S-matrix treatment of many overlapping resonances*

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We present a general S-matrix approach to the phenomenological treatment of scattering problems which involve many overlapping resonances in the presence of a constant background. The problem is treated systematically using the analyticity and unitarity of $S(E)$, and CPT and T invariance. We derive the possible factorization properties of the resonance residues for first-order and higher-order poles, the restriction imposed by CPT invariance, the restrictions imposed by CPT and T, and the constraints on the residues imposed by unitarity. The nature of the contraints is investigated in detail for overlapping first-order resonances, and in less detail for higher-order resonances. The results are illustrated by several examples, including the case of a single dipole resonance in the presence of background.

I. INTRODUCTION

The problem of overlapping resonances in scattering problems has been considered by many authors using a variety of methods. The subject has a long history in nuclear physics.[1] It has been of somewhat less general interest in particle physics, but has nevertheless been important in connection with such phenomena as ρ-ω mixing,[2,3] the mixing of the K_L and K_S systems,[4-10] and attempts to explain what was thought to be a split structure in the A_2 meson in terms of overlapping resonances or, alternatively, dipole resonances.[11-13] The recent discovery of a rich array of 1^- resonances in the 4.2-GeV region in e^+e^- annihilation into hadrons[14] suggests that the treatment of overlapping resonances may again become a subject of considerable interest.

The methods which have been used to treat overlapping resonances include the R-matrix approach[1] (mainly in nuclear theory), the Weisskopf-Wigner[15] or mass-matrix method,[9-12,16] propagator methods in field theory,[17,18] the K-matrix method,[19,20] and methods based purely on the S matrix.[4-8,13,16,21] All methods have advantages and disadvantages. However, for problems in which threshold effects are unimportant, and any background scattering may be taken as constant over the resonance region, the S-matrix method is probably the simplest. It has the considerable advantage of conceptual simplicity, and provides a parametrization of the resonances and background in terms of a minimum number of parameters—the background S matrix, the complex resonance energies, and the residues at the resonance poles (partial decay amplitudes). It is not necessary to make any assumptions about the underlying dynamics, for example, the assumptions about the existence of an unperturbed Hamiltonian with bound states, and the form of the perturbation which induces their decay, which appear in the Weisskopf-Wigner approach. In addition, the parameters which appear in the S-matrix approach have direct experimental significance, as opposed, say, to the locations and residues of R-matrix or K-matrix poles. It is nevertheless possible to carry out complete phenomenological analyses of such problems as mixing of resonances in the K_L-K_S (see Refs. 5 and 6) and ρ-ω[19] systems, and to reproduce the conventional mass-matrix phenomenology in purely S-matrix language. The point of view and details of the calculations, and the way some physical ideas are introduced, are of course different from the usual mass-matrix approach, and the methods are in this sense complementary.

It seems worthwhile for the foregoing reasons to give a systematic treatment of the S-matrix approach to the problem of many overlapping resonances, including the relatively unfamiliar but potentially interesting subject of higher-order poles in the S matrix.[11,12,22-24] We present such a treatment in this paper. Most of the individual results are of course known, if not always familiar, but with the exception of those for the case of two first-order poles with[21] or without[13] background, have apparently not been presented before in the present systematic fashion. Our general results on higher-order poles appear to be new.

In Sec. II of this paper, we use the general constraints imposed by the analyticity and unitarity of the S matrix to determine the most general form of the S matrix for N overlapping resonances in the presence of a constant background, and determine the constraints on the resonance parameters imposed by unitarity. In Sec. II A, we study the factorization properties of the resonance residues for first-order poles, and derive the constraints on the residues imposed by CPT invariance, and the combination of CPT and T invariance. In Sec. II B, we examine the unitarity constraints for many overlapping first-order resonances in detail. We derive the properties of a

matrix $\underline{\alpha}$, which describes the overlap of different resonances, and give sum rules and bounds for the partial decay widths. We generalize these results to the case of higher-order poles in Sec. II C. Finally, we use our general results in Sec. III to treat some simple examples involving first-order poles (Sec. III A) and second-order poles or dipoles (Sec. III B).

II. S-MATRIX TREATMENT OF MANY OVERLAPPING RESONANCES

A. General form of S for narrow resonances

1. General considerations

We will be concerned in the present section with the structure of the partial-wave S matrix in the situation in which there are N overlapping resonances in the range of energy which is of interest. We will assume throughout the discussion that the resonances are sufficiently narrow that any nonresonant background can be taken as constant over this energy range. We also neglect possible threshold effects associated with the opening of new channels in the resonance region.[1,19] The S matrix for the N resonances in the presence of background is then well approximated by an expression of the form

$$S = B + \sum_{n=1}^{N} \frac{R_n}{E - \mathcal{E}_n} , \tag{1}$$

where the background matrix B, the residue matrices R_n, and the complex resonance energies $\mathcal{E}_n$,

$$\mathcal{E}_n = E_n - i\Gamma_n/2 , \tag{2}$$

are all constants independent of the total energy E.[25] We will assume initially that the $\mathcal{E}_n$ are all distinct, and we will return to the questions of coincident and higher-order poles in Sec. II C.

The S matrix acts in the space of the open channels. When necessary, we will denote a given channel by a Greek index which will include continuous variables (subenergies, momenta, and angles) in the case of multiparticle channels. Sums over Greek indices are to be understood to include integrals over the corresponding phase space for multiparticle channels.

The conditions imposed on the S matrix by unitarity and causality are most conveniently discussed using the continued unitarity relation[20,26,27]

$$S^\dagger(E^*)S(E) = S(E)S^\dagger(E^*) = \underline{1} . \tag{3}$$

This is simply the analytic continuation to complex E of the familiar unitarity relation for real E, $S^\dagger(E)S(E) = S(E)S^\dagger(E) = \underline{1}$. All of our results can be derived somewhat less compactly using the latter.[13,21] We note also that the requirement of causality restricts the poles of $S(E)$ to the lower half of the energy plane for resonance poles, or to the real axis below the lowest threshold for bound-state poles.[1,26] Thus, all the resonance energies $\mathcal{E}_n$ in (1) must lie in the lower half plane, and the Γ_n are necessarily positive, $\Gamma_n > 0$. Correspondingly, $S^\dagger(E^*)$ can have no poles for E in the lower half plane.

2. Factorization of the residues at isolated poles

We can use the constraints imposed by unitarity and causality to demonstrate the familiar theorem, that the residues of the isolated poles of $S(E)$ factor.[1,27-29] The unitarity relation (3) can be written compactly for $S(E)$ of the form given in (1) as

$$S^\dagger(E^*)S(E) = B^\dagger B + \sum_{n=1}^{N} \frac{S^\dagger(\mathcal{E}_n^*)R_n}{E - \mathcal{E}_n} + \sum_{n=1}^{N} \frac{R_n^\dagger S(\mathcal{E}_n^*)}{E - \mathcal{E}_n^*} . \tag{4}$$

(This expansion can be obtained trivially by noting that the function $[S^\dagger(E^*)S(E) - B^\dagger B]$ is a meromorphic function of E which vanishes for $|E| \to \infty$. It can therefore be written as a sum of the contributions of its poles, with the constant residues indicated.) Since the matrix $S^\dagger(E^*)S(E) = \underline{1}$ has no poles, the residues of the apparent poles at $E = \mathcal{E}_n$ and $E = \mathcal{E}_n^*$ must vanish,[30]

$$S^\dagger(\mathcal{E}_n^*)R_n = \underline{0}, \quad R_n^\dagger S(\mathcal{E}_n^*) = \underline{0}, \quad n = 1, \ldots, N . \tag{5}$$

