Zero-mass plane waves in nonzero gravitational backgrounds

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The mathematical definition of what is intuitively called a "plane wave" on the curved background of a black hole is clarified and discussed from the viewpoints of potentials and fields. Because of the long-range Newtonian part of the gravitational field the asymptotic wave fronts of an incident "plane wave" (describing a radiative perturbation for a scattering experiment) are distorted in a manner analogous to the wave fronts of an electron beam in the quantum-mechanical Coulomb scattering problem. In addition, the electromagnetic and gravitational fields can be described with either a potential formalism (i.e., the vector potential and the metric perturbation) or a field formalism (i.e., the electromagnetic-field tensor and the Riemann tensor). In this paper we present a distorted "plane wave" prescription, necessary for the calculation of the scattering cross sections of electromagnetic and gravitational waves off of a black hole, which agrees with the accepted prescription for a massless scalar field and satisfies the intuitive notions of what constitutes a "plane wave" in terms of potentials and fields.

I. INTRODUCTION

In order to discuss the scalar, electromagnetic, and gravitational scattering cross sections of a black hole, it is necessary (as with any scattering problem) to find an asymptotic form for the field, a solution of the appropriate wave equation consisting of an incident plane wave and a term describing the scattered wave. In the mathematical formulation of the black-hole scattering problem, one must consider both the long-range Newtonian field, which distorts the shape of the wave fronts even infinitely far away from the scattering center, and the existence of a horizon, which forces the replacement of the usual boundary condition of regularity at the origin with an "ingoing flux" condition at the horizon. In this paper we illustrate how these features affect the definition of a plane wave and present formulas for the Weyl tensor components ψ_0 and ψ_4 for a plane wave incident along the symmetry axis of a Kerr black hole. (The electromagnetic case is treated in Appendix B.) These formulas will be used in a subsequent paper to find the gravitational-radiation scattering cross section of a Kerr black hole.[1]

We first review electron scattering in a Coulomb field, the archetype of the long-range-force scattering problem. The result provides guidance in Sec. III, where the gravitational-plane-wave problem is considered and partial-wave amplitudes for an incident plane wave are found. In the final section, we give an alternate derivation of the gravitational-plane-wave formulas for the purpose of illustrating an interesting property of black-hole scattering and the Bianchi identities.

II. ELECTRON SCATTERING IN A COULOMB FIELD

The problem of the scattering of (spinless) electrons off a point charge, as shown by Gordon,[2] has a closed-form analytic solution in parabolic cylindrical coordinates in terms of hypergeometric functions. The asymptotic form of this solution,

$$\psi \underset{r\to\infty}{\simeq} \exp\{i\omega[z - 2M\ln\omega(r-z)]\} + \frac{f_c(\theta)}{r}\exp\{i\omega[r + 2M\ln(2\omega r)]\}, \tag{2.1}$$

with

$$f_c = \frac{-M}{\sin^2(\frac{1}{2}\theta)}\exp\{2i\omega M\ln[\sin^2(\tfrac{1}{2}\theta)] + i\pi + 2i\eta_0\} \tag{2.2}$$

and

$$\eta_0 = \arg\Gamma(1-2i\omega M), \tag{2.3}$$

displays the existence of a logarithmic adjustment in the phase term of the second part (the "scattered piece") as well as very distorted phase fronts in the first piece (the incident plane wave). (This solution is for the attractive scattering between a fixed-force center and a scalar electron with the parameter choice $\hbar^2/2\mu = 1$, where μ is the mass of the scattered particle, and with attractive charges of magnitude $Ze^2 = 4M\omega^2$. In this form it directly gives the Newtonian gravitational limit of scattering of a scalar massless field of frequency $|\omega|$ in a field generated by a mass M.) This identification of the two parts of (2.1) is justified by the resultant cross section using $|f_c(\theta)|^2$, which gives the experimentally measured classical Rutherford cross section.[3]

Because the Coulomb potential is spherically symmetric, the scattering problem also can be approached in terms of an l, m mode sum for each incident positive-energy state. This is essential for the analysis of a plane wave on a Kerr background. No closed-form solution for a gravitational plane wave is known [cf. (2.1)], and the only analytical technique available in the Kerr geometry is based on the separability and decoupling of the perturbation equations. Such an approach amounts to a partial-wave analysis using the spin-weighted spheroidal harmonics.[4] If in the Coulomb case the angular functions in the mode sum are taken to be spherical harmonics so that

$$\psi_\omega = \sum_l u_l(r,\omega) P_l(\theta), \tag{2.4}$$

then the u_l are related to hypergeometric functions,

$$u_l = C_l r^l e^{i\omega r} F(l+1-2iM\omega, 2l+2, -2i\omega r), \tag{2.5}$$

and their asymptotic form is

$$u_l \sim r^{-1} \exp[\pm i\omega(r + 2M\ln 2\omega r - \tfrac{1}{2}l\pi + \eta_l)], \tag{2.6}$$

with

$$\eta_l = \arg\Gamma(l+1-2iM\omega). \tag{2.7}$$

To solve a scattering problem from this mode-sum approach, the incident plane wave must be extracted from the full solution (2.4). Such may be done by subtracting the scattered piece, as determined by (2.1), from (2.4) provided that $f_c(\theta)$ in (2.1) be expanded in terms of spherical harmonics. As shown by Schiff,[3] the expansion of $f_c(\theta)$ is

$$f_c(\theta) = \frac{1}{2i\omega} \sum_{l=0}^{\infty} (2l+1) P_l(\theta) e^{2i\eta_l}, \tag{2.8}$$

or

$$f_c(\theta) = \frac{1}{2i\omega} \sum_{l=0}^{\infty} (2l+1) P_l(\theta)(e^{2i\eta_l} - 1), \tag{2.9}$$

since $\sum_l (2l+1) P_l(\theta) = 4\delta(1-\cos\theta)$ vanishes for $\theta \neq 0$. Hence, we find from (2.1), (2.4), and (2.9) that the asymptotic incident plane wave is

$$\frac{1}{2i\omega r} \sum_l (2l+1)[e^{i\omega r_c} - (-1)^l e^{-i\omega r_c}] P_l(\theta), \tag{2.10}$$

where $r_c = r + 2M\ln 2\omega r$. This is identical to the asymptotic form for a flat-space plane wave with the single exception that r_c replaces r in the exponents (only).

