PHYSICAL REVIEW D VOLUME 14, NUMBER 10 15 NOVEMBER 1976

Meson theory with internal coordinates

F. C. Hoh
8931 N. La Crosse, Skokie, Illinois 60076
(Received 30 October 1975)

This paper is a continuation of the preceding paper in which a generalized Bethe-Salpeter equation in the ladder approximation for a quark-antiquark pair is presented. The generalized equation includes functions of a set of three complex coordinates, called internal coordinates, spanning an abstract three-dimensional complex space. These functions are essential in describing mesons. This generalized equation is treated in this paper. Upon consistent restrictions, it is reduced, in the case of pseudoscalar mesons, to one nine-component tensor equation involving internal coordinates only. Keeping the zero-order SU_3-symmetry-preserving interaction only, the tensor equation is further reduced to two coupled radial equations and finally to two algebraic recurrence relations for the η' meson. In the η-meson case, four such relations are obtained. Preliminary treatment of the η' case indicates that the internal interaction is strong. By including the interaction term transforming like the eighth component of an SU_3 octet vector as a first-order perturbation, the Gell-Mann–Okubo formula is reproduced with the coefficients determined by given relations. Necessary removal of possible degeneracy in the zero-order states leads to mixing of these states.

I. INTRODUCTION

The present paper is a continuation of the preceding paper,[1] hereafter referred to as I. In I, the free-particle Dirac wave function $\chi(x)$ was generalized to $\chi(x)\xi^a(z)$, where z denotes a set of three complex coordinates, called internal coordinates, in an abstract complex three-dimensional space or internal space; $\xi^a(z)$ was assumed to contain a description of the state of a quark triplet and a runs from 1 to 3. The mass in the free-particle Dirac equation was replaced by a second-order tensor operator $-\partial_a^b$ operating on $\xi^a(z)$. The so-generalized Dirac equation was assumed to include a description of a free-quark triplet. Then, interactions, in both space-time and in the internal space, between two quark triplets were introduced. These interactions preserve both Lorentz invariance and U_3 invariance in the internal space. Subsequently, two SU_3-symmetry-breaking interaction terms, one transforming like the eighth component of an SU_3 octet vector and the other like the SU_3 charge operator, were similarly introduced.

In describing the interactions between the two quark triplets so far, the total wave function of one of the quark triplets was assumed to be known which, however, is not the case. In quantum theory with given and constant masses, the Bethe-Salpeter equation[2] describes an interacting two-particle system. Thus, the Bethe-Salpeter equation in the so-called ladder approximation was similarly generalized to include functions in the internal space. The so-generalized Bethe-Salpeter equation was further modified to describe a quark triplet interacting with an antiquark triplet.

The purpose of this investigation is to apply the so-modified Bethe-Salpeter equation to mesons, in particular to two pseudoscalar mesons η' and η. In Sec. II, the generalized Bethe-Salpeter equation in the ladder approximation for a pair of quark-antiquark triplets, given in I, is presented together with the equations describing the interaction functions. Center-of-mass coordinates are introduced. In Sec. III, the space-time part of the meson equations is treated. Upon consistent restrictions, the meson equations for pseudoscalar mesons were reduced to one nine-component tensor equation involving internal coordinates only. Some mathematics to be used in treating this tensor equation is developed in Secs. IV and V. A set of eight-vector spherical harmonics in the internal space, corresponding to the usual vector spherical harmonics in space, is introduced. Some derivatives involving such harmonics are given.

The SU_3-symmetry-breaking interaction terms in the nine-component tensor equation are considered to be perturbations. Neglecting these terms, the nine-component tensor equation is regarded as a zero-order equation and it is shown in Sec. VI that the angular parts of this equation cancel out, leaving two coupled radial equations in the η' meson case and four such in the η meson case. In Sec. VII, the radial equations for the η' meson were reduced to two coupled algebraic recurrence relations. A preliminary calculation indicates that the internal SU_3-symmetry-preserving interaction is strong. A similar set of such recurrence relations for the η meson is mentioned in Sec. VIII.

In Sec. IX, the interaction term transforming like the eighth component of an SU_3 octet vector is included as a first-order perturbation, and the

Gell-Mann–Okubo formula[3] is reproduced with the coefficients determined by given relations involving eigenvalues and eigenfunctions of the zero-order equation. In Sec. X, necessary removal of possible degeneracy in the zero-order state is treated and leads to mixing of these states. Also, a second-order equation including the electromagnetic interaction term in the internal space is given but not solved. Finally, possible charmed mesons are briefly discussed in the present context.

Apart from the singlet mesons which have complex masses, the recurrence relations for the η meson have been worked out but neither reproduced nor solved numerically here. In order to make predictions about the masses of the pseudoscalar octet mesons, one needs to derive similar recurrence relations for the π and K mesons starting from (3.21) and following the same procedure used for the η' and η mesons here. These recurrence relations then need be solved numerically. In this process, the unknown interaction constants μ_{10} and μ_0 in (6.11) need be, and may hopefully be, determined.

II. THE GENERALIZED BETHE-SALPETER EQUATION

The generalized Bethe-Salpeter equation in the ladder approximation for a pair of quark-antiquark triplets, according to Eq. (5.10) of I, reads

$$
\begin{aligned}
&(i\gamma^\mu\partial_{\mu\mathrm{I}}\delta^a_b+\partial^a_{b\mathrm{I}})\Psi_{qa}(x_\mathrm{I},x_\mathrm{II})\Xi^b_d(z_\mathrm{I},z_\mathrm{II})(i\gamma^\mu\partial_{\mu\mathrm{II}}\delta^d_c+\partial^d_{c\mathrm{II}})\\
&\quad=i^2G_{Pm}(|x_\mathrm{I}-x_\mathrm{II}|)\gamma_5\Psi_{qa}(x_\mathrm{I},x_\mathrm{II})\gamma_5\Xi^a_c(z_\mathrm{I},z_\mathrm{II})+[G_\mathrm{ph}(|x_\mathrm{I}-x_\mathrm{II}|)+G_{Vm}(|x_\mathrm{I}-x_\mathrm{II}|)]\gamma_\mu\Psi_{qa}(x_\mathrm{I},x_\mathrm{II})\gamma^\mu\Xi^a_c(z_\mathrm{I},z_\mathrm{II})\\
&\qquad+\{\tau(|z_\mathrm{I}-z_\mathrm{II}|)\delta^a_b\delta^d_c+\omega(|z_\mathrm{I}-z_\mathrm{II}|)(\lambda_s)^a_b(\lambda_s)^d_c+[G_{mP}(|z_\mathrm{I}-z_\mathrm{II}|)+G_{mV}(|z_\mathrm{I}-z_\mathrm{II}|)][(\lambda_8)^a_b\delta^d_c+\delta^a_b(\lambda_8)^d_c]\\
&\qquad+G_\mathrm{em}(|z_\mathrm{I}-z_\mathrm{II}|)(Q^a_b\delta^d_c+\delta^a_bQ^d_c)\}\Psi_{qa}(x_\mathrm{I},x_\mathrm{II})\Xi^b_d(z_\mathrm{I},z_\mathrm{II}).
\end{aligned}
\tag{2.1}
$$