(The second equation is just the Hermitian conjugate of the first.) The residues are independent of E, so the pole terms vanish identically, and we conclude also that

$$B^\dagger B = \underline{1} . \tag{6}$$

Consider the first of the equations in (5). We will assume for the moment that S is a finite matrix of dimension $\mathfrak{N}$ ($\mathfrak{N}$ two-body channels). The unitarity relations (5) then lead to weak constraints on the ranks of the finite matrices R_n and $S^\dagger(\mathcal{E}_n^*)$. In particular, the matrix $S^\dagger(\mathcal{E}_n^*)R_n$ is a null matrix, hence has rank $\rho[S^\dagger(\mathcal{E}_n^*)R_n] = 0$, and a null-space of dimension $\nu[S^\dagger(\mathcal{E}_n^*)R_n] = \mathfrak{N}$. For general finite matrices, $\rho(A) = \mathfrak{N} - \nu(A)$ and $\nu(AB) \leq \nu(A) + \nu(B)$,[31] hence $\rho(AB) \geq \rho(A) + \rho(B) - \mathfrak{N}$. We conclude that the ranks of $S^\dagger(\mathcal{E}_n^*)$ and R_n are bounded by

$$0 = \rho[S^\dagger(\mathcal{E}_n^*)R_n] \geq \rho[S^\dagger(\mathcal{E}_n^*)] + \rho(R_n) - \mathfrak{N} . \tag{7}$$

Let the rank of $S^\dagger(\mathcal{E}_n^*)$ be $\mathfrak{N} - r$. Then

$$\rho(R_n) \leq r . \tag{8}$$

The case $r = 0$ is trivial, since a matrix of rank zero is null, and S would contain no resonances. Thus, $r \geq 1$. For a general rank s, R_n can be

written as[31]

$$R_n = \sum_{i=1}^{s} \alpha_n^i \beta_n^i \ . \tag{9}$$

The structure in (9) corresponds for $s=1$ to the presence of a single isolated pole in S at $E=\mathcal{E}_n$ with a factored residue.[32] For $s>1$, the structure is that of s poles with factored residues, all with the resonance energy $E=\mathcal{E}_n$. We will henceforth disregard this possibility of exactly coincident resonances (though it can be recovered easily later), and will assume that all poles are isolated. Then $s=1$, $r\geq 1$, and $\nu[S^\dagger(\mathcal{E}_n^*)]\geq 1$.[33] The matrices R_n factor, and S can be written in the form

$$S(E)=B-i\sum_{n=1}^{N}\frac{\gamma_n\tilde{\lambda}_n}{E-\mathcal{E}_n} \ . \tag{10}$$

Here $\gamma_n\tilde{\lambda}_n$ is an $\mathfrak{N}\times\mathfrak{N}$ matrix formed by taking the product of a column matrix γ_n with elements $\gamma_{n\alpha}$ with a row matrix $\tilde{\lambda}_n$ with elements $\lambda_{n\beta}$, $\alpha,\beta=1,\ldots,\mathfrak{N}$. The factor $-i$ in (10) is introduced in accordance with the usual conventions. Note that the quantities $|\gamma_{n\alpha}|^2$ and $|\lambda_{n\alpha}|^2$ are just the partial widths for decay and production of the nth resonance in the channel labeled by α.

Although we have demonstrated the factorization of the residues only for $\mathfrak{N}$ finite, a channel with continuous internal indices (a three- or more-body channel) can always be approximated by a finite set of channels with discrete indices, for which the factorization theorem holds. It is therefore clear physically (but perhaps not mathematically) that factorization will continue to hold in the continuous limit. Alternative proofs of factorization based on specific dynamical schemes, for example, the Schrödinger equation, are given elsewhere.[1,20,26,28,29]

3. *Discrete symmetries*

We will suppose in the present section that CPT is an exact symmetry, as proved for Lorentz covariant, causal field theories,[34] and we will examine the implications of CPT invariance, and later T invariance, for the general parametrization of the many-resonance S matrix given in (10).[5,6] The arguments are simplest if we choose for the channel states $\{\alpha\}$ eigenstates of the CP operation with eigenvalues ± 1. This can always be done. We will divide the channels into two groups with CP eigenvalues $+1$ and -1, and write S as a matrix in the $+-$ space

$$S=\begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} \ , \tag{11}$$

where the component matrices $S_{\pm\pm}$ act in the channel space.

The S matrix is invariant under the CPT transformation. However, the element $S_{\alpha\beta}$ transforms under the antiunitary CPT operation as

$$S_{\alpha\beta}\xrightarrow[CPT]{}\eta_\alpha\eta_\beta S_{\hat{\beta}\hat{\alpha}} \ , \tag{12}$$

where $\eta_\alpha,\eta_\beta=\pm 1$ are the CP eigenvalues of the channels α,β, and $\hat{\beta},\hat{\alpha}$ label the channels obtained by applying the time-reversal operation to α and β. The time-reversal operation involves the reversal of magnetic quantum numbers. However, if we use the helicity representation for S,[35] or, alternatively, organize S appropriately and omit the magnetic quantum numbers from the channel labels α,β, the labels $\alpha,\hat{\alpha}$ and $\beta,\hat{\beta}$ will be equivalent, and (12) reduces for CPT-invariant systems to the statement that

$$S_{\alpha\beta}=\eta_\alpha\eta_\beta S_{\beta\alpha}=\pm S_{\beta\alpha} \ , \tag{13}$$

where the sign is + for α and β in the same CP sector, and − for α and β in opposite sectors. Thus, from (11) and (13),

$$\tilde{S}_{++}=S_{++},\quad \tilde{S}_{--}=S_{--},\quad \tilde{S}_{+-}=-S_{-+},\quad \tilde{S}_{-+}=-S_{+-} \ . \tag{14}$$

The relations in (14) must hold separately for the constant background matrix B and the n energy-dependent pole terms in (10). We conclude that

$$\tilde{B}_{++}=B_{++},\quad \tilde{B}_{--}=B_{--},\quad \tilde{B}_{+-}=-B_{-+},\quad \tilde{B}_{-+}=-B_{+-} \ , \tag{15}$$

and that

$$\gamma_{n\alpha}\lambda_{n\beta}=\pm\gamma_{n\beta}\lambda_{n\alpha} \ . \tag{16}$$

The last relation can be written as

$$\gamma_{n\alpha}/\lambda_{n\alpha}=\pm\gamma_{n\beta}/\lambda_{n\beta} \ . \tag{17}$$

In this form, it is clear that the ratios $|\gamma_{n\alpha}/\lambda_{n\alpha}|$ are the same for all channels. We will fix the ratios by requiring the conventional normalization for elastic channels, $|\lambda_{n\alpha}|=|\gamma_{n\alpha}|$ (equivalent production and decay widths). Then $\gamma_{n\alpha}/\lambda_{n\alpha}=\pm 1$ for all channels with CP even, and $\gamma_{n\alpha}/\lambda_{n\alpha}=\mp 1$ for all channels with CP odd, where the choice of the overall sign is still free. We can use this information to write the vectors γ_n and λ_n and the matrix $\gamma_n\tilde{\lambda}_n$ in a notation corresponding to (10),[5,6]

$$\gamma_n=\begin{pmatrix}\gamma_{n+}\\ \gamma_{n-}\end{pmatrix},\quad \lambda_n=\begin{pmatrix}\pm\gamma_{n+}\\ \mp\gamma_{n-}\end{pmatrix}, \tag{18}$$

$$\gamma_n\tilde{\lambda}_n=\pm\begin{pmatrix}\gamma_{n+}\tilde{\gamma}_{n+} & -\gamma_{n+}\tilde{\gamma}_{n-}\\ \gamma_{n-}\tilde{\gamma}_{n+} & -\gamma_{n-}\tilde{\gamma}_{n-}\end{pmatrix} \ . \tag{19}$$

The additional assumption of T invariance leads to the requirement that S be symmetric, $\tilde{S}=S$, or $\tilde{S}_{++}=S_{++}$, $\tilde{S}_{--}=S_{--}$, $\tilde{S}_{+-}=S_{-+}$, and $\tilde{S}_{-+}=S_{+-}$. The last two conditions and the CPT relations in (14)

require that $S_{+-}=S_{-+}=0$. Equivalently, if we require CP and CPT invariance, S_{+-} and S_{-+} must vanish, and S_{++} and S_{--} must be symmetric. The extra requirements and the results in (19) imply that either γ_{n+} or γ_{n-} must vanish for a given resonance. The relation (16) thus becomes trivial, and we can choose the overall phase [the $\pm$ sign in (19)] independently for the + and − sectors. With the conventional choice of the + sign, S may be written for either sector as

$$S=B-i\sum_{n=1}^{N}\frac{\gamma_n\tilde{\gamma}_n}{E-\mathcal{E}_n}\,, \tag{20}$$

where $\tilde{B}=B$, and the sum includes only resonances of a definite CP. The further imposition of P invariance, with the channel states now taken as eigenstates of both CP and P (or C and P separately), eliminates transitions between channels with opposite P, but does not change the form of S given in (2) for transitions between channels with the same parity.