III. GRAVITATIONAL PLANE WAVES

The analysis of the previous section suggests that a partial-wave analysis of a plane wave in a long-range force should proceed by constructing the partial-wave decomposition of a free plane wave, replacing r by r_c in the exponents, and treating the problem like a short-range potential with r_c the natural variable (instead of r). Problems involving short-range forces superimposed on the Coulomb background have been treated in this way.[3]

Here we study the scattering of waves in black-hole backgrounds, considering both rotating and nonrotating black holes. In both cases the equations describing the propagation of massless scalar, electromagnetic, and gravitational waves in the background can be separated into a sum over spheroidal (or spherical) harmonics, with a harmonic time dependence and a radial (r) functional dependence which in general has to be integrated numerically.[5,6,7,8] In the text we concentrate on the gravitational problem; the electromagnetic case is treated in Appendix B (for an alternate treatment see Herlt and Stephani[9]); for the scalar case see Matzner.[5]

In outline, to describe the distorted incident plane wave in the case of a nonzero gravitational background, we choose the ingoing and outgoing pieces of a transverse-tracefree metric perturbation $h_{\mu\nu}$ to have the same values as the corresponding plane wave in a flat spacetime background. Then we modify the phases by making the replacement $r \to r^*$,

$$r^* = r + \frac{1}{r_+ - r_-}\left[(r_+{}^2 + a^2)\ln\left(\frac{r}{r_+} - 1\right) - (r_-{}^2 + a^2)\ln\left(\frac{r}{r_-} - 1\right)\right] + \text{const}, \tag{3.1}$$

where $r_\pm$ are the roots to $r^2 - 2Mr + a^2 = 0$, and a and M are the angular momentum parameter and the mass of a Kerr black hole.[10,11] For the Schwarzschild ($a = 0$) case, (3.1) reduces to the Regge-Wheeler coordinate[12]

$$r^*_{(\text{Sch})} = r + 2M\ln\left(\frac{r}{2M} - 1\right) + \text{const}, \tag{3.2}$$

a form suggestively similar to the radial coordinate in the Coulomb background. The WKB solutions of the Regge-Wheeler and Zerilli[13] equations for odd and even gravitational perturbations of a Schwarzschild black hole have solutions of the form $\exp(\pm i\omega r^* - i\omega t)$, so r^* is an appropriate radial variable.

The asymptotic form of $h_{\mu\nu}$ determined by the $r \to r^*$ replacement is matched to a solution to the Kerr metric perturbation equations. By differentiation we then obtain from the perturbed metric (the asymptotic form is not sufficient) the field quantities ψ_0 and ψ_4, which are taken to represent the distorted plane wave.

On a flat background a transverse-tracefree plane gravitational wave traveling up the z axis is given in Cartesian coordinates by

$$h_{\mu\nu}=h\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\omega(t-z) & \sin\omega(t-z) & 0 \\ 0 & \sin\omega(t-z) & -\cos\omega(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \qquad (3.3)$$

[Throughout this paper we consider only monochromatic waves of frequency $|\omega|$; a positive (negative) choice of ω corresponds to a left (right) circularly polarized wave.] To facilitate a comparison with the $r\to\infty$ limit of a Kerr metric perturbation we project this tensor onto the complex m^μ legs of a null tetrad[14] compatible with the flat background [see (A2)] and expand these scalars in spin-weighted spheroidal harmonics ${}_sZ_l^m(\theta;a\omega)$ of the appropriate spin weight[7]

$$h_{mm}=h_{\mu\nu}m^\mu m^\nu, \qquad (3.4)$$

$$h_{mm}\underset{r\to\infty}{\sim}\sum_l -\frac{2h\pi}{i\omega r}\{{}_2N_{l;0}^{-2}(-a\omega)\,{}_2Z_l^{-2}(-a\omega)\exp[-i\omega(r-t)] + {}_2N_{l;\pi}^{2}(a\omega)\,{}_2Z_l^{2}(a\omega)\exp[-i\omega(r+t)]\},$$

and

$$h_{m^*m^*}\underset{r\to\infty}{\sim}\sum_l \frac{2\pi h}{i\omega r}\{{}_{-2}N_{l;0}^{2}(a\omega)\,{}_{-2}Z_l^{2}(a\omega)\exp[i\omega(r-t)] + {}_{-2}N_{l;\pi}^{-2}(-a\omega)\,{}_{-2}Z_l^{-2}(-a\omega) \times\exp[i\omega(r+t)]\}, \qquad (3.5)$$

where ${}_sN_{l;\pi}^m$ and ${}_sN_{l;0}^m$ are the numerical coefficients discussed in Appendix A. The expressions given by (3.4) and (3.5) are transverse-tracefree for each l,m mode.

With the substitution of r^* for r in the exponents, we match these metric plane-wave perturbations to the Kerr metric perturbations at infinity. It can be shown that the transverse-tracefree components are gauge invariant under any gauge transformation which leaves the metric perturbation—as expressed in any frame with constant-norm basis vectors—falling off as r^{-1} or faster at infinity.[13,15] Hence, we can match coefficients in any gauge which satisfies this condition.

Chrzanowski[16] has given formulas for the perturbed Kerr metric (see Table I and Appendix A for an explanation of the notation) in two distinct gauges, neither of which, in general, is transverse-tracefree near spatial infinity. However, these metric perturbations are transverse-tracefree as $r\to\infty$ in the ingoing gauge (outgoing gauge) provided that the perturbation is pure ingoing (outgoing) at infinity.

To proceed in this situation, we must match the incident plane wave to two particular homogeneous metric perturbations labeled $h_{\mu\nu}^{\rm up}$ and $h_{\mu\nu}^{\rm down}$. As shown in Fig. 1, $h_{\mu\nu}^{\rm up}$ is the solution which vanishes on past null infinity ($\mathscr{I}^-$) and corresponds to flux emerging from the past horizon, traveling toward future null infinity ($\mathscr{I}^+$) with some flux scattered back toward the future horizon; and $h_{\mu\nu}^{\rm down}$ is the solution which vanishes on future null infinity. Hence, $h_{\mu\nu}^{\rm up}$ and $h_{\mu\nu}^{\rm down}$ are linearly independent solutions which correspond, respectively, to pure outgoing and pure ingoing radiation at spatial infinity.

The metric perturbation $h_{\mu\nu}^{\rm down}$ has the property that it is asymptotically transverse-tracefree at spatial infinity in the ingoing gauge, so it may be matched to the ingoing piece of the plane wave given in (3.4) and (3.5). $h_{\mu\nu}^{\rm down}$ may be expanded into a mode sum:

$$h_{\mu\nu}^{\rm down}=\int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} K_{lm\bar\omega P}^{\rm down}\, h_{\mu\nu}^{\rm down}(x,lm\bar\omega P), \qquad (3.6)$$

where $h_{\mu\nu}(x,lm\omega P)$ is the l,m,ω,P mode of the perturbed metric as given in Table I (in this case, in the ingoing gauge). The asymptotic form of the transverse-tracefree piece is

$$h_{mm}^{\rm down}\underset{r\to\infty}{\sim}\int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} K_{lm\bar\omega P}^{\rm down}\,{}_{+2}Z_l^m(a\bar\omega)\frac{\exp[-i\bar\omega(r^*+t)]}{r}, \qquad (3.7)$$

when one takes the normalization

$${}_{-2}R^{\rm down}e^{-i\bar\omega t}\underset{r\to\infty}{\sim}\frac{1}{4\bar\omega^2}\frac{\exp[-i\bar\omega(r^*+t)]}{r} \qquad (3.8)$$

for the radial function which appears in the ingoing-gauge expression for $h_{\mu\nu}^{\rm down}(x,lm\omega P)$ in Table I. The constants $K_{lm\bar\omega P}$, to be determined by comparison with (3.4), satisfy the crossing relation