Equations (5.4)–(5.8) in I are

$$(\partial^a_b\partial^b_a-m_P{}^2)\tau(|z_\mathrm{I}-z_\mathrm{II}|)=(\Delta_c-m_P{}^2)\tau(|z_\mathrm{I}-z_\mathrm{II}|)=\mu_0\delta(z_\mathrm{I}-z_\mathrm{II}),\tag{2.2}$$

$$(\Delta_c-m_V{}^2)\omega(|z_\mathrm{I}-z_\mathrm{II}|)=\mu_n\delta(z_\mathrm{I}-z_\mathrm{II}),\tag{2.3}$$

$$(\Delta_c-m_P{}^2)G_{mP}(|z_\mathrm{I}-z_\mathrm{II}|)=\mu_8\delta(z_\mathrm{I}-z_\mathrm{II}),\tag{2.4}$$

$$(\Delta_c-m_V{}^2)G_{mV}(|z_\mathrm{I}-z_\mathrm{II}|)=\mu_8\delta(z_\mathrm{I}-z_\mathrm{II}),\tag{2.5}$$

$$\Delta_cG_\mathrm{em}(|z_\mathrm{I}-z_\mathrm{II}|)=\mu_Q\delta(z_\mathrm{I}-z_\mathrm{II}).\tag{2.6}$$

The primes in Eqs. (5.4)–(5.8) and (5.10) of I have been dropped here. The Greek indices μ and ν generally refer to space-time and each runs from 0 to 3. The Latin indices a,b,c,d generally refer to the internal space and each runs from 1 to 3. The Latin indices s,t,u,v also refer to the internal space and each runs from 0 to 8. The notations and the definitions of the symbols are given in I. In (2.1), the pseudoscalar, photon, and vector interaction functions in space-time, G_{Pm}, G_ph, and G_{Vm}, respectively, have been included. As mentioned in I, the right-hand side of (2.1) can be supplemented by

$$
\begin{aligned}
&\{G_0(|x_\mathrm{I}-x_\mathrm{II}|)\Psi_{qa}(x_\mathrm{I},x_\mathrm{II})\\
&\quad+i^2G_A(|x_\mathrm{I}-x_\mathrm{II}|)\gamma^\mu\gamma_5\Psi_{qa}(x_\mathrm{I},x_\mathrm{II})\gamma_5\gamma_\mu\\
&\quad+G_T(|x_\mathrm{I}-x_\mathrm{II}|)\sigma^{\mu\nu}\Psi_{qa}(x_\mathrm{I},x_\mathrm{II})\sigma^{\nu\mu}\}\Xi^a_c(z_\mathrm{I},z_\mathrm{II}).
\end{aligned}
\tag{2.7}
$$

Here G_0 refers to a scalar interaction in space-time, G_A a pseudovector one, G_T a tensor one, and $\sigma^{\mu\nu}=(\frac{1}{2}i)(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$. The corresponding internal interaction functions associated with (2.7) are of the same nature as those shown in the last term in (2.1) and are assumed to obey (2.2)–(2.5) with $m_P{}^2$ and $m_V{}^2$ replaced by other mass-separation constants. For the present purposes, it is not necessary to include these functions separately and therefore they will be left out. Ψ_{qa} has 16 components and Ξ^a_c has 9, so that $\Psi_{qa}\Xi^a_c$ has $9\times16=144$ components.

One can transform the coordinates x_I, x_II, z_I, and z_II into the relative coordinates x_μ, also denoted simply by x, z^a and z_a, also denoted simply by z, center-of-mass coordinates X^μ, and analogous such in the internal space, Z^a and Z_a:

$$
\begin{aligned}
&x^\mu=x^\mu_\mathrm{I}-x^\mu_\mathrm{II},\quad X^\mu=\alpha_\mathrm{I}x^\mu_\mathrm{I}+\alpha_\mathrm{II}x^\mu_\mathrm{II},\quad \alpha_\mathrm{I}+\alpha_\mathrm{II}=1,\\
&z^a=z^a_\mathrm{I}-z^a_\mathrm{II},\quad Z^a=\beta_\mathrm{I}z^a_\mathrm{I}+\beta_\mathrm{II}z^a_\mathrm{II},
\end{aligned}
\tag{2.8}
$$

where the α's and β's are constants. It follows that

$$
\begin{aligned}
&\partial_{\mu\mathrm{I}}=\alpha_\mathrm{I}\partial/\partial X^\mu+\partial/\partial x^\mu,\quad \partial_{\mu\mathrm{II}}=\alpha_\mathrm{II}\partial/\partial X^\mu-\partial/\partial x^\mu,\\
&\partial^a_\mathrm{I}=\beta^*_\mathrm{I}\partial/\partial Z_a+\partial/\partial z_a,\quad \partial^a_\mathrm{II}=\beta^*_\mathrm{II}\partial/\partial Z_a-\partial/\partial z_a,\\
&\partial^a_{b\mathrm{I}}=\partial^a_b+\beta_\mathrm{I}\partial^2/\partial z_a\partial Z^b+\beta^*_\mathrm{I}\partial^2/\partial Z_a\partial z^b+\beta_\mathrm{I}\beta^*_\mathrm{I}\partial^2/\partial Z_a\partial Z^b,\\
&\partial^a_{b\mathrm{II}}=\partial^a_b-\beta_\mathrm{II}\partial^2/\partial z_a\partial Z^b-\beta^*_\mathrm{II}\partial^2/\partial Z_a\partial z^b+\beta_\mathrm{II}\beta^*_\mathrm{II}\partial^2/\partial Z_a\partial Z^b,\\
&\partial^a_b=\partial^2/\partial z_a\partial z^b.
\end{aligned}
\tag{2.9}
$$

In analogy with the known procedure[2] in treating the Bethe-Salpeter equation, one can look for a solution of the type

$$\Psi_{qa}(x_{\rm I}, x_{\rm II})\Xi_d^b(z_{\rm I}, z_{\rm II}) = \exp[-i(K_\mu X^\mu + L_a Z^a + L^a Z_a)] \times \psi(x)\xi_d^b(z), \qquad (2.10)$$

where K_μ and L^a are constants. It can be seen that (2.10) satisfies a Klein-Gordon equation with a squared mass equal to $K_\mu K^\mu$. It will be assumed that $K_1 = K_2 = K_3 = 0$, so that the mass appearing in the Klein-Gordon equation is K_0. In analogy with this assumption, it will also be assumed that $L_1 = L_2 = L_3 = 0$. (2.10) now becomes

$$\Psi_{qa}(x_{\rm I}, x_{\rm II})\Xi_d^b(z_{\rm I}, z_{\rm II}) = e^{-iK_0 x^0}\psi(x)\xi_d^b(z). \qquad (2.11)$$

By making use of (2.11) and (2.9) and assuming $\alpha_{\rm I} = \alpha_{\rm II} = \frac{1}{2}$, (2.1) with the addition (2.7) becomes

$$(i\gamma^\mu\partial_\mu\delta_b^a + \tfrac{1}{2}\gamma^0 K_0\delta_b^a + \partial_b^a)\psi(x)\xi_d^b(z)(-i\gamma^\mu\partial_\mu\delta_c^d + \tfrac{1}{2}\gamma^0 K_0\delta_c^d + \partial_c^d)$$
$$= \{G_0(|x|)\psi(x) + i^2 G_{Pm}(|x|)\gamma_5\psi(x)\gamma_5 + (G_{\rm ph}(|x|) + G_{Vm}(|x|))\gamma^\mu\psi(x)\gamma_\mu + i^2 G_A(|x|)\gamma^\mu\gamma_5\psi(x)\gamma_5\gamma_\mu + G_T(|x|)\sigma^{\mu\nu}\psi(x)\sigma^{\nu\mu}\}\,\xi_c^a(z)$$
$$+ \{\tau(r)\delta_b^a\delta_c^d + \omega(r)(\lambda_s)_b^a(\lambda_s)_c^d + G_m(r)[(\lambda_8)_b^a\delta_c^d + \delta_b^a(\lambda_8)_c^d] + G_{\rm em}(r)[Q_b^a\delta_c^d + \delta_b^a Q_c^d]\}\,\psi(x)\xi_d^b, \qquad (2.12)$$

where

$$G_m(r) = G_{mP}(r) + G_{mV}(r), \qquad (2.13)$$

$$r = |z| \equiv |z_1 z^1 + z_2 z^2 + z_3 z^3|^{1/2}. \qquad (2.14)$$

III. REDUCTION OF THE SPACE-TIME PART

The space-time part of (2.12) will be treated first following the known treatments[4,5] of the Bethe-Salpeter equation. First, Wick's[6] rotation is introduced:

$$x_0 \to -ix_0', \quad \partial_0 \equiv \partial/\partial x^0 \to i\partial/\partial x'^0 \equiv \partial_0'. \qquad (3.1)$$