The general form of S assuming only CPT invariance was used by McVoy[5] and by McVoy and this author[6] for the case of two overlapping resonances to give an analysis of the K_L-K_S system based purely on S-matrix considerations. This is the only system known at present for which the general formulation of the overlapping resonance problem is necessary. We will therefore restrict our attention in the remainder of the paper to T- and CPT-invariant interactions, and will use the form of S given in (20). Note that this does not assume P invariance, so is applicable to the potentially interesting case of overlapping resonances with parity-violating decays. The results in the following sections are easily generalized using similar methods to the case of T-violating interactions. Details are given in Refs. 5 and 6.

B. Unitarity constraints on S

1. Constraints on B and γ_n

We have so far used the unitarity constraints (5) on $S(E)$ only to establish that the residue matrices R_n factor for isolated poles, that $S^\dagger(\mathcal{E}_n^*)$ has a null-space of dimension $\nu[S^\dagger(\mathcal{E}_n^*)]\geq 1$, that is, that $S(\mathcal{E}_n^*)$ has $\nu\geq 1$ zero eigenvalues, and that the background matrix B is unitary,

$$B^\dagger B=BB^\dagger=\underline{1}\,. \tag{21}$$

These conditions are necessary, but not sufficient, for the full matrix $S(E)$ to be unitary. We must require also that (5) be satisfied as a matrix equation. This can be restated using the factored form of R_n as the condition that

$$S^\dagger(\mathcal{E}_n^*)\gamma_n\tilde{\gamma}_n=\underline{0},\quad n=1,\ldots,N\,. \tag{22}$$

Thus $[S^\dagger(\mathcal{E}_n^*)\gamma_n]_\alpha\gamma_{n\beta}=0$ for all choices of the channel indices α and β. If γ_n is nontrivial, (22) is equivalent to the set of vector equations

$$S^\dagger(\mathcal{E}_n^*)\gamma_n=B^\dagger\gamma_n+i\sum_{m=1}^{N}\frac{\gamma_m^*(\gamma_m^\dagger\gamma_n)}{\mathcal{E}_n-\mathcal{E}_m^*}=0,\quad n=1,\ldots,N\,. \tag{23}$$

These equations must hold for all n and for every channel.

The equations of constraint which arise from the second of the equations (5) are equivalent by Hermitian conjugation to (23). The equations which arise from considering the unitarity equation in the second of the forms given in (3) can also be reduced to (23) by using Hermitian conjugation and the symmetry of $B, \tilde{B}=B$. The entire phenomenological content of unitarity is thus expressed in the factored form of the resonance residues, (20), and by the equations of constraint, (21) and (23).

2. Reduction of the equations of constraint

It will be convenient to use (23) in a slightly different form obtained by taking the complex conjugate of that equation and using the symmetry of B,

$$B\gamma_n^*=i\sum_m\frac{(\gamma_n^\dagger\gamma_m)}{\mathcal{E}_n^*-\mathcal{E}_m}\gamma_m,\quad n=1,\ldots,N\,. \tag{24}$$

We will define quantities α_{nm} by

$$\alpha_{nm}=\frac{i(\gamma_n^\dagger\gamma_m)}{\mathcal{E}_n^*-\mathcal{E}_m}=\alpha_{mn}^*\,. \tag{25}$$

With this notation the equations of constraint assume the form

$$B^\dagger B=BB^\dagger=\underline{1}\,, \tag{26a}$$

$$B\gamma_n^*=\sum_m\alpha_{nm}\gamma_m\,. \tag{26b}$$

The normalized inner products $(\gamma_n^\dagger\gamma_m)/|\gamma_n||\gamma_m|$ determine the extent to which the various resonances overlap in the production or decay channels, and play the role for S-matrix theory of the overlap $\langle n|m\rangle$ of the right (or left) eigenvectors of the mass matrix in the Weisskopf-Wigner formalism.[10] For $|\gamma_n^\dagger\gamma_m|$ fixed and $|\mathcal{E}_n^*-\mathcal{E}_m|\to\infty$ (widely separated resonances), $\alpha_{nm}\to 0$, and γ_n and γ_m are not directly coupled in (26b). For all resonance spacings large, the γ_n decouple completely.

We can restate some of the constraints in a form which does not depend on B as follows. We first multiply (26b) on the left by $\gamma_l^\dagger$, $l=1,\ldots,N$, to obtain the set of scalar equations

$$\gamma_l^\dagger B\gamma_n^*=\sum_m\alpha_{nm}(\gamma_l^\dagger\gamma_m)$$

$$=i\sum_m(\mathcal{E}_m-\mathcal{E}_l^*)\alpha_{nm}\alpha_{lm}\,. \tag{27}$$

If we then use the identity

$$\gamma_l^\dagger B\gamma_n^* = \gamma_n^\dagger \tilde{B}\gamma_l^* = \gamma_n^\dagger B\gamma_l^* , \tag{28}$$

we find from (27) that

$$\sum_m (\mathcal{E}_m - \mathcal{E}_l^*)\alpha_{nm}\alpha_{lm} = \sum_m (\mathcal{E}_m - \mathcal{E}_n^*)\alpha_{lm}\alpha_{nm} , \tag{29}$$

or that

$$\sum_m (\mathcal{E}_n^* - \mathcal{E}_l^*)\alpha_{nm}\alpha_{lm} = 0 . \tag{30}$$

Since there are assumed to be no coincident poles, $\mathcal{E}_n^* - \mathcal{E}_l^* \neq 0$, $l \neq n$, and

$$\sum_m \alpha_{nm}\alpha_{lm} = 0, \quad l \neq n . \tag{31}$$

If we take the complex conjugate of (31) and use (25), we find also that

$$\sum_m \alpha_{mn}\alpha_{ml} = 0, \quad l \neq n . \tag{32}$$

We can obtain a second set of relations independent of B by multiplying (26b) by its Hermitian conjugate and using (26a),

$$\tilde{\gamma}_n\gamma_n^* = \sum_{l,m} \alpha_{nl}^*\alpha_{nm}\gamma_l^\dagger\gamma_m . \tag{33}$$

If we note that $\alpha_{nl}^* = \alpha_{ln}$, and that $\tilde{\gamma}_n\gamma_n^* = \gamma_n^\dagger\gamma_n$, and use the definition (25), we find that

$$\begin{aligned}(\mathcal{E}_n^* - \mathcal{E}_n)\alpha_{nn} &= \sum_{l,m} (\mathcal{E}_l^* - \mathcal{E}_m)\alpha_{ln}\alpha_{nm}\alpha_{lm} \\ &= \sum_{l,m} (\mathcal{E}_l^* - \mathcal{E}_n^*)\alpha_{ln}\alpha_{nm}\alpha_{lm} \\ &\quad + \sum_{l,m} (\mathcal{E}_n^* - \mathcal{E}_m)\alpha_{ln}\alpha_{lm}\alpha_{nm} .\end{aligned} \tag{34}$$

The first sum vanishes when summed on m for $l \neq n$ by (31), and is identically zero for $l = n$. The second sum vanishes by (32) when summed on l unless $m = n$. We are therefore left with the equation

$$(\mathcal{E}_n^* - \mathcal{E}_n)\alpha_{nn} = (\mathcal{E}_n^* - \mathcal{E}_n)\alpha_{nn}\sum_l \alpha_{ln}\alpha_{ln} . \tag{35}$$

Since

$$(\mathcal{E}_n^* - \mathcal{E}_n)\alpha_{nn} = i\gamma_n^\dagger\gamma_n \neq 0 ,$$

$$\sum_l \alpha_{ln}\alpha_{ln} = \sum_l \alpha_{nl}\alpha_{nl} = 1 . \tag{36}$$

The B-independent equations obtained by multiplying $B\gamma_n^*$ by $(B\gamma_l^*)^\dagger$ for $l \neq n$ give no new information.