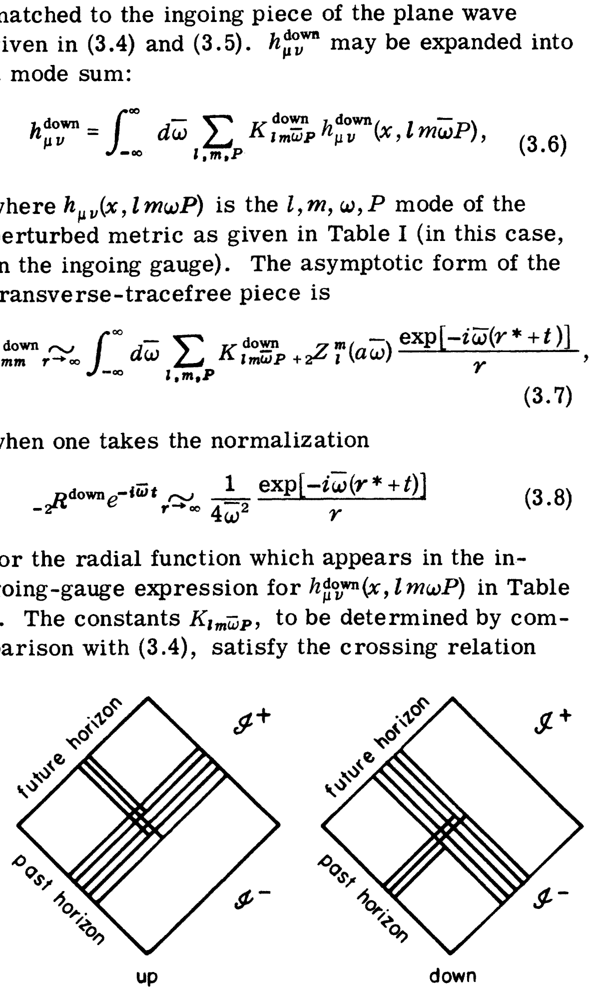

FIG. 1. The properties of the scattering solutions labeled "up" and "down" are illustrated above with the aid of Penrose conformal diagrams of a Kerr black hole. An "up" solution consists of a wave packet initially coming "up" from the past horizon, propagating out to $\mathscr{I}^+$, and scattering back through the future horizon. For a "down" solution the final state consists of waves going "down" the future horizon with no radiation arriving at $\mathscr{I}^+$

TABLE I. Perturbed potentials and field quantities.

Ingoing Radiation Gauge: $A_\mu l^\mu = h_{\mu\nu} l^\nu = h^\mu_\mu = 0$

$$A_\mu(x, lm\omega P=\pm) = [-l_\mu(\delta^* + 2\beta^* + \tau^*) + m^*_\mu(D + 2\epsilon^* + \rho^*)]\ {}_{-1}R_{lm\omega}(r)\ {}_{+1}Z^m_l(\theta, a\omega)e^{-i\omega t}$$
$$+ P[-l_\mu(\delta + 2\beta + \tau) + m_\mu(D + 2\epsilon + \rho)]\ {}_{-1}R_{lm\omega}(r)\ {}_{-1}Z^m_l(\theta, a\omega)e^{-i\omega t},$$

$$h_{\mu\nu}(x, lm\omega P=\pm) = \{-l_\mu l_\nu(\delta^* + \alpha + 3\beta^* - \tau^*)(\delta^* + 4\beta^* + 3\tau^*) - m^*_\mu m^*_\nu(D - \rho^* + 3\epsilon^* - \epsilon)(D + 3\rho^* + 4\epsilon^*)$$
$$+ l_{(\mu} m^*_{\nu)}[(D + \rho - \rho^* + \epsilon + 3\epsilon^*)(\delta^* + 4\beta^* + 3\tau^*)$$
$$+ (\delta^* + 3\beta^* - \alpha - \pi - \tau^*)(D + 3\rho^* + 4\epsilon^*)]\}\ {}_{-2}R_{lm\omega}(r)\ {}_{+2}Z^m_l(\theta, a\omega)e^{-i\omega t}$$
$$+ P\{-l_\mu l_\nu(\delta + \alpha^* + 3\beta - \tau)(\delta + 4\beta + 3\tau) - m_\mu m_\nu(D - \rho + 3\epsilon - \epsilon^*)(D + 3\rho + 4\epsilon)$$
$$+ l_{(\mu} m_{\nu)}[(D + \rho^* - \rho + \epsilon^* + 3\epsilon)(\delta + 4\beta + 3\tau)$$
$$+ (\delta + 3\beta - \alpha^* - \pi^* - \tau)(D + 3\rho + 4\epsilon)]\}\ {}_{-2}R_{lm\omega}(r)\ {}_{-2}Z^m_l(\theta, a\omega)e^{-i\omega t}.$$

Outgoing Radiation Gauge: $A_\mu n^\mu = h_{\mu\nu} n^\nu = h^\mu_\mu = 0$

$$A_\mu(x, lm\omega P=\pm) = \rho^{*-2}[n_\mu(\delta + \pi^* - 2\alpha^*) - m_\mu(\Delta + \mu^* - 2\gamma^*)]\ {}_{+1}R_{lm\omega}(r)\ {}_{-1}Z^m_l(\theta, a\omega)e^{-i\omega t}$$
$$+ P\rho^{-2}[n_\mu(\delta^* + \pi - 2\alpha) - m^*_\mu(\Delta + \mu - 2\gamma)]\ {}_{+1}R_{lm\omega}(r)\ {}_{+1}Z^m_l(\theta, a\omega)e^{-i\omega t},$$

$$h_{\mu\nu}(x, lm\omega P=\pm) = \rho^{*-4}\{-n_\mu n_\nu(\delta - 3\alpha^* - \beta + 5\pi^*)(\delta - 4\alpha^* + \pi^*) - m_\mu m_\nu(\Delta + 5\mu^* - 3\gamma^* + \gamma)(\Delta + \mu^* - 4\gamma^*)$$
$$+ n_{(\mu} m_{\nu)}[(\delta + 5\pi^* + \beta - 3\alpha^* + \tau)(\Delta + \mu^* - 4\gamma^*)$$
$$+ (\Delta + 5\mu^* - \mu - 3\gamma^* - \gamma)(\delta - 4\alpha^* + \pi^*)]\}\ {}_{+2}R_{lm\omega}(r)\ {}_{-2}Z^m_l(\theta, a\omega)e^{-i\omega t}$$
$$+ P\rho^{-4}\{-n_\mu n_\nu(\delta^* - 3\alpha - \beta^* + 5\pi)(\delta^* - 4\alpha + \pi) - m^*_\mu m^*_\nu(\Delta + 5\mu - 3\gamma + \gamma^*)(\Delta + \mu - 4\gamma)$$
$$+ n_{(\mu} m^*_{\nu)}[(\delta^* + 5\pi + \beta^* - 3\alpha + \tau^*)(\Delta + \mu - 4\gamma)$$
$$+ (\Delta + 5\mu - \mu^* - 3\gamma - \gamma^*)(\delta^* - 4\alpha + \pi)]\}\ {}_{+2}R_{lm\omega}(r)\ {}_{+2}Z^m_l(\theta, a\omega)e^{-i\omega t}.$$

$$\phi_0 = (D - \epsilon + \epsilon^* - \rho^*)A_m - (\delta + \pi^* - \beta - \alpha^*)A_l$$
$$\phi_2 = (\delta^* + \alpha + \beta^* - \tau^*)A_n - (\Delta + \mu^* + \gamma - \gamma^*)A_{m^*}$$

$$2\psi_0 = (\delta + \pi^* - 3\beta - \alpha^*)(\delta + \pi^* - 2\beta - 2\alpha^*)h_{ll} + (D - \rho^* - 3\epsilon + \epsilon^*)(D - \rho^* - 2\epsilon + 2\epsilon^*)h_{mm}$$
$$-[(D - \rho^* - 3\epsilon + \epsilon^*)(\delta + 2\pi^* - 2\beta) + (\delta + \pi^* - 3\beta - \alpha^*)(D - 2\rho^* - 2\epsilon)]h_{(lm)}$$
$$2\psi_4 = (\delta^* - \tau^* + 3\alpha + \beta^*)(\delta^* - \tau^* + 2\alpha + 2\beta^*)h_{nn} + (\Delta + \mu^* + 3\gamma - \gamma^*)(\Delta + \mu^* + 2\gamma - 2\gamma^*)h_{m^*m^*}$$
$$-[(\Delta + \mu^* + 3\gamma - \gamma^*)(\delta^* - 2\tau^* + 2\alpha) + (\delta^* - \tau^* + 3\alpha + \beta^*)(\Delta + 2\mu^* + 2\gamma)]h_{(nm^*)}.$$

$$[K_{lm\omega P}]^* = PK_{l\,-m\,-\omega P}, \tag{3.9}$$

required for the metric $h^{\rm down}_{\mu\nu}$ to be real. P takes on the values ± 1 according to the parity state.

