l+1} + \beta_{\rm III}\left(\frac{(\epsilon v r)^{-l}}{(2l+1)C_l(\epsilon(1+v^2)/2v)}\right). \tag{49}$$

Matching $G_{\rm II}$ and $G_{\rm III}$ then gives the following: For $k>0$ ($l=k$), dropping the arguments of C_l,

$$\alpha_{\rm II} = \beta_{\rm III}\frac{4^k}{(\epsilon v)^k(2k+1)C_k}, \qquad \beta_{\rm II} = \frac{\alpha_{\rm III}\, 4(\epsilon v)^{k+1}(C_k)(2k+1)}{(4)^{k+1}(1+\lambda)}. \tag{50}$$

For $k<0$ ($l=|k|-1$),

$$\alpha_{\rm II} = \alpha_{\rm III}\left(\frac{\epsilon v}{4}\right)^{|k|} C_{|k|-1}, \qquad \beta_{\rm II} = \frac{\beta_{\rm III}\, 4}{(1+\lambda)C_{|k|-1}}\left(\frac{4}{\epsilon v}\right)^{|k|-1}. \tag{51}$$

We can now determine $A_{\rm I}$ and $\alpha_{\rm I}$, which give the amplitude of the transmitted wave for an incoming wave of unit amplitude.

Scalar equation. From (28), (32), and (44) we obtain

$$A_{\rm I} = \frac{2(l!)^2}{(2l)!}(C_l)(\omega v)^{l+1}. \tag{52}$$

Dirac equation. From (28), (38), and (50) we obtain for $k>0$

$$\alpha_{\rm I} = \frac{8i}{\epsilon}\left(\frac{\epsilon v}{4}\right)^{k+1}\frac{(C_k)(2k+1)}{1+\lambda}. \tag{53}$$

From (28), (39), and(51) we obtain for $k<0$

$$\alpha_{\rm I} = 2\left(\frac{\epsilon v}{4}\right)^{|k|} C_{|k|-1}. \tag{54}$$

To determine the transmission coefficient (i.e., the percentage of the wave which goes down the black hole), we note that the Wronskian of the second-order equation for f and G is proportional to the radial flux of particles going into the black hole [see Eqs. (72) and (88)].

If the transmission coefficient were unity, the behavior of the scalar and Dirac waves near infinity would be

$$\tilde{f} \approx e^{-i\omega v r}, \qquad \tilde{G} \approx e^{-i\epsilon v r}, \tag{55}$$

with Wronskian near infinity of

$$\tilde{W}_S = r\tilde{f}^*\left(\frac{d}{dr^*}r\tilde{f}\right) - r\tilde{f}\left(\frac{d}{dr}r\tilde{f}^*\right) = -2i\omega v, \tag{56a}$$

$$\tilde{W}_D = \tilde{G}^*\frac{d}{dx}\tilde{G} - \tilde{G}\frac{d}{dx}\tilde{G}^* = \frac{\rho^2}{1+\lambda\rho}\left(G^*\frac{d}{dr}G - G\frac{d}{dr}G^*\right) = \frac{-2i}{1+\lambda}\epsilon v. \tag{56b}$$

The Wronskian for the actual wave propagating into the black hole can be calculated near $r=1$, where we have

$$f \approx A_{\rm I} e^{-i\omega r^*}, \tag{57a}$$

$$W_S \approx -2i\omega|A_{\rm I}|^2, \tag{57b}$$

$$G \approx \alpha_{\rm I} e^{-i\epsilon x}, \tag{58a}$$

$$W_D = -2i\epsilon|\alpha_{\rm I}|^2. \tag{58b}$$

The ratio of W_S to $\tilde{W}_S$ and W_D to $\tilde{W}_D$ is the transmission coefficient into the black hole:

$$(T_S)_l = \frac{|A_{\rm I}|^2}{v} = \frac{(4l)!^4}{(2l)!^2}\left|C_l\left(\frac{-\lambda^2\omega}{2v}\right)\right|^2\frac{(\omega v)^{2l+2}}{v}, \tag{59}$$

$$(T_D)_k = \frac{(1+\lambda)}{v}|\alpha_{\rm I}|^2$$

$$= \begin{cases} 4\left(\dfrac{\epsilon}{4}\right)^{2k} v^{2k+1}\dfrac{(2k+1)^2}{1+\lambda}|C_k(-(2\epsilon^2-\mu^2)/2\epsilon v)|^2, & k>0 \quad (60a)\\[2ex] 4\left(\dfrac{\epsilon}{4}\right)^{2|k|} v^{2|k|-1}|C_{|k|-1}(-(2\epsilon^2-\mu^2)/2\epsilon v)|^2(1+\lambda), & k<0. \quad (60b)\end{cases}$$