The background-independent equations (31), (32), and (36) can be summarized by noting that the matrix $\underline{\alpha}$ formed from the scalar parameters α_{nm} must be orthogonal, a result derived by quite different methods in Refs. 1 and 10,

$$\underline{\alpha}\underline{\tilde{\alpha}} = \underline{\tilde{\alpha}}\underline{\alpha} = \underline{1} . \tag{37}$$

From (25), $\underline{\alpha}$ is also Hermitian,

$$\underline{\alpha}^\dagger = \underline{\alpha} . \tag{38}$$

We note finally that the Schwarz inequality for the inner product of the vectors γ_n and γ_m,

$$|\gamma_n^\dagger\gamma_m|^2 \leq |\gamma_n^\dagger\gamma_n||\gamma_m^\dagger\gamma_m| , \tag{39}$$

leads to a set of inequalities for the matrix $\underline{\alpha}$,

$$|\mathcal{E}_n^* - \mathcal{E}_m|^2|\alpha_{nm}|^2 \leq \Gamma_n\Gamma_m\alpha_{nn}\alpha_{mm} . \tag{40}$$

Since $|\mathcal{E}_n^* - \mathcal{E}_m|^2 > \Gamma_n\Gamma_m$, $n \neq m$, the α's also satisfy a weaker set of inequalities,

$$|\alpha_{nm}|^2 < \alpha_{nn}\alpha_{mm} . \tag{41}$$

The diagonal elements of $\underline{\alpha}$ are of course real, $\alpha_{nn} = \gamma_n^\dagger\gamma_n/\Gamma_n$, and are necessarily positive because the Γ_n are positive (causality).

3. *Further deductions and constraints: Positivity and sum rules for the resonance widths*

The matrix $\underline{\alpha}$ is Hermitian, hence can be diagonalized and written in terms of its eigenvalues a_λ and eigenvectors x_λ as

$$\underline{\alpha} = \sum_\lambda a_\lambda x_\lambda x_\lambda^\dagger , \tag{42a}$$

$$\underline{\alpha}x_\lambda = a_\lambda x_\lambda , \tag{42b}$$

$$x_\lambda^\dagger x_{\lambda'} = \delta_{\lambda\lambda'} . \tag{42c}$$

$$\sum_\lambda x_\lambda x_\lambda^\dagger = \underline{1} , \tag{42d}$$

where the a_λ are all real. Since $\underline{\alpha}$ is orthogonal, $\underline{\alpha}\,\underline{\tilde{\alpha}} = \underline{1}$, $\det(\underline{\alpha}\,\underline{\tilde{\alpha}}) = (\det\underline{\alpha})^2 = 1$, none of the a_λ vanish, and $\underline{\alpha}$ has a unique inverse $\underline{\tilde{\alpha}}$ with eigenvalues a_λ^{-1}. However, the fact that $\underline{\alpha}$ is Hermitian implies that $\underline{\alpha}^{-1} = \underline{\tilde{\alpha}} = \underline{\tilde{\alpha}}^\dagger = \underline{\alpha}^*$. The matrix $\underline{\alpha}^*$ has (real) eigenvalues $a_\lambda^* = a_\lambda$ and normalized eigenvectors which differ at most by phases from x_λ^*. We conclude that the sets of eigenvalues $\{a_\lambda^{-1}\}$ and $\{a_\lambda\}$ are identical. As a consequence, the reciprocal of every eigenvalue of $\underline{\alpha}$ is also an eigenvalue. Furthermore, the normalized eigenvector corresponding to a_λ^{-1} differs at most by a phase from the complex conjugate of the eigenvector corresponding to a_λ.

We will supplement this general information about $\underline{\alpha}$ with a physical requirement that the S matrix for N overlapping resonances must go over smoothly to that appropriate for N-independent resonances, each interacting with its own background, if the resonances are widely separated relative to their total widths.[36] More

precisely, we let $|\mathcal{E}_n^* - \mathcal{E}_m| \to \infty$ with $\gamma_n^\dagger \gamma_n$, $\gamma_m^\dagger \gamma_m$, and $|\gamma_n^\dagger \gamma_m|$ bounded from above. In this limit, $\alpha_{nm} \to 0$, $n \neq m$, the different resonances decouple in (26b), and $\alpha_{nn} \to +1$. Thus, $\underline{\alpha}$ approaches the unit matrix as the poles are separated, $a_\lambda \to +1$ for all λ, and $\det \underline{\alpha} = +1$. The determinant is unchanged by any continuous motion of the poles, and in particular, cannot vanish. As a result, none of the (real) eigenvalues a_λ can change sign as we go from the case of N overlapping resonances to that of N independent resonances, and the a_λ must be positive in any configuration, $a_\lambda > 0$, all λ.

The condition $a_\lambda > 0$, all λ, is sufficient to ensure that the Schwarz inequalities for $\underline{\alpha}$ are satisfied in the weak form given in (41). We find from (42) that α_{nm} can be represented as

$$\alpha_{nm} = \sum_\lambda a_\lambda x_{\lambda n} x_{\lambda m}^* . \tag{43}$$

Thus

$$\begin{aligned} |\alpha_{nm}| &\leq \left(\sum_\lambda a_\lambda |x_{\lambda n} x_{\lambda m}^*| \right)^2 \\ &\leq \left(\sum_\lambda a_\lambda |x_{\lambda n}|^2 \right) \left(\sum_\lambda a_\lambda |x_{\lambda m}|^2 \right) \\ &= \alpha_{nn} \alpha_{mm} , \end{aligned} \tag{44}$$

where the second inequality follows from the use of the Schwarz inequality for series

$$\left(\sum_k |A_k B_k| \right)^2 \leq \left(\sum_k |A_k|^2 \right) \left(\sum_k |B_k|^2 \right), \tag{45}$$

with $A_\lambda = \sqrt{a_\lambda}\, x_{\lambda n}$ and $B_\lambda = \sqrt{a_\lambda}\, x_{\lambda m}^*$. However, the positivity of the a_λ is not sufficient to ensure that the Schwarz inequalities for $\underline{\alpha}$ are satisfied in the strong form (40).

We can use the foregoing information to obtain an upper bound on the sum of the partial widths of the individual resonances,

$$\sum_\beta \Gamma_{n\beta} = \gamma_n^\dagger \gamma_n = \Gamma_n \alpha_{nn} , \tag{46}$$

where Γ_n is the total width of the resonance. We will divide the eigenvectors into two groups, those with reciprocal eigenvalues, $a_\lambda, a_\lambda^{-1}$ with $a_\lambda > 1$, and those with eigenvalues $+1$. Then from (42)

$$\begin{aligned} \alpha_{nn} = &\sum_{a_\lambda > 1} \left[a_\lambda |x_n(a_\lambda)|^2 + a_\lambda^{-1} |x_n(a_\lambda^{-1})|^2 \right] \\ &+ \sum_{a_\lambda = 1} |x_n(a_\lambda)|^2 , \end{aligned} \tag{47}$$

where $x_n(a_\lambda)$ is the n component of the eigenvector with eigenvalue a_λ. We have already seen that the normalized eigenvector $x(a_\lambda^{-1})$ differs at most by a phase from the complex conjugate of the eigenvector $x(a_\lambda)$. Thus, $|x_n(a_\lambda)|^2 = |x_n(a_\lambda^{-1})|^2$, and (47) may be written as

$$\alpha_{nn} = \sum_{a_\lambda > 1} (a_\lambda + a_\lambda^{-1}) |x_n(a_\lambda)|^2 + \sum_{a_\lambda = 1} |x_n(a_\lambda)|^2 . \tag{48}$$

Now for $a_\lambda > 0$, $a_\lambda + a_\lambda^{-1} \geq 2$, so

$$\alpha_{nn} \geq \sum_{\text{all } a_\lambda} |x_n(a_\lambda)|^2 = 1, \tag{49}$$

where we have used the completeness relation for the eigenvectors of $\underline{\alpha}$, (42d). We therefore find from (46) that the sum of the partial widths of a resonance is generally greater than the total width [Ref. 1, Sec. IX, Eq. (2.13)],

$$\sum_\beta \Gamma_{n\beta} \geq \Gamma_n . \tag{50}$$

Equality is attained only in the trivial case of completely nonoverlapping resonances, $\underline{\alpha} = \underline{1}$.