The ingoing piece of (3.4), when matched to (3.7) after the $r \to r^*$ exponent substitution, implies that

$$\sum_P K^{\rm down}_{lm\bar\omega P} = -\frac{2h\pi}{i\omega}\ {}_2N^2_{l;\pi}(a\omega)\delta_{m2}\delta(\bar\omega - \omega). \tag{3.10}$$

The crossing relations then give

$$K^{\rm down}_{lm\bar\omega P} = -\frac{h\pi}{i\omega}\ {}_2N^2_{l;\pi}(a\omega)[\delta_{m2}\delta(\bar\omega - \omega) - P\delta_{m\,-2}\delta(\bar\omega + \omega)]. \tag{3.11}$$

The outgoing piece of our plane wave may be used to fix the amplitude of $h^{\rm up}_{\mu\nu}$ in the outgoing gauge, which is transverse-tracefree at infinity for an "up" solution. If we take

$${}_{+2}R^{\rm up}e^{-i\bar\omega t} \underset{r\to\infty}{\sim} \frac{1}{\bar\omega^2}\frac{\exp[i\bar\omega(r^* - t)]}{r^5} \tag{3.12}$$

in the outgoing gauge in Table I, then

$$h^{\rm up}_{\mu\nu} = \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} K^{\rm up}_{lm\bar\omega P} h^{\rm up}_{\mu\nu}(x, lm\bar\omega P) \tag{3.13}$$

and

$$h^{\rm up}_{m^*m^*} \underset{r\to\infty}{\sim} \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} K^{\rm up}_{lm\bar\omega P}\ {}_{-2}Z^m_l(a\bar\omega)\frac{\exp[i\bar\omega(r^* - t)]}{r}, \tag{3.14}$$

together with (3.9). Comparison of $h^{\rm up}_{m^*m^*}$ with the outgoing part of (3.5) gives

$$K^{\rm up}_{lm\bar\omega P} = -(-1)^{l+m}K^{\rm down}_{lm\bar\omega P}, \tag{3.15}$$

when (A10) is used.

The metric perturbations $h^{\rm up}_{\mu\nu}$ and $h^{\rm down}_{\mu\nu}$ together constitute the asymptotic plane wave, yet they cannot simply be added since they are in different gauges. However, the perturbed Riemann tensor is obtained by linear operations on $h_{\mu\nu}$, so we may calculate the perturbed ψ_0 and ψ_4—being gauge invariant—by taking the sum of each of the contributions of $h^{\rm up}_{\mu\nu}$ and $h^{\rm down}_{\mu\nu}$. This avoids worry about adding two metric perturbations in different gauges.

To obtain a formula for $\psi_4^{\rm down}$, for example, we insert (3.6) with $h^{\rm down}_{\mu\nu}(x, lm\bar{\omega}P)$ given by the ingoing-gauge expression in Table I into the equation for ψ_4 also listed in the table. The resultant expression for $\psi_4^{\rm down}$, consisting of a fourth-order differential operator acting on some combination of ${}_{-2}R^{\rm down}$ with ${}_2Z_l^m(a\omega)$ and ${}_{-2}Z_l^m(a\omega)$, may be simplified using the properties of the Teukolsky functions. The formulas, valid at all radii, obtained in this fashion are

$$-8\psi_0^{\rm down} = \int_{-\infty}^{\infty} d\bar{\omega} \sum_{lmP} K^{\rm down}_{lm\bar{\omega}P}\, {}_{+2}R^{\rm down}\, {}_{+2}Z_l^m(a\bar{\omega}) e^{-i\bar{\omega}t},$$

$$-8\rho^{-4}\psi_4^{\rm down} = \int_{-\infty}^{\infty} d\bar{\omega} \sum_{lmP} ({\rm Re}C + 12iM\bar{\omega}P) \times K^{\rm down}_{lm\bar{\omega}P}\, {}_{-2}R^{\rm down}\, {}_{-2}Z_l^m(a\bar{\omega}) e^{-i\bar{\omega}t}, \quad (3.16)$$

$$-8\psi_0^{\rm up} = \int_{-\infty}^{\infty} d\bar{\omega} \sum_{lmP} ({\rm Re}C - 12iM\bar{\omega}P) \times K^{\rm up}_{lm\bar{\omega}P}\, {}_{+2}R^{\rm up}\, {}_{+2}Z_l^m(a\bar{\omega}) e^{-i\bar{\omega}t},$$

$$-8\rho^{-4}\psi_4^{\rm up} = \int_{-\infty}^{\infty} d\bar{\omega} \sum_{lmP} |C|^2 K^{\rm up}_{lm\bar{\omega}P}\, {}_{-2}R^{\rm up}\, {}_{-2}Z_l^m(a\bar{\omega}) e^{-i\bar{\omega}t},$$

where

$$C = {\rm Re}C + 12iM\omega \quad (3.17)$$

is defined by (A16). With regards to the derivation of the above, it is important to note that the asymptotic form of the perturbed metric does not constitute sufficient information to derive the $r\to\infty$ limit of (3.16) needed to specify an incident plane wave. This is why the asymptotic perturbed metric has been matched to exact formulas for Kerr metric perturbations.

Rather than directly sum the coefficients as in (3.16) we choose to consider each l, m, ω, P mode separately since this is how they enter numerical calculations. Eqs. (3.11), (3.15), and (3.16) allow us to write

$$\psi_0^{\rm down} \sim -2|\omega|^2 N \frac{e^{-i\bar{\omega}(r^*+t)}}{r}\, {}_{+2}Z_l^m(a\bar{\omega}),$$

$$\psi_0^{\rm up} \sim \frac{(-1)^{l+m}N}{8|\omega|^2}({\rm Re}C - 12iPM\bar{\omega})\frac{e^{i\bar{\omega}(r^*-t)}}{r^5}\, {}_{+2}Z_l^m(a\bar{\omega}), \quad (3.18)$$

$$\psi_4^{\rm down} \sim \frac{-N}{32|\omega|^2}({\rm Re}C + 12iPM\bar{\omega})\frac{e^{-i\bar{\omega}(r^*+t)}}{r^5}\, {}_{-2}Z_l^m(a\bar{\omega}),$$

$$\psi_4^{\rm up} \sim \tfrac{1}{2}(-1)^{l+m}|\omega|^2 N \frac{e^{i\omega(r^*-t)}}{r}\, {}_{-2}Z_l^m(a\bar{\omega}),$$