As we have

$$|\Gamma(l+1+i\eta)|^2 = \left[\sum_{s=1}^{l}(s^2+\eta^2)\right]\frac{\pi\eta}{\sinh(\pi\eta)}, \tag{61}$$

we find that

$$|C_l(\eta)|^2 = \frac{2^{2l+1}(\pi\eta)}{(2l+1)!^2(1-e^{-2\pi\eta})}\prod_{s=1}^{l}(s^2+\eta^2). \tag{62}$$

We therefore have the following expressions for the transmission coefficients:

$$(T_S)_l \approx \frac{\pi l!^4 2^{2l+2}\omega(1+v^2)\omega^{2l+2}v^{2l}}{(2l)!^4(2l+1)^2\{1-\exp[-\pi\omega(1+v^2)/v]\}} \prod_{s=1}^{l}\left[s^2+\left(\frac{\omega(1+v^2)}{2v}\right)^2\right], \tag{63}$$

$$(T_D)_k = \begin{cases} \dfrac{\pi\epsilon^{2k+1}v^{2k}(1+v^2)\prod_{s=1}^{k}\{s^2+[\epsilon(1+v^2)/2v]^2\}}{2^{2k-2}(1+\lambda)(2k!)^2\{1-\exp[-\pi\epsilon(1+v^2)/v]\}}, & k>0 \quad (64)\\[2ex] \dfrac{\pi\epsilon^{2|k|+1}v^{2|k|-2}(1+\lambda)(1+v^2)\prod_{s=1}^{|k|-1}\{s^2+[\epsilon(1+v^2)/2v]^2\}}{2^{2|k|}(2|k|-1)!\{1-\exp[-\pi\epsilon(1+v^2)/v]\}}, & k<0. \quad (65)\end{cases}$$

The absorption cross section is given by the ratio of the total number of particles absorbed by the black hole to the incoming flux of particles. For scalar waves we will assume that the incoming wave is a plane wave which near infinity goes as

$$\phi = e^{-i\omega t}e^{-i\omega v z}. \tag{66}$$

The current for scalar particles is given by

$$J^\mu = \tfrac{1}{2}i\,\phi^*\overleftrightarrow{\partial}^\mu\phi\ . \tag{67}$$

For the above wave function this leads to a current in the z direction of

$$|J^z| = \omega v. \tag{68}$$

The number of particles absorbed by the black hole per unit time will be given by

$$N = -\int_S \sqrt{-g}\,J^r d\mathcal{S}, \tag{69}$$

where $\mathcal{S}$ is a surface of constant radius surrounding the black hole. Now J^r is given by

$$J^r = -\tfrac{1}{2}i(1-2M/r)\phi^*\frac{\overleftrightarrow{\partial}}{\partial r}\phi$$

$$= -\tfrac{1}{2}i\phi^*\frac{\overleftrightarrow{\partial}}{\partial r^*}\phi. \tag{70}$$

Any ϕ which is a solution to the wave equation in the Schwarzschild metric can be written as a sum of spherical modes,

$$\phi = e^{-i\omega t}\sum_{l\tilde{m}} K_{l\tilde{m}}f_l(r)Y_{l\tilde{m}}(\theta). \tag{71}$$

We therefore obtain

$$N = -i\sum_{l\tilde{m}}\frac{|K_{l\tilde{m}}|^2}{2}r^2\left(f_l^*\frac{\partial}{\partial r^*}f_l - f_l\frac{\partial}{\partial r^*}f_l^*\right)$$

$$= +i\sum_{l\tilde{m}}\frac{|K_{l\tilde{m}}|^2}{2}(W_S)_l, \tag{72}$$

where $(W_S)_l$ is given in Eq. (57b). By the way we have defined f_l (such that the incoming part of f has unit amplitude), we have

$$\sigma = \frac{N}{\omega v} = \sum_{l\tilde{m}}\frac{|K_{l\tilde{m}}|^2(W_S)_l}{-i2\omega v}$$

$$= \sum_{l\tilde{m}}|K_{l\tilde{m}}|^2(T_S)_l. \tag{73}$$

To determine the cross section, we need only determine the coefficients $K_{l\tilde{m}}$.