We note finally that we can obtain another background-dependent sum rule for the resonance widths directly from (26b) by multiplying on the left by $\gamma_n^\dagger$, summing on n, and using the definition (25) and the relations (31) and (32),

$$\begin{aligned} \sum_n \gamma_n^\dagger B \gamma_n^* &= i \sum_{m,n} (\mathcal{E}_m - \mathcal{E}_n^*) \alpha_{nm} \alpha_{nm} \\ &= i \sum_n (\mathcal{E}_n - \mathcal{E}_n^*) . \end{aligned} \tag{51}$$

That is,

$$\sum_n \Gamma_n = \sum_n \gamma_n^\dagger B \gamma_n^* . \tag{52}$$

C. Higher-order poles in the S matrix

We can easily generalize the foregoing results to the case in which S contains higher-order or coincident poles. The case of coincident poles is essentially trivial. This corresponds to the choice of a rank $s > 1$ in (9), and leads to an expression for S in which some subset of the poles have exactly coincident energies, $\mathcal{E}_n = \mathcal{E}_0$, $n = 1, \ldots, s$. We can of course achieve this configuration directly as a limit of the general N-pole configuration, and will not consider it further.

The case of higher-order poles in S is more interesting,[11,22-24] and, as was noted in the Introduction, the possibility that the A_2 meson should be described by a second-order resonance pole (dipole) was the subject of lively debate a few years ago. At least two models are known which lead to second-order poles in the S matrix,[12,23] so the subject is not vacuous.

We will assume that S can be approximated by a sum of pole terms,

$$S(E)=B+\sum_{n=1}^{N}\sum_{k=1}^{l_n}\frac{R_{n,k}}{(E-\mathcal{E}_n)^k}. \tag{53}$$

The consequence of the unitarity relation (3) for S can be expressed in a compact form by noting that $[S^\dagger(E^*)S(E)-B^\dagger B]$ is a meromorphic function of E which vanishes for $E\to\infty$. It can therefore be written as a sum over its poles at $\{\mathcal{E}_n\}$ and $\{\mathcal{E}_n^*\}$. The residues are easily determined using Cauchy's theorem, and we find that

$$\begin{aligned}S^\dagger(E^*)S(E)&=\underline{1}\\&=B^\dagger B+\sum_{n=1}^{N}\sum_{k=1}^{l_n}\frac{1}{(E-\mathcal{E}_n)^k}\sum_{m=0}^{l_n-k}\frac{1}{m!}[S^\dagger(\mathcal{E}_n^*)]^{(m)}R_{n,k+m}\\&\quad+\sum_{n=1}^{N}\sum_{k=1}^{l_n}\frac{1}{(E-\mathcal{E}_n^*)^k}\sum_{m=0}^{l_n-k}\frac{1}{m!}R^\dagger_{n,k+m}[S(\mathcal{E}_n^*)]^{(m)},\end{aligned} \tag{54}$$

where

$$[S^\dagger(\mathcal{E}_n^*)]^{(m)}=\left[\frac{d^m}{dE^m}S^\dagger(E^*)\right]_{E=\mathcal{E}_n} \tag{55a}$$

and

$$[S(\mathcal{E}_n^*)]^{(m)}=\left[\frac{d^m}{dE^m}S(E)\right]_{E=\mathcal{E}_n^*}. \tag{55b}$$

Note that a pole of order k in the expansion (53) contributes in general to the poles of order $k, k-1,\ldots,1$ in the pole (or Mittag-Leffler) expansion of $S^\dagger(E^*)S(E)$.

The product $S^\dagger(E^*)S(E)$ actually has no poles, so the residues of the apparent poles in (54) at $\{\mathcal{E}_n\}$ and $\{\mathcal{E}_n^*\}$ must vanish. Thus,

$$\sum_{m=0}^{l_n-k}\frac{1}{m!}[S^\dagger(\mathcal{E}_n^*)]^{(m)}R_{n,k+m}=\underline{0}, \tag{56a}$$

$$n=1,\ldots,N,\quad k=1,\ldots,l_n$$

and

$$\sum_{m=0}^{l_n-k}\frac{1}{m!}R^\dagger_{n,k+m}[S(\mathcal{E}_n^*)]^{(m)}=\underline{0}, \tag{56b}$$

$$n=1,\ldots,N,\quad k=1,\ldots,l_n.$$

The second relation is just the Hermitian conjugate of the first, and can be dropped. We will assume that, at each of the points $E=\mathcal{E}_n$, the highest-order pole in the Mittag-Leffler expansion of $S^\dagger(E^*)S(E)$ is isolated in the sense of Ref. 32. In this case, R_{n,l_n} must factor (R_{n,l_n} is of rank 1). Since the residue matrices are also symmetric for CPT- and T-invariant interactions, we can write R_{n,l_n} as

$$R_{n,l_n}=\gamma_{n,l_n}\gamma_{n,l_n}. \tag{57}$$

It is now straightforward to determine the ranks of the residue matrices for the lower-order poles at $E=\mathcal{E}_n$. We first rewrite the expression in (56a) as

$$S^\dagger(\mathcal{E}_n^*)R_{n,k}=-\sum_{m=1}^{l_n-k}\frac{1}{m!}[S^\dagger(\mathcal{E}_n^*)]^{(m)}R_{n,k+m}, \tag{58}$$

and use the relations[31] $\rho(A+B)\le\rho(A)+\rho(B)$, $\rho(AB)\ge\rho(A)+\rho(B)-\mathfrak{N}$, and $\rho(AB)\le\min\{\rho(A),\rho(B)\}$, to obtain the inequalities

$$\begin{aligned}\rho(R_{n,k})+\rho(S^\dagger)-\mathfrak{N}\le\rho(S^\dagger R_{n,k})&=\rho\left(\sum_{m=1}^{l_n-k}S^{\dagger(m)}R_{n,k+m}\right)\\&\le\sum_{m=1}^{l_n-k}\rho(S^{\dagger(m)}R_{n,k+m})\\&\le\sum_{m=1}^{l_n-k}\min\{\rho(S^{\dagger(m)}),\rho(R_{n,k+m})\}\\&\le\sum_{m=1}^{l_n-k}\rho(R_{n,k+m}).\end{aligned} \tag{59}$$

A simple recursive argument beginning with $k=l_n-1$ then shows that

$$\begin{aligned}\rho(R_{n,l_n-m})&\le\nu(S^\dagger)+\sum_{k=1}^{m-1}\rho(R_{n,l_n-k})\\&\le 2^{m-1}[\nu(S^\dagger)+\rho(R_{n,l_n})].\end{aligned} \tag{60}$$

The most general form of the symmetrical matrix $R_{n,k}$ is therefore a sum of $2^{l_n-k-1}(\nu+\rho)$ factored terms. We will introduce factors of i and Γ_n for later convenience, and write $R_{n,k}$ as

$$R_{n,k}=(-i)^k\Gamma_n^{k-1}\sum_{j=1}^{j_k}\gamma^j_{n,k}\tilde\gamma^j_{n,k}, \tag{61}$$

$$j_k=2^{l_n-k-1}(\nu+\rho).$$

With this convention, the residue vectors $\gamma^i_{n,k}$ all have the dimensions of partial decay amplitudes.

Now $\rho(R_{n,l_n})=1$ by our assumption that the highest-order pole at $E=\mathcal{E}_n$ is isolated. The null-space of $S^\dagger(\mathcal{E}_n^*)$ will generally have the lowest allowable dimension consistent with this assumption and (56a) for $k=l_n$, $\nu(S^\dagger)=\mathfrak{N}-\rho(S^\dagger)=1$. In this case, the most general S matrix of the form assumed in (53) which is consistent with unitarity and CPT and T conservation is given by

$$S(E)=B+\sum_{n=1}^{N}\sum_{k=1}^{l_n}\sum_{j=1}^{j_k}(-i)^k\Gamma_n^{k-1}\frac{\gamma^j_{n,k}\tilde\gamma^j_{n,k}}{(E-\mathcal{E}_n)^k}, \tag{62}$$

$$j_k=2^{l_n-k}.$$

The remaining unitarity constraints on $S(E)$ are given by (56) and the condition $B^\dagger B = \underline{1}$.