with

$$N = -\frac{h\pi}{i\omega}\, {}_2N^2_{l;\pi}(a\omega)\left(\frac{\bar{\omega}}{\omega}\right)^{(1+P)/2} \quad (3.19)$$

for the nonzero modes: ($m=2$; $\bar{\omega}=\omega$; $P=+1, -1$; $l\geq 2$) and ($m=-2$; $\bar{\omega}=-\omega$; $P=+1,-1$; $l\geq 2$). To obtain (3.18) we have used the fact that

$$ {}_{+2}R^{\rm down} e^{-i\bar{\omega}t} \sim 16\bar{\omega}^2 \frac{\exp[-i\bar{\omega}(r^*+t)]}{r},$$
$$ {}_{-2}R^{\rm up} e^{-i\bar{\omega}t} \sim \frac{4r^3\bar{\omega}^2}{|C|^2}\exp[i\bar{\omega}(r^*-t)] \quad (3.20)$$

are normalizations consistent with (3.8) and (3.12). Our representation of a distorted plane wave in ψ_4, then, is the sum (for the accessible l, m, ω, P modes) $\psi_4^{\rm up}+\psi_4^{\rm down}$; similarly, $\psi_0^{\rm up}+\psi_0^{\rm down}$ is the corresponding distorted plane wave in ψ_0.

The prescription that we have followed to obtain the above, namely substituting $r\to r^*$ in the exponents, can be seen to give, in fact, a minimally distorted plane wave from the following argument. A typical component of the flat-space metric is

$$h_{xx} = h\,{\rm Re}\{\exp[i\omega(z-t)]\}, \quad (3.21)$$

or asymptotically

$$h_{xx} \sim h{\rm Re}\left\{\frac{1}{2i\omega r}e^{-i\omega t}\sum_l (2l+1)[e^{i\omega r} - (-1)^l e^{-i\omega r}]P_l(\theta)\right\}, \quad (3.22)$$

since

$$e^{i\omega z} \sim \frac{1}{2i\omega r}\sum_l (2l+1)[e^{i\omega r} - (-1)^l e^{-i\omega r}]P_l(\theta). \quad (3.23)$$

According to our prescription, we then make the substitution $r\to r^*$ in the exponents in (3.22). (The metric $h_{\mu\nu}$ should strictly be expressed in spherical coordinates before the spherical harmonic decomposition, but the terms arising in this way do not affect the phase exponents.) The sum in (3.22) becomes precisely the sum which gives the distorted scalar plane wave appearing in (2.1). Hence, the metric is

$$h_{xx} = -h_{yy} = h\,\mathrm{Re}\{\exp[i\omega(z - 2M\ln\omega(r-z) - t)]\},$$
$$h_{xy} = h\,\mathrm{Im}\{\exp[i\omega(z - 2M\ln\omega(r-z) - t)]\}, \tag{3.24}$$

which is the flat-space expression with the distorted phase.

This minimally distorted plane wave (3.24) is transverse to lowest order in M/r in the incident region ($z<0$), and it suffers all the peculiarities of the scalar expression for $z=r$ downstream of the scattering. However, it must be remembered that these are only asymptotic expressions, so the singularity near $r-z=0$ (angle of scattering near zero) is masked because of the long-range small-angle tail of the Newtonian scattering.

IV. A RIEMANN-TENSOR APPROACH TO GRAVITATIONAL PLANE WAVES

The Riemann tensor components ψ_0 and ψ_4 for a distorted plane wave may be derived by more direct alternate means that do not necessitate the introduction of formulas for the Kerr metric perturbations. This second method for deriving formulas for ψ_0 and ψ_4 entails making the substitution $r \to r^*$ in the flat-space formulas for $\psi_0^{\rm down}$ and $\psi_4^{\rm up}$ and using the Newman-Penrose equations to determine the r^{-5} terms $\psi_0^{\rm up}$ and $\psi_4^{\rm down}$. By following this procedure, we implicitly build in a phase relation between the ingoing and outgoing pieces of the distorted plane wave, which is difficult to justify as being the correct choice of phase. However, the resultant formulas are identical to those found in Sec. III (i.e. the ingoing-outgoing phase relation coincides with our previous choice), thereby giving us added confidence in (3.18) for ψ_0 and ψ_4.

For a plane gravitational wave traveling up the z axis on a flat background, it is straightforward to show that

$$\psi_0 = b(1-\cos\theta)^2 e^{2i\varphi}\exp[i\omega(z-t)], \tag{4.1}$$

$$\psi_4 = \tfrac{1}{4} b(1+\cos\theta)^2 e^{2i\varphi}\exp[i\omega(z-t)], \tag{4.2}$$

($b = -\frac{1}{2}h\omega^2$), which may be expanded in terms of spheroidal harmonics to find the asymptotic forms

$$\psi_0 \sim \sum_l 8\pi\frac{b}{i}\left[\frac{24\,{}_2N^2_{l;0}(a\omega)}{(\omega r)^5}e^{i\omega r} - \frac{{}_2N^2_{l;\pi}(a\omega)e^{-i\omega r}}{\omega r}\right] \times {}_2Z^2_l(a\omega)e^{-i\omega t} \tag{4.3}$$

and

$$\psi_4 \sim \sum_l 2\pi\frac{b}{i}\left[{}_{-2}N^2_{l;0}(a\omega)\frac{e^{i\omega r}}{\omega r} - 24\frac{{}_{-2}N^2_{l;\pi}(a\omega)e^{-i\omega r}}{(\omega r)^5}\right] \times {}_{-2}Z^2_l(a\omega)e^{-i\omega t}. \tag{4.4}$$

If we follow the prescription of setting $r \to r^*$ in the exponents and taking the resulting expressions as the asymptotic values of ψ_0 and ψ_4 on the Kerr background, then the power flux per l mode of ingoing and outgoing radiation at infinity (as computed from ψ_4) does not balance. Using (4.4) and the Press and Teukolsky formulas[8] for power flux in terms of ψ_4, we obtain

$$\frac{dE_{\rm in}}{dt} = \frac{128\omega^6}{|C|^2}\frac{4\pi^2 b^2}{\omega^{10}}(24)^2({}_{-2}N^2_{l;\pi})^2 \tag{4.5}$$

and

$$\frac{dE_{\rm out}}{dt} = \frac{1}{2\omega^2}\frac{4\pi^2 b^2}{\omega^2}({}_{-2}N^2_{l;0})^2. \tag{4.6}$$

As shown in Appendix A, their ratio is given by

$$\frac{dE_{\rm out}}{dE_{\rm in}} = \frac{({}_{-2}N^2_{l;0})^2}{256}|C|^2\frac{1}{(24)^2}({}_{-2}N^2_{l;\pi})^{-2} = \frac{|C|^2}{|\mathrm{Re}C|^2}, \tag{4.7}$$

and is unity only if $M=0$, i.e. on a flat background. A similar discrepancy arises when the ψ_0 ingoing and outgoing power-flux formulas are compared; however, if we use the r^{-1} part of ψ_0 to describe the ingoing flux and the r^{-1} part of ψ_4 to describe the outgoing flux, then we find their ratio to be unity. Therefore it must be concluded that the coefficients of the r^{-5} terms are incorrect as given by (4.3) and (4.4).