For large r, we can expand[4] the ingoing part of $e^{-i\omega z}$ as

$$e^{-i\omega v z} = \sum_{l=0}^{\infty}\frac{i^l e^{-i\omega v r}}{2\omega v r}[(4\pi)(2l+1)]^{1/2}Y_{l0}(\theta,\phi). \tag{74}$$

Therefore the coefficients $K_{l\tilde{m}}$ are given by

$$K_{l\tilde{m}} = \frac{i^l}{2\omega v}[4\pi(2l+1)]^{1/2}\delta_{\tilde{m}0}\ . \tag{75}$$

From (73) we therefore obtain

$$\sigma_S = \sum_{l=0}^{\infty}\frac{\pi}{\omega^2 v^2}(2l+1)(T_S)_l. \tag{76}$$

As $(T_S)_l$ goes as ω^{2l+2} and as $\omega \ll 1$ by assumption, the only term which will contribute to the cross section will be the $l=0$ term. We therefore obtain

$$\sigma_S(\omega, v) = \frac{(2\pi)^2(1+v^2)\omega}{v^2(1-\exp[-\pi\omega(1+v^2)/v])}. \tag{77}$$

If we replace the factors of $2M$ which had been set equal to 1, this becomes

$$\sigma_S = \frac{(4\pi M)^2(1+v^2)(2M\omega)}{v^2(1-\exp[-2\pi M\omega(1+v^2)/v])}\ . \tag{78}$$

A similar analysis can be done for Dirac particles, with the presence of the spinors adding some complications. If ψ is a solution to the spinor equations, then the current is given by

$$J^{\mu}=\bar{\psi}\gamma^{\mu}\psi, \tag{79}$$

where $\bar{\psi}$ is the Dirac adjoint and the γ^{μ} are the Dirac matrices.

Assume that the form of the wave traveling toward the black hole from infinity is

$$\psi=(e^{-i\epsilon t}e^{-i\omega vz})S, \tag{80}$$

where S is a constant spinor obeying

$$-i\epsilon(\gamma^0 S+v\gamma^3 S)+i\mu S=0. \tag{81}$$

The representation for the γ matrices, which is implicit in the solutions for the radial modes of the Dirac field in Eq. (4), is

$$\gamma^0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \gamma^i=\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix}. \tag{82}$$

Any spinor S obeying (81) is a linear combination of two independent components,

$$S^+=\begin{bmatrix}\left(\frac{1+\lambda}{1-\lambda}\right)^{1/4}\\0\\-\left(\frac{1-\lambda}{1+\lambda}\right)^{1/4}\\0\end{bmatrix},\quad S^-=\begin{bmatrix}0\\\left(\frac{1+\lambda}{1-\lambda}\right)^{1/4}\\0\\\left(\frac{1-\lambda}{1+\lambda}\right)^{1/4}\end{bmatrix}. \tag{83}$$

For both of these spin components the flux in the z direction is

$$\bar{S}^+\gamma^3 S^+=\bar{S}^-\gamma^3 S^-=-2. \tag{84}$$

Now the ingoing part of the spinor function $S^+e^{-i\epsilon z}$ can be expressed in radial modes for large r as

$$S^+e^{-i\epsilon z}=S^+\sum_{l=0}^{\infty}\frac{i^l[4\pi(2l+1)]^{1/2}e^{-i\epsilon vr}}{2i\epsilon vr}\,Y_{l0}(\theta,\phi). \tag{85}$$

Recalling that our radial mode solutions [Eq. (4)] were defined so that $G_k(r)\approx(e^{-i\omega vr}+\text{outgoing portion})$ near infinity, we can now write the solution ψ^+, which goes as $S^+e^{-i\epsilon t}e^{-i\epsilon vz}$ before interacting with the black hole, in terms of the radial modes $\psi_{\epsilon k 1/2}$ as

$$\psi^+=\sum_k\frac{i^k\sqrt{\pi}}{\epsilon v}\left(\frac{1+\lambda}{1-\lambda}\right)^{1/4}(|k|)^{1/2}\psi_{\epsilon k\,1/2}\;. \tag{86}$$