III. EXAMPLES

A. Simple resonances

1. *Single resonance in the presence of background*

The simplest nontrivial problem is that of the single resonance in the presence of background in the many-channel case. The S matrix is of the form

$$S(E) = B - i\frac{\gamma\tilde{\gamma}}{E - \mathcal{E}}, \tag{63}$$

where $B^\dagger B = BB^\dagger = \underline{1}$, and $\tilde{B} = B$ (CPT and T). The unitarity constraints (26b) can be reduced immediately to the set of equations

$$B\gamma^* = \gamma, \quad \gamma^\dagger\gamma = \Gamma. \tag{64}$$

In the trivial case of no significant background, $B = \underline{1}$, the only restrictions are that the γ's be real with $\gamma^\dagger\gamma = \Gamma$. The general case can be reduced to the case of no background by noting that a symmetric unitarity matrix can be diagonalized by an orthogonal transformation,

$$B = Oe^{2i\Delta}\tilde{O} = b\tilde{b}, \quad b = Oe^{i\Delta}, \tag{65}$$

where Δ is the real diagonal matrix of eigenphase shifts for the background scattering. If we define new decay amplitudes g by $\gamma = bg$, $g = b^\dagger\gamma = e^{-i\Delta}\tilde{O}\gamma$, we can rewrite (63) as[21]

$$S(E) = b\left(\underline{1} - i\frac{g\tilde{g}}{E - \mathcal{E}}\right)\tilde{b}, \quad g^* = g, \quad g^\dagger g = \Gamma. \tag{66}$$

The matrix $\mathcal{S}(E) = b^\dagger S(E) b^*$ obtained by eliminating the background term entirely is still unitary and symmetric, and satisfies constraints analogous to those satisfied by $S(E)$, but with B replaced by $\underline{1}$.

2. *Two overlapping resonances*

The simplest and most familiar case of the general results of Sec. II is that of two overlapping resonances.[5,6,21] This problem has been treated elsewhere by a variety of methods.[1,9,13,16,19] The S matrix for two poles in the presence of a constant background is of the form given in (20) (we assume CPT and T invariance, but not necessarily P invariance),

$$S(E) = B - i\frac{\gamma_1\tilde{\gamma}_1}{E - \mathcal{E}_1} - i\frac{\gamma_2\tilde{\gamma}_2}{E - \mathcal{E}_2}, \tag{67}$$

where $B^\dagger B = BB^\dagger = \underline{1}$, and $B = \tilde{B}$. The unitarity constraints (26b) are

$$B\gamma_n^* = \sum_{m=1}^{2} \alpha_{nm}\gamma_m, \tag{68a}$$

$$\alpha_{nm} = \frac{i\gamma_n^\dagger\gamma_m}{\mathcal{E}_n^* - \mathcal{E}_m}. \tag{68b}$$

The fact that $\underline{\alpha}$ is an orthogonal matrix, (37), implies that

$$\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} = 0, \tag{69a}$$

$$\alpha_{11}^2 + \alpha_{12}^2 = 1, \tag{69b}$$

$$\alpha_{22}^2 + \alpha_{21}^2 = 1, \tag{69c}$$

hence, from (69a), that

$$\alpha_{11}/\alpha_{22} = -\alpha_{12}/\alpha_{21}. \tag{70}$$

Since $\underline{\alpha}^\dagger = \underline{\alpha}$, $\alpha_{21} = \alpha_{12}^*$, and the right-hand side of (70) has unit magnitude. In addition, α_{11} and α_{22} are real and positive. We conclude that $\alpha_{11} = \alpha_{22}$, that is,

$$\gamma_1^\dagger\gamma_1/\Gamma_1 = \gamma_2^\dagger\gamma_2/\Gamma_2, \tag{71}$$

and that α_{12} is purely imaginary,

$$\alpha_{12} = i\frac{\gamma_1^\dagger\gamma_2}{\mathcal{E}_1^* - \mathcal{E}_2} = -\alpha_{12}^*. \tag{72}$$

Equation (72) determines the phase of the inner product $\gamma_1^\dagger\gamma_2$,

$$\arg(\gamma_1^\dagger\gamma_2) = -\tan^{-1}\frac{\Gamma_1 - \Gamma_2}{2(E_1 - E_2)} \pmod{\pi}. \tag{73}$$

Equations (69b) and (69c) are equivalent, and give

$$\alpha_{11} = \alpha_{22} = (1 - \alpha_{12}^2)^{1/2} = (1 + |\alpha_{12}|^2)^{1/2}, \tag{74}$$

or

$$\sum_\beta \Gamma_{1\beta} = (1 + |\alpha_{12}|^2)^{1/2}\Gamma_1, \tag{75a}$$

$$\sum_\beta \Gamma_{2\beta} = (1 + |\alpha_{12}|^2)^{1/2}\Gamma_2. \tag{75b}$$

The sums of the partial widths $\Gamma_{n\beta}$ are therefore greater than the total widths Γ_n, as expected from (50).

Since $\alpha_{11} = \alpha_{22} > |\alpha_{12}|$, the weak form of the Schwarz inequality (41) is satisfied automatically. Furthermore, the two eigenvalues of $\underline{\alpha}$ are always positive. We can also obtain an upper bound on $|\alpha_{12}|$ by combining (74) with the strong form of the Schwarz inequality, (40),

$$|\alpha_{12}|^2 \leq |\mathcal{E}_1 - \mathcal{E}_2|^{-2}. \tag{76}$$

We note finally that the vector equations (67) must still be solved for the given background and the given number of channels. These equations may be rewritten using the results above as

$$\begin{aligned} B\gamma_1^* &= (1 + |\alpha_{12}|^2)^{1/2}\gamma_1 + i|\alpha_{12}|\gamma_2, \\ B\gamma_2^* &= (1 + |\alpha_{12}|^2)^{1/2}\gamma_2 - i|\alpha_{12}|\gamma_1. \end{aligned} \tag{77}$$

The solution of these equations is discussed by McVoy.[21] One can also reduce the problem to that of scattering in the absence of any background by using the method noted after (64).[21] The latter problem is treated in detail in Refs. 13 and 16. A thorough discussion of the slightly more complicated $K_L - K_S$ problem which illustrates how these equations may be used for phenomenology is given in Ref. 6.

3. Three overlapping resonances

We will not give a full treatment of the algebraically complicated problem of three overlapping resonances, but will simply note that the most general form of the overlap matrix $\underline{\alpha}$ consistent with the constraints of Secs. II 2 and II 3 can be specified in terms of three parameters: a real hyperbolic angle θ and a complex angle ϕ,

$$\underline{\alpha} = \begin{pmatrix} \cosh\theta\,|\cos\phi|^2 + |\sin\phi|^2, & \cosh\theta\cos\phi\sin^*\phi - \sin\phi\cos^*\phi, & -i\sinh\theta\cos\phi \\ \cosh\theta\sin\phi\cos^*\phi - \cos\phi\sin^*\phi, & \cosh\theta\,|\sin\phi|^2 + |\cos\phi|^2, & -i\sinh\theta\sin\phi \\ i\sinh\theta\cos^*\phi, & i\sinh\theta\sin^*\phi, & \cosh\theta \end{pmatrix}. \tag{78}$$

In this parametrization, $\underline{\alpha}$ has eigenvalues e^{θ}, $e^{-\theta}$, and 1, that is, a reciprocal pair and one unit eigenvalue. The general constraints on the resonance parameters γ_n, $n = 1, 2, 3$ are determined by the three-pole form of (26b).

B. Dipole resonances

1. Single isolated dipole resonance: Rank one

We will restrict our attention to the case of a single isolated dipole resonance in the presence of a nontrivial background. The general form of the S matrix is then that given in (62) with the sum restricted to a single value of n. We will rewrite (62) in a simpler notation as

$$S(E) = B - i\frac{\gamma\tilde{\gamma} + \lambda\tilde{\lambda}}{E - \mathcal{E}_n} - \Gamma\frac{\sigma\tilde{\sigma}}{(E - \mathcal{E}_n)^2}. \tag{79}$$

The resonance parameters are subject to the unitarity constraints (56),

$$[B^\dagger - \Gamma^{-1}(\gamma^*\gamma^\dagger + \lambda^*\lambda^\dagger + \sigma^*\sigma^\dagger)]\sigma\tilde{\sigma} = 0, \tag{80}$$

$$[B^\dagger - \Gamma^{-1}(\gamma^*\gamma^\dagger + \lambda^*\lambda^\dagger + \sigma^*\sigma^\dagger)](\gamma\tilde{\gamma} + \lambda\tilde{\lambda}) + \Gamma^{-1}(\gamma^*\gamma^\dagger + \lambda^*\lambda^\dagger - 2\sigma^*\sigma^\dagger)\sigma\tilde{\sigma} = 0, \tag{81}$$

$$BB^\dagger = B^\dagger B = \underline{1}. \tag{82}$$

Equation (81) shows immediately that the matrix $(\gamma\tilde{\gamma} + \lambda\tilde{\lambda})$ cannot vanish, since if it did, $(\sigma^\dagger\sigma)$ would have to vanish, σ would be a null vector, and the dipole term in (79) would be absent, contrary to our assumption.