Following this rotation, a new four-dimensional Euclidean space is obtained. Next, spherical coordinates are introduced in this Euclidean space:

$$x_1 = R\sin\Theta\cos\Phi, \quad x_2 = R\sin\Theta\sin\Phi,$$
$$x_3 = R\cos\Theta, \quad x_0' = R'\cos\Gamma, \qquad (3.2)$$
$$R = R'\sin\Gamma, \quad x_0'^2 + x_1^2 + x_2^2 + x_3^2 = R'^2.$$

For $\psi(x)$ in (2.12), a solution of the following type[4] is looked for:

$$\psi(x) = \begin{pmatrix} s(x) + \vec\sigma\cdot\vec s(x) & t(x) + \vec\sigma\cdot\vec t(x) \\ u(x) + \vec\sigma\cdot\vec u(x) & v(x) + \vec\sigma\cdot\vec v(x)\end{pmatrix}. \qquad (3.3)$$

Here, $\vec\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli spin matrices. The function $s(x)$ is now expanded in terms of hyperspherical harmonics[4,5]:

$$s(x) = \sum_{N=0}^{\infty}\sum_{l=0}^{N}\sum_{m=-l}^{l} s(R', N, l, m)\mathcal{Y}_{lN}^m(\Gamma, \Theta, \Phi). \qquad (3.4)$$

Similar expansions are also carried out for $t(x)$, $u(x)$, and $v(x)$. The function $\vec s(x)$ is expanded in terms of vector hyperspherical harmonics:

$$\vec s(x) = \sum_{N=0}^{\infty}\sum_{l=0}^{N}\sum_{L=l-1}^{l+1}\sum_{m=-l}^{l} s(R', N, L, l, m) \times \vec{\mathcal{Y}}_{NlL1}^m(\Gamma, \Theta, \Phi). \qquad (3.5)$$

Again, similar expansions are also carried out for $\vec t(x)$, $\vec u(x)$, and $\vec v(x)$. We can write

$$\mathcal{Y}_{lN}^m(\Gamma, \Theta, \Phi) = Y_{lm}(\Theta, \Phi)\mathcal{C}_l^N(\Gamma). \qquad (3.6)$$

Here, $Y_{lm}(\Theta, \Phi)$ are the usual spherical harmonics and

$$\mathcal{C}_l^N(\Gamma) = \left[\frac{2^{2l+1}}{\pi}\,\frac{(N+1)(N-l)!\,(l!)^2}{(N+l+1)!}\right]^{1/2} \times \sin^l\Gamma\, C_{N-l}^{l+1}(\cos\Gamma), \qquad (3.7)$$

where $C_{N-l}^{l+1}(\cos\Gamma)$ is a Gegenbauer polynomial. Defining $\vec\chi_\beta$ as

$$\chi_0 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \chi_{-1} = \frac{1}{\sqrt2}\begin{pmatrix}1\\-i\\0\end{pmatrix}, \quad \chi_{+1} = \frac{1}{\sqrt2}\begin{pmatrix}1\\i\\0\end{pmatrix}, \qquad (3.8)$$

the vector hyperspherical harmonics are defined as

$$\vec{Y}^{m}_{NlL1}(\Gamma,\Theta,\Phi)=\sum_{\beta=-1}^{1} C_{L1}(l,m;m-\beta,\beta)\mathcal{Y}^{m-\beta}_{LN}\vec{\chi}_{\beta}\,, \tag{3.9}$$

where $C_{L1}(l,m;m-\beta,\beta)$ denotes the Clebsch-Gordan coefficients.

The following definitions are introduced:

$$\begin{aligned}
\Box_N &= \partial_{R'}{}^2+\frac{3}{R'}\partial_{R'}-\frac{N(N+2)}{R'^2}\\
&=\partial_{R'N+}\partial_{R'N-}\\
&=\partial_{R'N-}\partial_{R'N-1+}\,,\\
\Box_{N+} &= \partial_{R'}{}^2-\frac{2N+7}{R'}\partial_{R'}+\frac{(N+2)(N+4)}{R'^2}\\
&=\partial_{R'N+}\partial_{R'N+1+}\,,\\
\Box_{N-} &= \partial_{R'}{}^2-\frac{2N-3}{R'}\partial_{R'}+\frac{N(N-2)}{R'^2}\\
&=\partial_{R'N-}\partial_{R'N-1-}\,,\\
\partial_{R'N+} &= \partial_{R'}+\frac{N+3}{R'}\,,\quad \partial_{R'N-}=\partial_{R'}-\frac{N-1}{R'}\,,\\
\partial_{Rl+} &= \partial_R+\frac{l}{R}\,,\quad \partial_{Rl-}=\partial_R-\frac{l}{R}\,.
\end{aligned} \tag{3.10}$$

Further, the following relations are defined:

$$\begin{aligned}
S&=S(R',N,l,m)\\
&=s(R',N,l,m)+v(R',N,l,m)\,,\\
V&=V(R',N,l,m)\\
&=s(R',N,l,m)-v(R',N,l,m)\,,\\
U&=U(R',N,l,m)\\
&=u(R',N,l,m)+t(R',N,l,m)\,,\\
T&=T(R',N,l,m)\\
&=u(R',N,l,m)-t(R',N,l,m)\,,\\
S_+&=S_+(R',N,l,m)\\
&=s(R',N,l+1,l,m)+v(R',N,l+1,l,m)\,,\\
S_-&=S_-(R',N,l,m)\\
&=s(R',N,l-1,l,m)+v(R',N,l-1,l,m)\,,\\
S^0&=S^0(R',N,l,m)\\
&=s(R',N,l,l,m)+v(R',N,l,l,m)\,.
\end{aligned} \tag{3.11}$$

A set of quantities V_+, V_-, V^0, U_+, U_-, U^0, T_+, T_-, and T^0 is defined in a similar way as the quantities in (3.11) are.