Similarly we define ψ^- as the solution which has an incoming part going as $S^-e^{-i\epsilon t}e^{-i\epsilon vz}$, and we obtain

$$\psi^-=\sum_k\frac{i^k\sqrt{\pi}}{\epsilon v}\left(\frac{1+\lambda}{1-\lambda}\right)^{1/4}\frac{(|k|)^{3/2}}{k}\psi_{\epsilon k\,-1/2}. \tag{87}$$

To determine the cross section, we need now only determine the value of terms of the form

$$\int r^2\bar{\psi}_{\epsilon k\tilde{m}}\gamma^r\psi_{\epsilon k'\tilde{m}'}\,d\cos\theta\,d\phi\;,$$

where γ^r is given by

$$\gamma^r=(1-1/r)^{1/2}(\sin\theta\cos\phi\,\gamma^1+\sin\theta\sin\phi\,\gamma^2+\cos\theta\,\gamma^3).$$

Using the expressions for the γ^i and for the radial modes $\psi_{\epsilon k\tilde{m}}$ this becomes

$$-\delta_{kk'}\delta_{\tilde{m}\tilde{m}'}(G_k^*F_k-F^*G). \tag{88}$$

But we have [from (4b)]

$$F_k=\frac{i}{\epsilon}\left[\frac{dG_k}{dx}+\frac{k\rho}{(1+\lambda\rho)r}G_k\right]. \tag{89}$$

So the above expression (88) becomes

$$\frac{-i}{\epsilon}\delta_{kk'}\delta_{\tilde{m}\tilde{m}'}\left(G_k^*\frac{dG_k}{dx}-G_k\frac{dG_k^*}{dx}\right)=\delta_{kk'}\delta_{\tilde{m}\tilde{m}'}\left[\frac{-2v}{1+\lambda}(T_D)_k\right]. \tag{90}$$

The current flows into the black hole for ψ^+, ψ^- are then given by

$$\begin{aligned}N_r^+&=\int_S\bar{\psi}^+\gamma^r\psi^+d\mathcal{S}\\&=-2\sum_k\frac{\pi}{(\epsilon v)^2}|k|(T_D)_k,\\N_r^-&=\int_S\bar{\psi}^-\gamma^r\psi^-d\mathcal{S}\\&=-2\sum_k\frac{\pi}{(\epsilon v)^2}|k|(T_D)_k.\end{aligned} \tag{91}$$

Therefore the cross section for either helicity state + or − on a Schwarzschild black hole is given by

$$\sigma_D=\frac{\pi}{(\epsilon v)^2}\sum_k|k|(T_D)_k. \tag{92}$$

An examination of $(T_D)_k$ shows that only $k=+1$ or $k=-1$ will contribute to the cross section for $\epsilon\ll 1$:

$$\begin{aligned}(T_D)_1&=\frac{\pi\epsilon^3v^2(1+v^2)(1+[\epsilon(1+v^2)/2v]^2)}{4(1+\lambda)(1-\exp[-\pi\epsilon(1+v^2)/v])}\;,\\(T_D)_{-1}&=\pi\frac{\epsilon^3}{4}\frac{(1+\lambda)(1+v^2)}{1-\exp[-\pi\epsilon(1+v^2)/v]}.\end{aligned} \tag{93}$$

We finally obtain

$$\sigma_D(\epsilon,v)=\frac{\pi^2}{2v^2}\frac{\epsilon(1+v^2)}{1-\exp[-\pi\epsilon(1+v^2)/v]}=\tfrac{1}{8}\sigma_S(\epsilon,v). \tag{94}$$

The Dirac absorption cross section is exactly $\frac{1}{8}$ of the scalar absorption cross section for all velocities v under the approximation scheme used here. Furthermore, the scalar cross section comes only from absorption of the $l=0$ wave, whereas the Dirac cross section results from a combination of the two $j=\frac{1}{2}$ $(l=0,1)$ modes.

It is instructive to compare the quantum absorption cross section with the classical result in which the particles are treated as point particles. For that case one obtains[5]

$$\sigma_C=\frac{\pi M^2}{v^2}\left\{\frac{[8(1-v^2)]^3}{4(1-4v^2+(1+8v^2)^{1/2})(3-(1+8v^2)^{1/2})^2}\right\}. \tag{95}$$

The quantity in curly brackets is a slowly varying function of v going from 16 for $v=0$ to 27 for $v=1$.