To correct the r^{-5} pieces in the expressions for ψ_0 and ψ_4, we must use the Bianchi identities together with the presumably correct r^{-1} terms $\psi_4^{\rm up}$ and $\psi_0^{\rm down}$. Eqs. (4.3) and (4.4) lead to the (asymptotic) identification

$$\psi_0^{\rm down} \sim \sum_l -\frac{8\pi b}{i}\,{}_2N^2_{l;\pi}\frac{\exp[-i\omega(r^*+t)]}{\omega r}\,{}_2Z^2_l(a\omega) \tag{4.8}$$

and

$$\psi_4^{\rm up} \sim \sum_l \frac{2\pi b}{i}\,{}_{-2}N^2_{l;0}\frac{\exp[i\omega(r^*-t)]}{\omega r}\,{}_{-2}Z^2_l(a\omega). \tag{4.9}$$

Since ψ_0 and ψ_4 completely determine each other,[17] we may find $\psi_0^{\rm up}$ from $\psi_4^{\rm up}$ and $\psi_4^{\rm down}$ from $\psi_0^{\rm down}$. Specifically, the Bianchi identities together with the spin-coefficient equations lead to the connection formulas

$$D^4\rho^{-4}\psi_4=\rho^{-4}(\delta^*+3\alpha+\beta^*)(\delta^*+2\alpha+2\beta^*)(\delta^*+\alpha+3\beta^*)(\delta^*+4\beta^*)\psi_0-3M\frac{\partial}{\partial t}\psi_0^* \tag{4.10}$$

and

$$\rho^{-4}(\Delta+5\mu^*+3\gamma-\gamma^*)(\Delta+5\mu^*+2\gamma-2\gamma^*)(\Delta+5\mu^*+\gamma-3\gamma^*)(\Delta+\mu^*-4\gamma^*)\psi_0$$
$$=(\delta+5\pi^*-3\beta-\alpha^*)(\delta+5\pi^*-2\beta-2\alpha^*)(\delta+5\pi^*-\beta-3\alpha^*)(\delta+\pi^*-4\alpha^*)\rho^{-4}\psi_4+3M\frac{\partial}{\partial t}\psi_4^*. \tag{4.11}$$

Substitution of the "down" and "up" formulas into, respectively, (4.10) and (4.11) gives (3.18) summed over parity states.

The result summed over parities has a remarkable feature: Only one value of m appears in the ingoing (outgoing) part of ψ_0 (ψ_4), while both m and its negative appear in the outgoing (ingoing) parts. For instance, consider the composite form of ψ_0:

$$\psi_0\sim\sum_l\frac{4\pi h\omega^2}{i}\,{}_2N^2_{l;\pi}(a\omega)\left[\frac{\exp[-i\omega(r^*+t)]}{\omega r}\,{}_2S^2_l(a\omega)e^{2i\varphi}\right.$$
$$\left.-(-1)^{l+m}\left(\frac{\mathrm{Re}C}{16}\,\frac{e^{+i\omega(r^*-t)}}{\omega^5r^5}\,{}_2S^2_l(a\omega)e^{2i\varphi}-\frac{3iM\omega}{4}\,\frac{\exp[-i\omega(r^*-t)]}{\omega^5r^5}\,{}_2S_l^{-2}(-a\omega)e^{-2i\varphi}\right)\right]. \tag{4.12}$$

An $m=-2$ term occurs in the outgoing piece only. This anomalous outgoing piece has no convenient interpretation nor do the connection formulas (4.10), (4.11) used to derive (4.12). In addition, this extra term makes the summed-over parity-states result unsuitable for the purpose of numerical calculations.

To make sense out of the connection formulas and the resultant expressions for the distorted plane wave, one must consider separately the scattering of each parity state. Notice in (3.18) that each parity solution has both ingoing and outgoing parts for each m; connection formulas (4.10) and (4.11) relate in a comprehensible fashion ψ_0 and ψ_4 for the ingoing or outgoing piece of either parity state. It is an unfortunate interference between the two parity solutions which renders the peculiar form of (4.12).

The fact that the two parity solutions scatter differently (and, accordingly, have different reference phase behavior) is a somewhat surprising result but is the same type of phenomenon as the dependence of cross sections on the value of m. $P=\pm 1$ describe two different eigenstates of the system and have to be treated separately.

The obvious source of this complicating feature of gravitational wave scattering is that the constant C is complex, a result whose origins, unfortunately, are not quite so obvious. Investigation of the Bianchi identities leading to the connection formulas shows that the presence of ImC is attributable to the fact that one is perturbing a field whose background value is nonzero. The appearance of ImC in the potential formulation used in Sec. III is more obscure because ImC arises only at the end of a rather lengthy calculation.

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APPENDIX A: THE TEUKOLSKY FUNCTIONS AND THEIR SYMMETRIES AND ASYMPTOTIC BEHAVIOR.

As shown by Teukolsky,[4] the equations for the Weyl tensor components and electromagnetic field components

$$\begin{aligned}
&\psi_0=-C_{\mu\nu\alpha\beta}l^\mu m^\nu l^\alpha m^\beta \quad (s=2)\\
&\phi_0=F_{\mu\nu}l^\mu m^\nu \quad (s=1)\\
&(r-ia\cos\theta)^2\phi_2=(r-ia\cos\theta)^2F_{\mu\nu}m^{*\mu}n^\nu \quad (s=-1)\\
&(r-ia\cos\theta)^4\psi_4=-(r-ia\cos\theta)^4C_{\mu\nu\alpha\beta}n^\mu m^{*\nu}n^\alpha m^{*\beta} \quad (s=-2)
\end{aligned} \tag{A1}$$

decouple and separate provided that the null tetrad (in Boyer-Lindquist $[t,r,\theta,\varphi]$ coordinates) coordinates)

$$\begin{aligned}
&l^\alpha=[(r^2+a^2)/\Delta,1,0,a/\Delta]\\
&n^\alpha=\frac{[r^2+a^2,-\Delta,0,a]}{2(r^2+a^2\cos^2\theta)},\\
&m^\alpha=\frac{[ia\sin\theta,0,1,i/\sin\theta]}{\sqrt{2}\,(r+ia\cos\theta)}
\end{aligned} \tag{A2}$$

is used. In this tetrad, the only nonzero spin coefficients are

$$\rho=\frac{-1}{r-ia\cos\theta},\quad \beta=\frac{-\rho^*\cot\theta}{2\sqrt{2}},\quad \alpha=\pi-\beta^*,$$
$$\pi=\frac{ia\rho^2\sin\theta}{\sqrt{2}},\quad \tau=\frac{-ia\rho\rho^*\sin\theta}{\sqrt{2}},\quad \mu=\rho^2\rho^*\tfrac{1}{2}\Delta, \tag{A3}$$
$$\gamma=\mu+\rho\rho^*\tfrac{1}{2}(r-M)$$

with $\Delta=r^2-2Mr+a^2$. (These spin coefficients and the directional derivatives $D=l^\mu\partial_\mu$, $\Delta=n^\mu\partial_\mu$, and $\delta=m^\mu\partial_\mu$ appear in Table I.)