For a given l and a given m, (2.12) can now be transformed to 16 linear partial differential equations. These 16 equations have been obtained, but only two of them are reproduced below:

$$\begin{aligned}
&[(-\Box_N+\tfrac14K_0{}^2+G_0+G_P-4G_V-4G_A-6G_T)\xi^a_c-D^{ad}_{bc}\xi^b_d]U\mathcal{C}^N_l-(\tfrac12K_0d^{ad}_{bc-}+d^{ad}_{bc+}\partial_0')\xi^b_dT\mathcal{C}^N_l\\
&\quad+i\left(\frac{l}{2l+1}\right)^{1/2}\partial_{Rl-1-}(K_0S_-\xi^a_c+d^{ad}_{bc+}\xi^b_dV_-)\mathcal{C}^N_{l-1}-i\left(\frac{l+1}{2l+1}\right)^{1/2}\partial_{Rl+2+}(K_0S_+\xi^a_c+d^{ad}_{bc+}\xi^b_dV_+)\mathcal{C}^N_{l+1}=0\,,
\end{aligned} \tag{3.12a}$$

$$\begin{aligned}
&\left[\left(-\frac{1}{2l+1}(\Box_N+2l\partial_0'^2)+\tfrac14K_0{}^2-G_0+G_P-2G_V+2G_A\right)\xi^a_c+D^{ad}_{bc}\xi^b_d\right]V_-\mathcal{C}^N_{l-1}\\
&\quad+(\tfrac12K_0d^{ad}_{bc+}+d^{ad}_{bc-}\partial_0')\xi^b_dS_-\mathcal{C}^N_{l-1}-2\frac{[l(l+1)]^{1/2}}{2l+1}\partial_{Rl+1+}\partial_{Rl+2+}\xi^a_cV_+\mathcal{C}^N_{l+1}\\
&\quad+i\left(\frac{l}{2l+1}\right)^{1/2}\partial_{Rl+1+}(2\partial_0'T\xi^a_c+d^{ad}_{bc+}\xi^b_dU)\mathcal{C}^N_l-i\left(\frac{l+1}{2l+1}\right)^{1/2}\partial_{Rl+1+}(K_0T^0\xi^a_c-d^{ad}_{bc-}\xi^b_dU^0)\mathcal{C}^N_l=0\,,
\end{aligned} \tag{3.12b}$$

where

$$G_V=G_{\text{ph}}(|x|)+G_{Vm}(|x|)\,,\quad G_P=G_{Pm}(|x|),\quad D^{ad}_{bc}=\partial^a_b\partial^d_c-M^{ad}_{bc}\,, \tag{3.13}$$

$$\begin{aligned}
M^{ad}_{bc}&=\tau(r)\delta^a_b\delta^d_c+\omega(r)(\lambda_s)^a_b(\lambda_s)^d_c+G_m(r)[(\lambda_8)^a_b\delta^d_c+\delta^a_b(\lambda_8)^d_c]+G_{\text{em}}(r)[Q^a_b\delta^d_c+\delta^a_bQ^d_c]\,,\\
d^{ad}_{bc\pm}&=\partial^a_b\delta^d_c\pm\delta^a_b\partial^d_c\,.
\end{aligned} \tag{3.14}$$

The 16 equations represented by (3.12) are assumed to describe mesons with a total angular momentum l and its third component m. For instance, a vector meson or a pseudovector meson is associated with $l=1$ and a tensor meson with $l=2$. In what follows, only scalar and pseudoscalar mesons corresponding to $l=0$ in the 16 equations represented by (3.12) will be treated. Noting that $\mathfrak{C}_{-1}^N=0$ it can be shown that the quantities S^0, V^0, U^0, T^0, S_-, V_-, U_-, and T_-, in the 16 equations represented by (3.12) all drop out and only eight equations survive. Four of these surviving equations are now multiplied by $\int_0^\pi d\Gamma \sin^2\Gamma\, \mathfrak{C}_0^N$ from the left, and the remaining four by $\int_0^\pi d\Gamma \sin^2\Gamma\, \mathfrak{C}_1^N$ from the left. The notations

$$S_{N(P)+M}=\begin{cases} S(R', N+M, l, m), & N\geq P\,, \\ 0, & N<P\,, \end{cases}$$
$$S_{+N(P)+M}=\begin{cases} S_+(R', N+M, l, m), & N\geq P\,, \\ 0, & N<P\,, \end{cases} \tag{3.15}$$

and similar notations when S is replaced by T, U, and V are then introduced. Only two of the eight equations for the case of $l=0$ are written down below:

$$\begin{aligned}&[(-\Box_N+\tfrac{1}{4}K_0{}^2+G_0+G_P-4G_V-4G_A-6G_T)\xi_c^a-D_{bc}^{ad}\,\xi_d^b]\,U_{N(0)}\\ &\quad-\tfrac{1}{2}\,d_{bc+}^{ad}\,\xi_d^b\left\{\partial_{R'N+}\left[T_{N(0)+1}+i\left(\frac{N+3}{N+1}\right)^{1/2}V_{+N(0)+1}\right]+\partial_{R'N-}\left[T_{N(1)-1}-i\left(\frac{N-1}{N+1}\right)^{1/2}V_{+N(2)-1}\right]\right\}-\tfrac{1}{2}K_0 d_{bc-}^{ad}\,\xi_d^b\,T_{N(0)}\\ &\quad-i\,\tfrac{1}{2}K_0\xi_c^a\left[\left(\frac{N+3}{N+1}\right)^{1/2}\partial_{R'N+}S_{+N(0)+1}-\left(\frac{N-1}{N+1}\right)^{1/2}\partial_{R'N-}S_{+N(2)-1}\right]=0\,,\end{aligned}\tag{3.16}$$

$$\begin{aligned}&[(-\Box_N+\tfrac{1}{4}K_0{}^2-G_0-G_P-4G_V-4G_A+6G_T)\xi_c^a+D_{bc}^{ad}\,\xi_d^b]\,S_{N(0)}\\ &\quad+\tfrac{1}{2}K_0 d_{bc+}^{ad}\,\xi_d^b\,V_{N(0)}+\tfrac{1}{2}d_{bc-}^{ad}\,\xi_d^b\left\{\partial_{R'N+}\left[V_{N(0)+1}+i\left(\frac{N+3}{N+1}\right)^{1/2}T_{+N(0)+1}\right]+\partial_{R'N-}\left[V_{N(1)-1}-i\left(\frac{N-1}{N+1}\right)^{1/2}T_{+N(2)-1}\right]\right\}\\ &\quad+i\tfrac{1}{2}K_0\xi_c^a\left[-\left(\frac{N+3}{N+1}\right)^{1/2}\partial_{R'N+}U_{+N(0)+1}+\left(\frac{N-1}{N+1}\right)^{1/2}\partial_{R'N-}U_{+N(2)-1}\right]=0\,.\end{aligned}\tag{3.17}$$

One can show that the four equations represented by (3.16) are decoupled from the other four by (3.17). With the help of (3.15), (3.11), and (3.3), one can associate (3.16) with scalar mesons and (3.17) with pseudoscalar mesons.

Since N runs from 0 to ∞, the eight equations represented by (3.16) and (3.17) consist of two infinite sets of coupled equations. It was found that these chains of equations can be suitably truncated if all the functions with a subscript N' have a magnitude of the order of $\epsilon^{N'}$, where $\epsilon \ll 1$, and if $S_0 \equiv S(R', 0, 0, 0)$, V_0, U_0, and T_0 all depend weakly upon R', the Wick-rotated relative four-distance between the two quark triplets, i.e., the derivative of these quantities with respect to R' is of the order of ϵ or higher. Making these assumptions, the four equations represented by (3.16) become, to the ϵ^0 order,

$$[(\tfrac{1}{4}K_0{}^2+G_0+G_P-4G_V-4G_A-6G_T)\xi_c^a-D_{bc}^{ad}\,\xi_d^b]\,U_0 -\tfrac{1}{2}K_0 d_{bc-}^{ad}\,\xi_d^b\,T_0=0\,,\tag{3.18a}$$

$$[(\tfrac{1}{4}K_0{}^2+G_0-G_P+2G_V-2G_A)\xi_c^a-D_{bc}^{ad}\,\xi_d^b]\,T_0 -\tfrac{1}{2}K_0 d_{bc-}^{ad}\,\xi_d^b\,U_0=0\,,\tag{3.18b}$$

which are assigned to scalar mesons. Similarly, the four equations represented by (3.17) become, to the ϵ^0 order,