We will consider first the case in which the first-order pole in (79) has a residue matrix of rank one, $\lambda \equiv 0$. The relations in (80) must hold for each element of $\tilde{\sigma}$ separately. Thus, if $\tilde{\sigma}$ is nonzero, (80) is equivalent to the vector equation.

$$B^\dagger\sigma = \Gamma^{-1}\gamma^*(\gamma^\dagger\sigma) - \Gamma^{-1}\sigma^*(\sigma^\dagger\sigma). \tag{83}$$

We can also reduce (81) quite easily. If we assume that σ and γ are linearly independent, the coefficients of $\tilde{\gamma}$ and $\tilde{\sigma}$ must vanish independently. However, the vanishing of the coefficient of $\tilde{\sigma}$ in (81) would imply that

$$\gamma^*(\gamma^\dagger\sigma) = 2\sigma^*(\sigma^\dagger\sigma), \tag{84}$$

hence that σ is a multiple of γ, contrary to our assumption. We conclude that $\gamma = a\sigma$ for some nonzero constant a (γ cannot be null, by the argument above). Equations (81) and (83) then require respectively that

$$B^\dagger\sigma = \Gamma^{-1}[a^{*2} - 1 - a^{-2}(a^{*2} - 2)]\sigma^*(\sigma^\dagger\sigma) \tag{85a}$$

and

$$B^\dagger\sigma = \Gamma^{-1}(a^{*2} - 1)\sigma^*(\sigma^\dagger\sigma). \tag{85b}$$

Thus, $a^{*2} = 2$, and a is real and equal to $\sqrt{2}$.

The final equation of constraint is now

$$B^\dagger\sigma = \sigma^*(\sigma^\dagger\sigma)/\Gamma. \tag{86}$$

If we take the Hermitian square of this equation and use the unitarity of B, we find that $\sigma^\dagger\sigma = \Gamma$. The S matrix is therefore given by a typical dipole expression

$$S(E) = B - i\frac{2\sigma\tilde{\sigma}}{E - \mathcal{E}} - \Gamma\frac{\sigma\tilde{\sigma}}{(E - \mathcal{E})^2} = B - \frac{2i(E - E_R)}{(E - \mathcal{E})^2}\sigma\tilde{\sigma}, \tag{87}$$

where $\mathcal{E} = E_R - i\Gamma/2$. The remaining unitarity constraints (86) are now completely equivalent to those for a single first-order resonance in the presence of background,

$$B^\dagger\sigma = \sigma^*, \quad \sigma^\dagger\sigma = \Gamma. \tag{88}$$

The expression for S in (87) can be obtained rather simply as a singular limit of the two-pole formula (67) as the two poles merge. We will consider the case in which $\gamma_2 \to e^{i\phi}\gamma_1$ as $\mathcal{E}_2 \to \mathcal{E}_1$. The combined residue functions will then continue to factor as in (88). Equation (72) implies that $\gamma_1^\dagger\gamma_2/(\mathcal{E}_1^* - \mathcal{E}_2)$ is real, hence, that

$$e^{i\phi}=\left(\frac{\mathcal{E}_1^*-\mathcal{E}_2}{\mathcal{E}_1-\mathcal{E}_2^*}\right)^{1/2},\ \mathrm{mod}\,\pi. \tag{89}$$

Moreover, for $\gamma_2=e^{i\phi}\gamma_1$, the inequality for $|\alpha_{12}|$ in (76) becomes an equality, so α_{12} becomes singular for $\mathcal{E}_2\to\mathcal{E}_1$,

$$|\alpha_{12}|\to|\mathcal{E}_1-\mathcal{E}_2|^{-1}. \tag{90}$$

We can take both (89) and (90) into account by introducing a new residue vector σ related to γ_1 and γ_2 by

$$\gamma_1=\left(\frac{\mathcal{E}_1-\mathcal{E}_2^*}{\mathcal{E}_1-\mathcal{E}_2}\right)^{1/2}\sigma,\qquad \gamma_2=\left(\frac{\mathcal{E}_1^*-\mathcal{E}_2}{\mathcal{E}_1-\mathcal{E}_2}\right)^{1/2}\sigma. \tag{91}$$

If we substitute the definitions (91) into (66), we find that the resulting expression reduces exactly to (87) for $\mathcal{E}_2\to\mathcal{E}_1=\mathcal{E}$. Note that the original residue vectors diverge in this limit.

2. *Single isolated dipole resonance: Rank two*

The reduction of the unitarity constraints in (80) and (81) is somewhat more complicated if the residue matrix for the first-order pole is of rank two, that is, for γ and λ independent, nonzero vectors. One can again show that the assumption that the residue vectors are all linearly independent is inconsistent, hence, that

$$\sigma=a\gamma+b\lambda \tag{92}$$

for some constants a and b. Since γ and λ are independent, the coefficients of $\tilde{\gamma}$ and $\tilde{\lambda}$ must vanish independently in (80) and (81), and we find that

$$B^\dagger(a\gamma+b\lambda)=\Gamma^{-1}\gamma^*[(\gamma^\dagger\sigma)-a^*(\sigma^\dagger\sigma)]+\Gamma^{-1}\lambda^*[(\lambda^\dagger\sigma)-b^*(\sigma^\dagger\sigma)], \tag{93a}$$

$$B^\dagger\gamma=\Gamma^{-1}\gamma^*[(\gamma^\dagger\gamma)-a^*(\sigma^\dagger\gamma)-a(\gamma^\dagger\sigma)+2a^*a(\sigma^\dagger\sigma)]+\Gamma^{-1}\lambda^*[(\lambda^\dagger\gamma)-b^*(\sigma^\dagger\gamma)-a(\lambda^\dagger\sigma)+2ab^*(\sigma^\dagger\sigma)], \tag{93b}$$

$$B^\dagger\lambda=\Gamma^{-1}\gamma^*[(\gamma^\dagger\lambda)-a^*(\sigma^\dagger\lambda)-b(\gamma^\dagger\sigma)+2a^*b(\sigma^\dagger\sigma)]+\Gamma^{-1}\lambda^*[(\lambda^\dagger\lambda)-b^*(\sigma^\dagger\lambda)-b(\lambda^\dagger\sigma)+2bb^*(\sigma^\dagger\sigma)]. \tag{93c}$$

If we combine the last two equations to obtain a new equation for $B^\dagger(a\gamma+b\lambda)$, we find it is consistent with (93a) only if

$$\gamma^*(a^2+b^2)[(\gamma^\dagger\sigma)-2a^*(\sigma^\dagger\sigma)]+\lambda^*(a^2+b^2)[(\lambda^\dagger\sigma)-2b^*(\sigma^\dagger\sigma)]=0. \tag{94}$$

The coefficients of γ^* and λ^* must vanish independently, so either

$$a^2+b^2=0 \tag{95}$$

or

$$2a^*(\sigma^\dagger\sigma)=\gamma^\dagger\sigma,\quad 2b^*(\sigma^\dagger\sigma)=\lambda^\dagger\sigma. \tag{96}$$

The conditions in (96) and the definition (92) require that $a^2+b^2=\frac{1}{2}$. The solutions to equations (93) which correspond to the constraints (95) and (96) are therefore distinct.