The separable radial and angular functions, ${}_sR_{lm\omega}(r)$ and ${}_sZ_l^m(\theta,a\omega)$, are labeled by the spin weight s of the field function; ψ_0, for example, has $s=2$ and separates as follows:

$$\psi_0=\sum_{l,m}\int d\omega\,{}_2R_{lm\omega}(r)\,{}_2Z_l^m(\theta,a\omega)e^{-i\omega t}. \tag{A4}$$

The differential equation satisfied by ${}_sR_{lm\omega}(r)$ is given by Teukolsky[4], and the angular function ${}_sS_l^m(\theta,a\omega)$, defined by

$${}_sZ_l^m(\theta,a\omega)={}_sS_l^m(\theta,a\omega)e^{im\varphi}, \tag{A5}$$

is a solution to the second-order differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}S\right)+\left(a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}-2a\omega s\cos\theta-\frac{2ms}{\sin^2\theta}\cos\theta-s^2\cot^2\theta+E-s^2\right)S=0 \tag{A6}$$

with E an eigenvalue equal to $l(l+1)$ when $a\omega=0$. The harmonics are normalized to give

$$\int {}_sZ_l^m(\theta,a\omega)\,{}_sZ_l^m(\theta,a\omega)^*d\Omega=1.$$

The constants introduced in the asymptotic expansions (3.4) and (3.5) are coefficients in the limiting behavior of ${}_sS_l^m$ near $\theta=0$ and $\theta=\pi$. Investigation of the limiting form of (A6) near the poles gives

$${}_sS_l^m \underset{\theta\to 0}{\sim} \theta^{|m+s|}\,{}_sN_{l;0}^m(a\omega),$$
$${}_sS_l^m \underset{\theta\to\pi}{\sim} (\pi-\theta)^{|m-s|}\,{}_sN_{l;\pi}^m(a\omega), \tag{A7}$$

where ${}_sN_{l;0}^m(a\omega)$ and ${}_sN_{l;\pi}^m(a\omega)$ are constants.

Because of the symmetries of the angular functions[18]

$${}_{-s}S_l^m(\theta,a\omega)=(-1)^{l+m}\,{}_sS_l^m(\pi-\theta,a\omega),$$
$${}_sS_l^m(\theta,-a\omega)=(-1)^{l+s}\,{}_sS_l^{-m}(\pi-\theta,a\omega), \tag{A8}$$

we have

$${}_{-s}N_{l;0}^m=(-1)^{l+m}\,{}_sN_{l;\pi}^m, \tag{A9}$$

$${}_sN_{l;0}^m(a\omega)=(-1)^{l+s}\,{}_sN_{l;\pi}^{-m}(-a\omega). \tag{A10}$$

Additional relationships among the constants follow with the aid of the operators

$$\mathfrak{L}_n(m,\omega)=\partial_\theta+m\csc\theta-a\omega\sin\theta+n\cot\theta \tag{A11}$$

and

$$\mathfrak{L}^\dagger(m,\omega)=\mathfrak{L}(-m,-\omega) \tag{A12}$$

introduced by Teukolsky and Press.[8] They act as the analog of raising and lowering operators connecting angular functions ${}_sS_l^m$ with those of opposite spin weight. Specifically, one has

$$\mathfrak{L}_0\mathfrak{L}_1\,{}_1S_l^m=B\,{}_{-1}S_l^m \tag{A13}$$

and

$$\mathfrak{L}_{-1}\mathfrak{L}_0\mathfrak{L}_1\mathfrak{L}_2\,{}_2S_l^2=(\mathrm{Re}C)\,{}_{-2}S_l^2, \tag{A14}$$

where B and $\mathrm{Re}C>0$ are constants shown by Teukolsky and Press[8] and Starobinsky[19] to take the values

$$B=+[(E+a^2\omega^2-2am\omega)^2+4am\omega-4a^2\omega^2]^{1/2} \tag{A15}$$

and

$$|\mathrm{Re}C|^2=(Q^2+4a\omega m-4a^2\omega^2)[(Q-2)^2+36a\omega m-36a^2\omega^2]+(2Q-1)(96a^2\omega^2-48a\omega m)-144\omega^2a^2, \tag{A16}$$

with $Q\equiv E+a^2\omega^2-2a\omega m$.

Consider (A13) for $\theta=0$ and suppose $m>0$. Keeping only the dominant terms in this limit, we find

$${}_1N_{l;0}^m(2m+2)2m\theta^{|m-1|}=B\,{}_{-1}N_{l;0}^m\theta^{|m-1|},$$
$$4(m+1)m\frac{{}_1N_{l;0}^m}{{}_{-1}N_{l;0}^m}=B. \tag{A17}$$

A similar analysis in the gravitational case leads to the result

$$384\,{}_2N_{l;0}^2=(\mathrm{Re}C)\,{}_{-2}N_{l;0}^2. \tag{A18}$$

APPENDIX B: ELECTROMAGNETIC PLANE WAVE IN A KERR BACKGROUND

Derivation of formulas for the electromagnetic-field components ϕ_0 and ϕ_2 describing a plane wave incident along the z axis of a Kerr black hole proceeds in the same fashion as the derivation of the ψ_0 and ψ_4 expressions in the text. We make the substitution $r\to r^*$ in the exponents of a transverse vector potential A_μ, match the amplitudes of the ingoing and outgoing pieces to formulas for the vector potential in a Kerr background, and use these formulas to find ϕ_0 and ϕ_2 for the plane wave.

We start with a flat-space plane wave traveling up the z axis:

$$A_\mu = A \begin{bmatrix} 0 \\ \cos\omega(t-z) \\ \sin\omega(t-z) \\ 0 \end{bmatrix}. \tag{B1}$$

The asymptotic ($r\to\infty$) behavior of the spherical transverse pieces of the above are

$$A_m = A_\mu m^\mu \underset{r\to\infty}{\simeq} \frac{A}{2\sqrt{2}}\{(\cos\theta-1)\exp[i\omega(z-t)+i\varphi]+(\cos\theta+1)\exp[-i\omega(z-t)-i\varphi]\}$$

$$=\sum_l \frac{\sqrt{2}A}{i\omega r}\pi\{{}_1N^1_{l;\pi}(a\omega)\,{}_1Z^1_l(\theta,a\omega)\exp[-i\omega(r+t)]-{}_1N^{-1}_{l;0}(-a\omega)\,{}_1Z^{-1}_l(\theta,-a\omega)\exp[-i\omega(r-t)]\} \tag{B2}$$

and

$$A_{m^*} \underset{r\to\infty}{\simeq} \sum_l \frac{\sqrt{2}A\pi}{i\omega r}\{-{}_{-1}N^{-1}_{l;\pi}(-a\omega)\,{}_{-1}Z^{-1}_l(\theta,-a\omega)\exp[i\omega(r+t)]+{}_{-1}N^1_{l;0}(a\omega)\,{}_{-1}Z^1_l(\theta,a\omega)\exp[i\omega(r-t)]\}. \tag{B3}$$

After the substitution $r\to r^*$ in the exponents, (B2) and (B3) may be matched to the asymptotic forms of vector-potential perturbations of the Kerr metric.