$$[(\tfrac{1}{4}K_0{}^2-G_0-G_P-4G_V-4G_A+6G_T)\xi_c^a+D_{bc}^{ad}\,\xi_d^b]\,S_0 +\tfrac{1}{2}K_0 d_{bc+}^{ad}\,\xi_d^b\,V_0=0\,,\tag{3.19a}$$

$$[(\tfrac{1}{4}K_0{}^2-G_0+G_P+2G_V-2G_A)\xi_c^a+D_{bc}^{ad}\,\xi_d^b]\,V_0 +\tfrac{1}{2}K_0 d_{bc+}^{ad}\,\xi_d^b\,S_0=0\,,\tag{3.19b}$$

which are assigned to pseudoscalar mesons. Since U_0, T_0, S_0, and V_0 all depend weakly upon R', the linear combinations of G_0, G_P, G_V, G_A, and G_T, all assumed to be functions of R' only as indicated in (5.2) of I and in (2.7), appearing in (3.18) and (3.19) must depend weakly upon R', or essentially cancel out, or are all of order ϵ or higher to allow for consistency. From (3.13) and the forms of $G_{\rm ph}$, G_{Pm}, and G_{Vm} following (5.2) in I, it is seen that the G's in general depend rather strongly upon R'. Further, if the linear combinations of the G's in (3.18) and (3.19) all vanish, then one can show that $G_V=G_A=0$ and $G_0=G_P=3G_T$. These relations are very restrictive and are assumed not to hold.

Thus, one is left with the assumption that all the G's, the interaction functions in space-time, in (3.18) and (3.19) are of order ϵ or higher and are small compared to M^{ad}_{bc}, the interaction function in the internal space. With this assumption, (3.18) and (3.19) are now consistent with the present truncation scheme. In the limit of $\epsilon \to 0$, U_0, T_0, S_0, V_0 are constants and the G's can be neglected so that (3.18) becomes

$$\tfrac{1}{4}K_0{}^2\xi^a_c(z) \mp \tfrac{1}{2}K_0[\partial^a_b\,\xi^b_c(z) - \partial^b_c\,\xi^a_b(z)] - (\partial^a_b\partial^d_c - M^{ad}_{bc})\xi^b_d(z) = 0\,, \quad (3.20)$$

which refers to scalar mesons, and (3.19) becomes

$$\tfrac{1}{4}K_0{}^2\xi^a_c(z) \mp \tfrac{1}{2}K_0[\partial^a_b\,\xi^b_c(z) + \partial^b_c\,\xi^a_b(z)] + (\partial^a_b\partial^d_c - M^{ad}_{bc})\xi^b_d(z) = 0\,, \quad (3.21)$$

which refers to the pseudoscalar mesons. Here, the $\mp$ signs in (3.20) and (3.21) refer to the solutions $U_0 = \pm T_0$ and $S_0 = \mp V_0$, respectively.

Thus, the relative meson wave function $\psi(x)\xi^a_c(z)$ as well as the interaction between the quark and the antiquark are, to ϵ^0 order, wholly internal. Their space-time dependence enters as corrections to their internal dependence and is of order ϵ or higher. In the following, only (3.21) and (3.20) will be treated further. In this connection, some useful mathematics is developed in the following two sections.

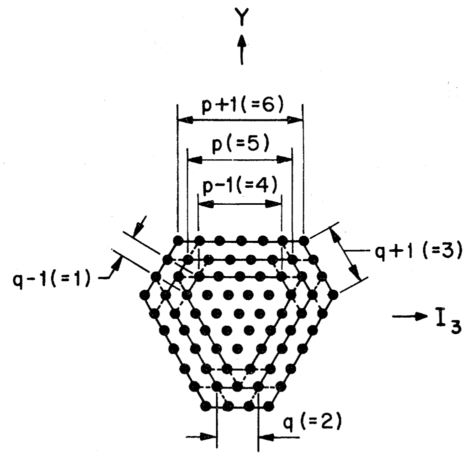

FIG. 1. Weight diagram used to obtain Eq. (4.6).

IV. EIGHT-VECTOR SPHERICAL HARMONICS

Corresponding to the vector spherical harmonics in space, $Y_{lm}(\Theta, \Phi)$ in (3.6), are the eight-vector spherical harmonics in the internal space defined by

$$\underline{Y}^{p''q''}_{YII_3}(p, q, \vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = \sum_{Y'I'I'_3Y''I''I''_3} \begin{pmatrix} 1\,1 & p''q'' & pq \\ Y'I'I'_3 & Y''I''I''_3 & YII_3 \end{pmatrix} Y^{p''q''}_{Y''I''I''_3}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3)\underline{\Lambda}^{1\,1}_{Y'I'I'_3}\,, \quad (4.1)$$

where the Y expression was defined in (6.14) of I and is proportional to that of Bég and Ruegg,[7] $\underline{\Lambda}^{1\,1}_{Y'I'I'_3}$ is the SU_3 spin vector in octet state, corresponding to $\vec{\chi}_\beta$ in (3.9), and is, dropping the superscripts 1 1,

$$\begin{aligned}
\underline{\Lambda}_{011} &= -\frac{1}{\sqrt{2}}\,(1i000000),\\
\underline{\Lambda}_{01-1} &= \frac{1}{\sqrt{2}}\,(1-i000000),\\
\underline{\Lambda}_{010} &= (00100000),\\
\underline{\Lambda}_{1\frac{1}{2}\frac{1}{2}} &= -\frac{1}{\sqrt{2}}\,(0001i000),\\
\underline{\Lambda}_{-1\frac{1}{2}-\frac{1}{2}} &= \frac{1}{\sqrt{2}}\,(0001-i000),\\
\underline{\Lambda}_{1\frac{1}{2}-\frac{1}{2}} &= -\frac{1}{\sqrt{2}}\,(000001i0),\\
\underline{\Lambda}_{-1\frac{1}{2}\frac{1}{2}} &= \frac{1}{\sqrt{2}}\,(000001-i0),\\
\underline{\Lambda}_{000} &= (00000001),
\end{aligned} \quad (4.2)$$

and the parentheses next to the summation sign denote the SU_3 Clebsch-Gordan coefficients.[8,9] The last coefficients can be written as

$$\begin{pmatrix} 1\,1 & p''q'' & pq \\ Y'I'I'_3 & Y''I''I''_3 & YII_3 \end{pmatrix} = \left(\begin{matrix} 1\,1 & p''q'' \\ Y'I' & Y''I'' \end{matrix}\middle|\begin{matrix} pq \\ YI \end{matrix}\right) C(I'I''I; I'_3I''_3I_3)\,, \quad (4.3)$$

where $C(I'I''I; I'_3I''_3I_3)$ denotes the Clebsch-Gordan coefficients in a notation different from the one in (3.9) and the parentheses on the right-hand side denote the isoscalar factor.[8] It is conventional to introduce the dimension of a multiplet,[9]

$$d(p, q) = \tfrac{1}{2}(p+1)(q+1)(p+q+2)\,. \quad (4.4)$$

Next, in analogy with the expansion of $\psi(x)$ in (3.3)–(3.5), $\xi^a_c(z)$ is expanded in terms of eight-vector spherical harmonics in the following way:

$$\xi_c^a(z)=\sum_{p,q,Y,I,I_3} A(p,q,Y,I,I_3)[\xi_c^a(z)]_{YII_3}^{pq},$$
$$[\xi_c^a(z)]_{YII_3}^{pq}=g_{p+q}(p-q,Y,I,I_3,r)Y_{YII_3}^{pq}(\vartheta,\xi,\varphi_1,\varphi_2,\varphi_3)+\underline{X}_{YII_3}^{pq}(z)\underline{\lambda}_c^a, \tag{4.5}$$