If we choose $b=ia$ corresponding to the constraint in (95), we obtain a solution to the unitarity equations which generalizes that given by Rebbi and Slansky[13] and Dothan and Horn[16] for the case of no background scattering, $B=\underline{1}$,

$$S(E)=B-i\frac{\gamma\tilde{\gamma}+\lambda\tilde{\lambda}}{E-\mathcal{E}}-a^2\Gamma\frac{(\gamma+i\lambda)(\tilde{\gamma}+i\tilde{\lambda})}{(E-\mathcal{E})^2}, \tag{97a}$$

where

$$a^2=\frac{1}{2}\frac{(\gamma^\dagger-i\lambda^\dagger)(\gamma-i\lambda)}{|\gamma+i\lambda|^2}. \tag{97b}$$

The vectors γ and λ must be relatively real,

$$\gamma^\dagger\lambda=(\gamma^\dagger\lambda)^*=\lambda^\dagger\gamma, \tag{98a}$$

and are normalized so that

$$\gamma^\dagger\gamma+\lambda^\dagger\lambda=2\Gamma. \tag{98b}$$

The unitarity equations (93b) and (93c) can be reduced finally to simple constraints on the phases of γ and λ,

$$B^\dagger\gamma=\gamma^*,\quad B^\dagger\lambda=\lambda^*. \tag{99}$$

The choice $b=-ia$ in (95) does not lead to a new solution, since the phase of λ is only determined modulo π.

The algebraic manipulations which lead from (93b), (93c), and (95) to the results in (97)–(99) are straightforward, but lengthy. The method used involves taking the Hermitian squares and products of the expressions in (93b) and (93c), and also using the identity $\tilde{\lambda}B^\dagger\gamma=\gamma B^\dagger\lambda$, to obtain a set of equations independent of B. These are sufficient to determine the constant a^2 given in (97b) and give the reality and normalization conditions in (98a) and (98b). When these conditions are used in (93b) and (93c), the unitarity equations collapse to the very simple form given in (99). We note finally that these results can also be obtained from those of Rebbi and Slansky[13] or Dothan and Horn[16] by using the method discussed following (64) to eliminate the background. (The Rebbi-Slansky solution for the dipole S matrix was obtained as a limit of the S matrix for two first-order poles in the absence of any background scattering. The Dothan-Horn solution was obtained by the mass-matrix method, again in the absence of back-

ground. The generalization to higher-order poles has been considered by Ruuskanen[11] and by Chan and Slansky[24] using the mass-matrix method.)

The second independent solution to the rank-two dipole problem corresponding to the conditions (96) is also easily constructed. In this case, it is convenient to use σ and γ as the independent vectors. The resulting S matrix is quite simple,

$$S = B - ic\,\frac{\gamma\tilde{\gamma} - \Gamma^{-1}(\sigma^\dagger\gamma)(\gamma\tilde{\sigma} + \sigma\tilde{\gamma}) + 2\sigma\tilde{\sigma}}{E - \mathcal{E}} - \Gamma\,\frac{\sigma\tilde{\sigma}}{(E-\mathcal{E})^2}\,, \tag{100a}$$

where

$$c^{-1} = \gamma^\dagger\gamma - \Gamma^{-1}|\sigma^\dagger\sigma|^2\,. \tag{100b}$$

The vectors σ and γ are normalized so that

$$\sigma^\dagger\sigma = \Gamma, \qquad \gamma^\dagger\gamma - \Gamma^{-1}|\sigma^\dagger\gamma|^2 + \tfrac{1}{2}\Gamma^{-1}(\sigma^\dagger\gamma)^2 = \Gamma\,. \tag{101}$$

The last condition implies that $(\sigma^\dagger\gamma)$ is real. The remaining unitarity constraints assume the relatively simple form

$$B^\dagger\sigma = \sigma^*\,, \tag{102a}$$

$$B^\dagger\gamma = \gamma^* + \Gamma^{-1}\sigma^*(\sigma^\dagger\gamma - \gamma^\dagger\sigma)\,. \tag{102b}$$

The S matrix in (100a) corresponds to a situation in which a first-order pole is coincident with an isolated dipole resonance of the form given in (87). We can make this correspondence clear by introducing a residue vector γ' which is orthogonal to σ, $\gamma' = \gamma - \Gamma^{-1}(\sigma^\dagger\gamma)\sigma$. Then

$$S(E) = B - i\,\frac{\gamma'\tilde{\gamma}' + 2\sigma\tilde{\sigma}}{E - \mathcal{E}} - \Gamma\,\frac{\sigma\tilde{\sigma}}{(E-\mathcal{E})^2}\,, \tag{103a}$$

where

$$B^\dagger\sigma = \sigma^*, \quad \sigma^\dagger\sigma = \Gamma\,, \qquad B^\dagger\gamma' = \gamma'^*, \quad \gamma'^\dagger\gamma' = \Gamma\,, \tag{103b}$$

and

$$\sigma^\dagger\gamma' = 0\,. \tag{104}$$

The last condition requires that the first-order and the dipole resonances be nonoverlapping even though they decay into the same channels.

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[25]In some commonly used parametrizations for resonances, for example, the effective-range expansions, R and $\mathcal{E}$ are taken as functions of E. However, if the resonance in question does not occur near an important threshold, one can expand the energy-dependent residue function and pole positions in Taylor series in the

quantity $(E - \mathcal{E})$. The result in (1) is recovered if quadratic and higher terms in the expansion of $R(E)$, and cubic and higher terms in the expansion of $\mathcal{E}(E)$, can be neglected.

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[27]We will use standard notation for operations on a matrix A, with $A^\dagger$ the Hermitian conjugate of A, A^* its complex conjugate, and $\tilde{A}$ its transpose. $\underline{1}$ denotes the unit matrix, and $\underline{0}$ the null matrix.

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[30]Equation (5) is a special case of a more general theorem. For any allowable energy dependence, the S matrix satisfies the continued unitarity relation $S(E^*)S(E) = \underline{1}$. This implies that $S(E) = [S^\dagger(E^*)]^{-1} = [\mathrm{cof}S^\dagger(E^*)]/\det S^\dagger(E^*)$, where $\mathrm{cof}S^\dagger(E^*)$ is the matrix of cofactors of $S^\dagger(E^*)$. Since $S^\dagger(E^*)$ has no poles for E in the lower half plane, $\mathrm{cof}S^\dagger(E^*)$ is finite for $E \to \mathcal{E}_n$, and the poles of S at $E = \mathcal{E}_n$ arise from the zeros of $\det S^\dagger(E^*)$. However, if $\det S^\dagger(E^*)$ vanishes for $E = \mathcal{E}_n$, the matrix $S^\dagger(E^*)S(E)\det S^\dagger(E^*) = \underline{1}\det S^\dagger(E^*) = S^\dagger(E^*)[\mathrm{cof}S^\dagger(E^*)]$ also vanishes for $E = \mathcal{E}_n$. Since R_n is proportional to $\mathrm{cof}S^\dagger(\mathcal{E}_n^*)$, this reproduces the first of the equations in a more general setting. The second equation is obtained by applying a similar argument at the poles of $S^\dagger(E^*)$ at $E = \mathcal{E}_n^*$.

[31]P. R. Halmos, *Finite Dimensional Vector Spaces* (Princeton Univ. Press, Princeton, New Jersey, 1955), Secs. 36–38.

[32]In the absence of a dynamical model, this is partly a matter of definition. We follow the usual definition, based on the assumption that the case of "exactly coincident poles," $s > 1$, is accidental, and that small perturbations of the interaction would in fact split the single pole (with a nonfactored residue) into s isolated poles.

[33]More generally, the vanishing of $\det S^\dagger(E^*)$ at the poles $E = \mathcal{E}_n$ of $S(E)$ (see Ref. 30) implies that $S^\dagger(\mathcal{E}_n^*)$ has at least one zero eigenvalue, hence $\nu(S^\dagger(\mathcal{E}_n^*)) \geq 1$. Application of the identity in (7) to the product $S^\dagger(\mathcal{E}_n^*)[\mathrm{cof}S^\dagger(\mathcal{E}_n^*)]$ with $\nu = r$ leads to the result above.

[34]See, for example, R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (Benjamin, New York, 1964).

[35]M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).

[36]This is equivalent to the dynamical assumption that the overlapping configuration of the N resonances can be changed continuously to a nonoverlapping configuration by an appropriate continuous change in the Hamiltonian. This is certainly the expected situation. It is interesting to note that the weak form of the Schwarz inequalities given in (41) is actually sufficient to guarantee that $\underline{\alpha}$ is a positive matrix in the cases of two and three resonances. We do not know if this result can be extended to arbitrary n independently of any dynamical assumptions.