First consider the ingoing piece of the vector potential $A^{\rm down}_\mu$. In the ingoing gauge listed in Table I, $A^{\rm down}_\mu$ is transverse at infinity and may be expanded as follows:

$$A^{\rm down}_\mu = \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} K^{\rm down}_{lm\bar\omega P} A^{\rm down}_\mu(x, lm\bar\omega P). \tag{B4}$$

If we normalize the radial function in the ingoing-gauge vector potential in Table I to give

$${}_{-1}R^{\rm down}e^{-i\bar\omega t} \sim \frac{1}{2i\bar\omega}\frac{\exp[-i\bar\omega(r^*+t)]}{r}, \tag{B5}$$

then asymptotically $A^{\rm down}_m$ behaves like

$$A^{\rm down}_m \sim \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} K^{\rm down}_{lm\bar\omega P}\,{}_{+1}Z^m_l(\theta,a\bar\omega)\frac{\exp[-i\bar\omega(r^*+t)]}{r}. \tag{B6}$$

As in the gravitational case, imposition of the reality condition for the vector potential leads to the crossing relation (3.9). Hence when (B6) is matched to the ingoing piece of (B2), we find

$$K^{\rm down}_{lm\bar\omega P} = \frac{A\pi}{\sqrt{2}i\omega}\,{}_1N^1_{l;\pi}(a\omega)[\delta_{m1}\delta(\omega-\bar\omega)-P\delta_{m-1}\delta(\omega+\bar\omega)]. \tag{B7}$$

The outgoing piece of the vector potential $A^{\rm up}_\mu$ is transverse at infinity in the outgoing gauge listed in Table I, so its amplitude is fixed by comparison with the outgoing part of (B3). If $A^{\rm up}_\mu$ is expanded in spheroidal harmonics

$$A^{\rm up}_\mu = \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} K^{\rm up}_{lm\bar\omega P} A^{\rm up}_\mu(x, lm\bar\omega P), \tag{B8}$$

then the asymptotic form of $A^{\rm up}_{m^*}$ is

$$A^{\rm up}_{m^*} \sim \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} K^{\rm up}_{lm\bar\omega P}\,{}_{-1}Z^m_l(\theta,a\bar\omega) \times \frac{\exp[i\bar\omega(r^*-t)]}{r}, \tag{B9}$$

if

$${}_{+1}R^{\rm up}e^{-i\bar\omega t} \underset{r\to\infty}{\sim} -\frac{1}{i\bar\omega}\frac{\exp[i\bar\omega(r^*-t)]}{r^3}. \tag{B10}$$

Equation (B9), when matched to the outgoing piece of (B4), gives

$$K^{\rm up}_{lm\bar\omega P} = (-1)^{l+m} K^{\rm down}_{lm\bar\omega P}\,. \tag{B11}$$

With the aid of Table I, we obtain formulas for the electromagnetic field,

$$-2\phi_0^{\rm down} = \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} K^{\rm down}_{lm\bar\omega P}\,{}_{+1}R^{\rm down}\,{}_{+1}Z^m_l(a\bar\omega)e^{-i\bar\omega t},$$

$$-2\rho^{-2}\phi_2^{\rm down} = \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} BK^{\rm down}_{lm\bar\omega P}\,{}_{-1}R^{\rm down}\,{}_{-1}Z^m_l(a\bar\omega)e^{-i\bar\omega t}, \tag{B12}$$

$$-2\phi_0^{\rm up} = \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} BK^{\rm up}_{lm\bar\omega P}\,{}_{+1}R^{\rm up}\,{}_{+1}Z^m_l(a\bar\omega)e^{-i\bar\omega t},$$

$$-2\rho^{-2}\phi_2^{\rm up} = \int_{-\infty}^{\infty} d\bar\omega \sum_{l,m,P} B^2K^{\rm up}_{lm\bar\omega P}\,{}_{-1}R^{\rm up}\,{}_{-1}Z^m_l(a\bar\omega)e^{-i\bar\omega t},$$

where the constant B is given by (A15). Taken together with (B7) and (B11), these formulas yield the asymptotic forms

$$\phi_0^{\text{down}} \sim -2i\bar{\omega}N\frac{\exp -i\bar{\omega}(r^*+t)}{r}\,{}_{+1}Z_l^m(\theta,a\bar{\omega}),$$
$$\phi_0^{\text{up}} \sim \frac{(-1)^{l+m}BN}{2i\bar{\omega}}\frac{\exp[i\bar{\omega}(r^*-t)]}{r^3}\,{}_{+1}Z_l^m(\theta,a\bar{\omega}),$$
$$\phi_2^{\text{down}} \sim -\frac{BN}{4i\bar{\omega}}\frac{\exp[-i\bar{\omega}(r^*+t)]}{r^3}\,{}_{-1}Z_l^m(\theta,a\bar{\omega}),\tag{B13}$$
$$\phi_2^{\text{up}} \sim (-1)^{l+m}Ni\bar{\omega}\frac{\exp[i\bar{\omega}(r^*-t)]}{r}\,{}_{-1}Z_l^m(\theta,a\bar{\omega}),$$

with

$$N=\left(\frac{\omega}{\bar{\omega}}\right)^{(1+P)/2}\frac{A\pi}{\sqrt{2}\,i\omega}\,{}_1N_{l;\pi}^1(a\omega)\tag{B14}$$

for the nonzero modes: ($m=1$; $\omega=\bar{\omega}$; $P=+1,-1$; $l\geq 1$) and ($m=-1$; $\bar{\omega}=-\omega$; $P=+1,-1$; $l\geq 1$). The limits

$${}_{+1}R^{\text{down}}e^{-i\omega t} \sim 4i\omega\frac{\exp[-i\omega(r^*+t)]}{r},$$
$${}_{-1}R^{\text{up}}e^{-i\omega t} \sim \frac{-2i\omega r}{B^2}\exp[i\bar{\omega}(r^*-t)]\tag{B15}$$

are used to obtain the above.

An alternative method for deriving (B13), modeled after the discussion in Sec. IV for the gravitational case, is much simpler. Start with the formulas for ϕ_0 and ϕ_2 for a flat-space plane wave

$$\phi_0=\frac{i\omega A}{\sqrt{2}}(1-\cos\theta)\exp[i\omega(z-t)+i\varphi],$$
$$\phi_2=\frac{i\omega A}{2\sqrt{2}}(1+\cos\theta)\exp[i\omega(z-t)+i\varphi].\tag{B16}$$

Then the expansion of (B16) in spheroidal harmonics

$$\phi_0\sim\sum_l -2\sqrt{2}\pi\omega A\left[{}_1N_{l;\pi}^1(a\omega)\frac{\exp[-i\omega(r+t)]}{\omega r}\,{}_1Z_l^1(\theta,a\omega)+2\,{}_1N_{l;0}^1(a\omega)\frac{\exp i\omega(r-t)}{(\omega r)^3}\,{}_1Z_l^1(\theta,a\omega)\right]\tag{B17}$$

and

$$\phi_2\sim\sum_l \sqrt{2}\pi\omega A\left[{}_{-1}N_{l;0}^1(a\omega)\frac{\exp[i\omega(r-t)]}{\omega r}\,{}_{-1}Z_l^1(\theta,a\omega)+2\,{}_{-1}N_{l;\pi}^1(a\omega)\frac{\exp[-i\omega(r+t)]}{(\omega r)^3}\,{}_{-1}Z_l^1(\theta,a\omega)\right]\tag{B18}$$

gives the correct asymptotic forms for ϕ_0 and ϕ_2 when the substitution $r\rightarrow r^*$ is made in the exponents.

Equations (B17) and (B18) are in a form suitable for numerical calculations, although the parity states have been summed over. Unlike the gravitational case, the two parity solutions do not scatter differently [i.e., (B13) do not explicitly depend on P], so no anomalous term appears in the parity-summed form of the solution.

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