where $\underline{\lambda}_c^a$ denotes the eight Gell-Mann matrices and A a set of expansion coefficients. Just as there are three different kinds of vector spherical harmonics (3.5), there are a number of different eight-vector spherical harmonics. One can write

$$\begin{aligned}\underline{X}_{YII_3}^{pq}(z)&=\sum_{p''q''} f_{p''+q''}(p''-q'',Y,I,I_3,r)\underline{Y}_{YII_3}^{p''q''}(p,q,\vartheta,\xi,\varphi_1,\varphi_2,\varphi_3)\\&=f_{p+q+2}(p-q,Y,I,I_3,r)\underline{Y}_{YII_3}^{p+1\,q+1}+f_{p+q+1}(p-q+3,Y,I,I_3,r)\underline{Y}_{YII_3}^{p+2\,q-1}\\&\quad+f_{p+q+1}(p-q-3,Y,I,I_3,r)\underline{Y}_{YII_3}^{p-1\,q+2}\\&\quad+f_{p+q}(p-q,Y,I,I_3,r)\underline{Y}_{YII_3}^{pq}+f_{p+q-1}(p-q+3,Y,I,I_3,r)\underline{Y}_{YII_3}^{p+1\,q-2}\\&\quad+f_{p+q-1}(p-q-3,Y,I,I_3,r)\underline{Y}_{YII_3}^{p-2,q+1}+f_{p+q-2}(p-q,Y,I,I_3,r)\underline{Y}_{YII_3}^{p-1\,q-1},\end{aligned} \tag{4.6}$$

where $p+q-2\leqslant p''+q''\leqslant p+q+2$. This expression can be constructed from Fig. 1 as follows. The dots on and within the innermost full line belong to the multiplet $(p-1,q-1)$. The dots on and within the middle full line belong to the multiplet (p,q). The dots on and within the outermost full line belong to the multiplet $(p+1,q+1)$, and so on. Combinations of the dotted lines and portions of the full lines give rise to closed lines confining the multiplets $(p+2,q-1)$, $(p-1,q+2)$, $(p+1,q-2)$, and $(p-2,q+1)$. Each dot represents at least one member of a given multiplet containing it. It can be seen from Fig. 1 that each member of the multiplet (p,q) is one step away from at least one member belonging to the multiplets mentioned after (4.6). Each of these last mentioned multiplets contributes a term to (4.6). Members belonging to multiplets smaller than $(p-1,q-1)$ cannot reach the outer members of the multiplet (p,q) by one step. The inner members belonging to multiplets larger than $(p+1,q+1)$ can of course reach each member belonging to the multiplet (p,q) by one step. These members, however, also belong to the multiplets indicated in (4.6) and therefore there is no need to include the multiplets larger than $(p+1,q+1)$ which include these members. The upper indices in the eight-vector spherical harmonics in (4.6) must each be $\geqslant 0$; otherwise the corresponding harmonics vanish.

If the full expansion (4.5) is used, expressions like $\partial_b^a\,\xi_c^b(z)$, which occurs in this investigation and involves derivatives of the d function, may become rather bulky. Instead, only the singlet term with $p=q=0$ and denoted by $[\xi_c^a(z)]_{000}^{00}$ and the octet term with $p=q=1$ and denoted by $[\xi_c^a(z)]_{YII_3}^{11}$ in (4.5) will be kept. One finds, with the help of (4.1)–(4.6), (6.14) of I, the expression for the d function, and tables of Clebsch-Gordan coefficients and of isoscalar factors, that

$$[\xi_c^a(z)]_{000}^{00}=\begin{pmatrix}\xi_1^1 & \xi_1^2 & \xi_1^3\\ \xi_2^1 & \xi_2^2 & \xi_2^3\\ \xi_3^1 & \xi_3^2 & \xi_3^3\end{pmatrix}_{000}^{00}=g_s+f_s[-\pi(\pi/3)^{1/2}]\,\underline{Y}_{000}^{11\,(1)}\underline{\lambda}_c^a, \tag{4.7}$$

where

$$\underline{Y}_{000}^{11\,(1)}\underline{\lambda}_c^a=-\frac{1}{\pi}\left(\frac{3}{\pi}\right)^{1/2}\begin{pmatrix}-\frac12\sin^2\vartheta\cos2\xi+\frac16(1-3\cos^2\vartheta) & \frac12\sin^2\vartheta\sin2\xi\,e^{i(\varphi_2-\varphi_1)} & \frac12\sin2\vartheta\cos\xi\,e^{i(\varphi_3-\varphi_1)}\\ \frac12\sin^2\vartheta\sin2\xi\,e^{i(\varphi_1-\varphi_2)} & -\frac12\sin^2\vartheta\cos2\xi+\frac16(1-3\cos^2\vartheta) & \frac12\sin2\vartheta\sin\xi\,e^{i(\varphi_3-\varphi_2)}\\ \frac12\sin2\vartheta\cos\xi\,e^{i(\varphi_1-\varphi_3)} & \frac12\sin2\vartheta\sin\xi\,e^{i(\varphi_2-\varphi_3)} & -\frac13(1-3\cos^2\vartheta)\end{pmatrix}, \tag{4.8}$$

$$g_s=-\frac{1}{\pi\sqrt{\pi}}\,g_0(0,0,0,0,r)=Y_{000}^{00}\,g_0(0,0,0,0,r), \tag{4.9a}$$

$$f_s=-\frac{1}{\pi}\left(\frac{3}{\pi}\right)^{1/2} f_2(0,0,0,0,r). \tag{4.9b}$$

The superscript (1) in (4.7) and (4.8) denotes singlet or $p=q=0$. In obtaining (4.8) the phase factor in

(6.8) of I, δ, has been put to 0 in all cases except in the expressions for Y^{11}_{010} and $Y^{11}_{-1\frac{1}{2}-\frac{1}{2}}$ in which $\delta=\pi$. If this is not done, (4.8) would not be Hermitian and the derivatives of (4.7) occurring in this investigation would not have the relatively simple forms shown later. For the octet term, only the $Y=I=I_3=0$ component is worked out. In a manner analogous to that in obtaining (4.7) and (4.8), one finds that

$$\begin{aligned}[\xi_c^a(z)]^{11}_{000} = {} & g_2(0,0,0,0,r)Y^{11}_{000} + f_0(0,0,0,0,r)\underline{Y}^{00}_{000}\underline{\lambda}^a_c + f_2^{(8_1)}(0,0,0,0,r)\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^a_c \\ & + f_2^{(8_2)}(0,0,0,0,r)\underline{Y}^{11(8_2)}_{000}\underline{\lambda}^a_c + f_3(-3,0,0,0,r)\underline{Y}^{03}_{000}\underline{\lambda}^a_c + f_3(3,0,0,0,r)\underline{Y}^{30}_{000}\underline{\lambda}^a_c \\ & + f_4(0,0,0,0,r)\underline{Y}^{22}_{000}\underline{\lambda}^a_c ,\end{aligned} \tag{4.10}$$

where

$$Y^{11}_{000} = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{1/2}(1-3\cos^2\vartheta) \tag{4.11}$$

$$\underline{Y}^{00}_{000}\underline{\lambda}^a_c = -\frac{1}{\pi\sqrt{3\pi}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \tag{4.12}$$