This is also the cross section in the high-energy limit $(\omega,\epsilon\gg 1)$ for scalar and Dirac particles.[6] The cross section essentially goes as $1/v^2$.

For the low-energy scalar and Dirac particles $(\omega,\epsilon\ll 1)$ one finds that the behavior of the cross section as a function of v, the velocity at infinity, is quite different. Taking the scalar cross section (as the Dirac cross section is exactly $\frac{1}{8}$ of this) we recall that

$$\sigma_S=\frac{(4\pi M)^2(1+v^2)(2Mm)}{v^2(1-v^2)^{1/2}\{1-\exp[-2\pi Mm(1+v^2)/v(1-v^2)^{1/2}]\}}. \tag{96}$$

For low velocities $(v\lesssim 2\pi Mm)$ the cross section is given by

$$\sigma_S\approx\frac{(4\pi M)^2(2Mm)}{v^2}. \tag{97}$$

The cross section goes as $1/v^2$ as for σ_C but is suppressed by a factor of Mm.

However, as v becomes greater than $2\pi Mm$, we have

$$1-\exp\{-2\pi Mm(1+v^2)/[v(1-v^2)]\}\approx 2\pi Mm(1+v^2)/[v(1-v^2)], \tag{98}$$

and the cross section becomes

$$\sigma_S\approx\frac{\pi(4M)^2}{v}\approx\frac{16\pi M^2}{v}. \tag{99}$$

This holds up to velocities such that $1-v^2\approx 2\pi Mm$, at which point the approximation that the wavelength is larger than the Schwarzschild radius breaks down (i.e., $2M\omega\sim 1$). Note that for $v\approx 1$, this cross section (99) becomes $16\pi M^2$, as compared with the classical cross section of $27\pi M^2$.

The behavior of the various cross sections is indicated in Fig. 1, where the relative contributions of the S and P waves $(j=\frac{1}{2},\ l=0,1)$ to σ_D are also indicated. It is only for high velocities that the $l=1$ (P wave) state contributes.

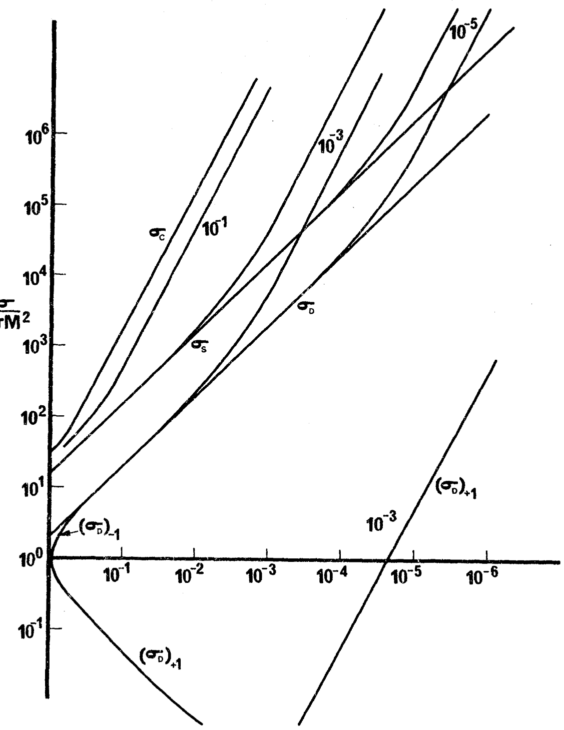

FIG. 1. Absorption cross sections for small black holes vs velocity for various values of $2Mm$ [σ_C for classical, σ_S for scalar particle, σ_D for Dirac particle, $(\sigma_D)_{+1}$, $(\sigma_D)_{-1}$ for P_1, S_1 contributions].

Another interesting point is that for $v\approx 1$, the various transmission coefficients [Eq. (65)] for Dirac particles are comparable for waves with equal $|k|$ (i.e., equal angular momentum) and not for equal l (total orbital angular momentum). This is a striking illustration of the effect of gravitational spin-orbit coupling on the motion of a particle with quantum spin.

The fact that both σ_D and σ_S are comparable to the classical cross section for $v\approx 1$ means that naive arguments about particles not being able to be absorbed by a black hole if their Compton wavelengths are much larger than the Schwarzschild radius are wrong. All particles with $v\approx 1$ see a black hole as having roughly the same size, whether they are quantum particles or classical particles.

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