$$\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^a_c = \frac{1}{\pi}\left(\frac{6}{5\pi}\right)^{1/2}\begin{pmatrix} \sin^2\vartheta\cos 2\xi - \frac{1}{3}(1-3\cos^2\vartheta) & \sin^2\vartheta\sin 2\xi\, e^{i(\varphi_2-\varphi_1)} & -\cos\vartheta\sin\vartheta\cos\xi\, e^{i(\varphi_3-\varphi_1)} \\ \sin^2\vartheta\sin 2\xi\, e^{i(\varphi_1-\varphi_2)} & -\sin^2\vartheta\cos 2\xi - \frac{1}{3}(1-3\cos^2\vartheta) & -\cos\vartheta\sin\vartheta\sin\xi\, e^{i(\varphi_3-\varphi_2)} \\ -\cos\vartheta\sin\vartheta\cos\xi\, e^{i(\varphi_1-\varphi_3)} & -\cos\vartheta\sin\vartheta\sin\xi\, e^{i(\varphi_2-\varphi_3)} & \frac{2}{3}(1-3\cos^2\vartheta) \end{pmatrix}, \tag{4.13}$$

$$\underline{Y}^{11(8_2)}_{000}\underline{\lambda}^a_c = \frac{1}{\pi}\left(\frac{6}{\pi}\right)^{1/2}\begin{pmatrix} 0 & 0 & -\cos\vartheta\sin\vartheta\cos\xi\, e^{i(\varphi_3-\varphi_1)} \\ 0 & 0 & -\cos\vartheta\sin\vartheta\sin\xi\, e^{i(\varphi_3-\varphi_2)} \\ \cos\vartheta\sin\vartheta\cos\xi\, e^{i(\varphi_1-\varphi_3)} & \cos\vartheta\sin\vartheta\sin\xi\, e^{i(\varphi_2-\varphi_3)} & 0 \end{pmatrix}, \tag{4.14}$$

$$\underline{Y}^{30}_{000}\underline{\lambda}^a_c = \frac{2}{\pi}\left(\frac{3}{\pi}\right)^{1/2}\begin{bmatrix} \sin^2\vartheta\cos\vartheta\sin\xi\cos\xi & -\sin^2\vartheta\cos\vartheta\sin^2\xi & -\sin\vartheta\cos^2\vartheta\sin\xi \\ \times e^{i(\varphi_1+\varphi_2+\varphi_3)} & \times e^{i(2\varphi_2+\varphi_3)} & \times e^{i(\varphi_2+2\varphi_3)} \\ \sin^2\vartheta\cos\vartheta\cos^2\xi & -\sin^2\vartheta\cos\vartheta\sin\xi\cos\xi & \sin\vartheta\cos^2\vartheta\cos\xi \\ \times e^{i(2\varphi_1+\varphi_3)} & \times e^{i(\varphi_1+\varphi_2+\varphi_3)} & \times e^{i(\varphi_1+2\varphi_3)} \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.15}$$

$$\underline{Y}^{03}_{000}\underline{\lambda}^a_c = \frac{2}{\pi}\left(\frac{3}{\pi}\right)^{1/2}\begin{bmatrix} \sin^2\vartheta\cos\vartheta\sin\xi\cos\xi & \sin^2\vartheta\cos\vartheta\cos^2\xi & \\ \times e^{-i(\varphi_1+\varphi_2+\varphi_3)} & \times e^{-i(2\varphi_1+\varphi_3)} & 0 \\ -\sin^2\vartheta\cos\vartheta\sin^2\xi & -\sin^2\vartheta\cos\vartheta\sin\xi\cos\xi & \\ \times e^{-i(2\varphi_2+\varphi_3)} & \times e^{-i(\varphi_1+\varphi_2+\varphi_3)} & 0 \\ -\sin\vartheta\cos^2\vartheta\sin\xi & \sin\vartheta\cos^2\vartheta\cos\xi & \\ \times e^{-i(\varphi_2+2\varphi_3)} & \times e^{-i(\varphi_1+2\varphi_3)} & 0 \end{bmatrix}, \tag{4.16}$$

$$\underline{Y}^{22}_{000}\underline{\lambda}^a_c = \frac{2}{\pi\sqrt{5\pi}}\begin{bmatrix} (5\sin^4\vartheta-4\sin^2\vartheta)\cos 2\xi & (5\sin^4\vartheta-4\sin^2\vartheta)\sin 2\xi\, e^{i(\varphi_2-\varphi_1)} & -2(3\cos^3\vartheta\sin\vartheta-2\cos\vartheta\sin^3\vartheta) \\ +\frac{1}{2}(3\cos^4\vartheta-6\cos^2\vartheta\sin^2\vartheta+\sin^4\vartheta) & & \times\cos\xi\, e^{i(\varphi_3-\varphi_1)} \\ (5\sin^4\vartheta-4\sin^2\vartheta)\sin 2\xi & -(5\sin^4\vartheta-4\sin^2\vartheta)\cos 2\xi & -2(3\cos^3\vartheta\sin\vartheta-2\cos\vartheta\sin^3\vartheta) \\ \times e^{i(\varphi_1-\varphi_2)} & +\frac{1}{2}(3\cos^4\vartheta-6\cos^2\vartheta\sin^2\vartheta+\sin^4\vartheta) & \times\sin\xi\, e^{i(\varphi_3-\varphi_2)} \\ -2(3\cos^3\vartheta\sin\vartheta-2\cos\vartheta\sin^3\vartheta) & -2(3\cos^3\vartheta\sin\vartheta-2\cos\vartheta\sin^3\vartheta) & -(3\cos^4\vartheta-6\cos^2\vartheta\sin^2\vartheta+\sin^4\vartheta) \\ \times\cos\xi\, e^{i(\varphi_1-\varphi_3)} & \times\sin\xi\, e^{i(\varphi_2-\varphi_3)} & \end{bmatrix}. \tag{4.17}$$

Again, in obtaining (4.13)–(4.17), the phase factor δ in (6.8) of I has been put to 0 in all cases except in the expressions for Y^{11}_{010}, $Y^{11}_{-1\frac{1}{2}-\frac{1}{2}}$, Y^{22}_{010}, and $Y^{22}_{-1\frac{1}{2}-\frac{1}{2}}$ in which $\delta=\pi$ for a similar reason as that given between (4.9) and (4.10). The superscripts (8_1) and (8_2) refer to the two kinds of octet, $\underline{8}_1$ and $\underline{8}_2$, respectively. Further, the following relation for the contracted quantity:

$$\int (\underline{Y}^{p'q'(d')}_{Y'I'I_3'}\underline{\lambda}^a_c)^\dagger \underline{Y}^{pq(d)}_{YII_3}\underline{\lambda}^a_c \frac{1}{r^5 dr}\, d^8 z \propto \delta_{pp'}\delta_{qq'}\delta_{dd'}\delta_{YY'}\delta_{II'}\delta_{I_3 I_3'}, \tag{4.18}$$

holds for the quantities given in (4.8), (4.9a), and (4.11)–(4.17).

V. EXPRESSIONS FOR SOME DERIVATIVES

By means of (6.4) of I, the following derivatives can be obtained:

$$\partial^a_b[\xi^b_c(z)]^{00}_{000} = \partial^b_c[\xi^a_b(z)]^{00}_{000} = g_k + f_k[-\pi(\pi/3)^{1/2}]\underline{Y}^{11(1)}_{000}\underline{\lambda}^a_c, \tag{5.1}$$

$$g_k = \tfrac{1}{12}\left(g_s'' + \frac{5}{r}g_s' + \frac{2}{3}f_s'' + \frac{22}{3r}f_s'\right), \tag{5.2}$$

$$f_k = \tfrac{1}{4}\left(g_s'' - \frac{1}{r}g_s' + \frac{2}{3}f_s'' + \frac{10}{3r}f_s' - \frac{8}{r^2}f_s\right), \tag{5.3}$$

$$\partial^d_c[\xi^b_d(z)]^{00}_{000}\partial^a_b = \partial^d_c\{g_k + f_k[-\pi(\pi/3)^{1/2}]\underline{Y}^{11(1)}_{000}\underline{\lambda}^a_c\} = g_l + f_l[-\pi(\pi/3)^{1/2}]\underline{Y}^{11(1)}_{000}\underline{\lambda}^a_c, \tag{5.4}$$

$$g_l = \frac{1}{12}\left(g_k'' + \frac{5}{r}g_k' + \frac{2}{3}f_k'' + \frac{22}{3r}f_k\right), \tag{5.5}$$

$$f_l = \frac{1}{4}\left(g_k'' - \frac{1}{r}g_k' + \frac{2}{3}f_k'' + \frac{10}{3r}f_k' - \frac{8}{r^2}f_k\right). \tag{5.6}$$

In this section the prime denotes $\partial/\partial r$. These results refer to the singlet or $p=q=0$ case. In the case of the octet or $p=q=1$ with $Y=I=I_3=0$, one finds that

$$\begin{aligned}2\partial^a_c\pi\left(\frac{\pi}{2}\right)^{1/2}Y^{11}_{000}F(r) = {} & \frac{\pi\sqrt{5\pi}}{8}\underline{Y}^{22}_{000}\underline{\lambda}^a_c\frac{3}{5}\left(F'' - \frac{5}{r}F' + \frac{8}{r^2}F\right)\\ & + \frac{\pi}{4}\left(\frac{\pi}{2}\right)^{1/2}\left[\left(\frac{5}{3}\right)^{1/2}\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^a_c + Y^{11}_{000}\frac{5}{3}\right]\frac{2}{5}\left(F'' + \frac{5}{r}F' - \frac{12}{r^2}F\right)\\ & - \frac{\pi\sqrt{3\pi}}{4}\underline{Y}^{00}_{000}\underline{\lambda}^a_c\frac{1}{6}\left(F'' + \frac{11}{r}F' + \frac{24}{r^2}F\right),\end{aligned} \tag{5.7}$$

$$\begin{aligned}\partial^b_c\pi\left(\frac{\pi}{2}\right)^{1/2}Y^{11}_{000}F(r)\partial^a_b = {} & \frac{\pi\sqrt{5\pi}}{8}\underline{Y}^{22}_{000}\underline{\lambda}^a_c\frac{3}{40}\left(F'''' - \frac{39}{r^2}F'' + \frac{183}{r^3}F' - \frac{288}{r^4}F\right)\\ & + \frac{\pi}{4}\left(\frac{\pi}{2}\right)^{1/2}\left[\left(\frac{5}{3}\right)^{1/2}\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^a_c + Y^{11}_{000}\frac{5}{3}\right]\frac{1}{20}\left(F'''' + \frac{10}{r}F''' - \frac{9}{r^2}F'' - \frac{87}{r^3}F' + \frac{192}{r^4}F\right)\\ & - \frac{\pi}{4}\sqrt{3\pi}\,\underline{Y}^{00}_{000}\underline{\lambda}^a_c\frac{1}{48}\left(F'''' + \frac{16}{r}F''' + \frac{57}{r^2}F'' - \frac{9}{r^3}F' - \frac{96}{r^4}F\right),\end{aligned} \tag{5.8}$$

$$(\partial^b_c\underline{\lambda}^a_b + \partial^a_b\underline{\lambda}^b_c)(-\pi\sqrt{3\pi})\underline{Y}^{00}_{000}F(r) = \frac{\pi}{4}\left(\frac{\pi}{2}\right)^{1/2}\left[\left(\frac{5}{3}\right)^{1/2}\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^a_c + Y^{11}_{000}\frac{2}{3}\right]\left(F'' - \frac{1}{r}F'\right) - \frac{\pi}{4}\sqrt{3\pi}\,\underline{Y}^{00}_{000}\underline{\lambda}^a_c\frac{2}{3}\left(F'' + \frac{5}{r}F'\right), \tag{5.9}$$

$$\begin{aligned}\partial^d_c(-\pi\sqrt{3\pi})\underline{Y}^{00}_{000}\underline{\lambda}^b_d F(r)\partial^a_b = {} & \frac{\pi}{8}\sqrt{5\pi}\,\underline{Y}^{22}_{000}\underline{\lambda}^a_c\frac{3}{40}\left(F'''' - \frac{6}{r}F''' + \frac{15}{r^2}F'' - \frac{15}{r^3}F'\right)\\ & + \frac{\pi}{4}\left(\frac{\pi}{2}\right)^{1/2}\left[\left(\frac{5}{3}\right)^{1/2}\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^a_c + Y^{11}_{000}\frac{5}{3}\right]\frac{1}{20}\left(F'''' + \frac{4}{r}F''' - \frac{15}{r^2}F'' + \frac{15}{r^3}F'\right)\\ & - \frac{\pi}{4}\sqrt{3\pi}\,\underline{Y}^{00}_{000}\underline{\lambda}^a_c\frac{1}{48}\left(F'''' + \frac{10}{r}F''' + \frac{15}{r^2}F'' - \frac{15}{r^3}F'\right),\end{aligned} \tag{5.10}$$

$$\begin{aligned}(\partial^b_c\underline{\lambda}^a_b + \partial^a_b\underline{\lambda}^b_c)\left(\frac{5\pi}{6}\right)^{1/2}\underline{Y}^{11(8_1)}_{000}F(r) = {} & \frac{\pi}{8}\sqrt{5\pi}\,\underline{Y}^{22}_{000}\underline{\lambda}^a_c\frac{1}{5}\left(F'' - \frac{5}{r}F' + \frac{8}{r^2}F\right)\\ & + \frac{\pi}{4}\left(\frac{\pi}{2}\right)^{1/2}\left[\left(\frac{5}{3}\right)^{1/2}\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^a_c + Y^{11}_{000}\frac{20}{21}\right]\frac{7}{15}\left(F'' + \frac{5}{r}F' - \frac{12}{r^2}F\right)\\ & - \frac{\pi}{4}\sqrt{3\pi}\,\underline{Y}^{00}_{000}\underline{\lambda}^a_c\frac{5}{36}\left(F'' + \frac{11}{r}F' + \frac{24}{r^2}F\right),\end{aligned} \tag{5.11}$$

$$\partial^b_c\,\pi\left(\frac{5\pi}{6}\right)^{1/2}\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^d_b F(r)\partial^a_d=\frac{2}{3}\,\partial^b_c\,\pi\left(\frac{\pi}{2}\right)^{1/2}Y^{11}_{000}F(r)\,\partial^a_b\ , \tag{5.12}$$

$$(\partial^b_c\underline{\lambda}^a_b+\partial^a_b\underline{\lambda}^b_c)\frac{\pi}{2}\sqrt{5\pi}\,\underline{Y}^{22}_{000}F(r)=\frac{\pi}{8}\sqrt{5\pi}\,\underline{Y}^{22}_{000}\frac{6}{5}\left(F''+\frac{5}{r}F'-\frac{32}{r^2}F\right)\underline{\lambda}^a_c$$
$$+\frac{\pi}{4}\left(\frac{\pi}{2}\right)^{1/2}\left[\left(\frac{5}{3}\right)^{1/2}\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^a_c+Y^{11}_{000}\frac{10}{3}\right]\frac{3}{10}\left(F''+\frac{15}{r}F'+\frac{48}{r^2}F\right), \tag{5.13}$$

$$\partial^b_c\frac{\pi}{2}\sqrt{5\pi}\,\underline{Y}^{22}_{000}\underline{\lambda}^d_b F(r)\partial^a_d=\frac{\pi}{4}\sqrt{5\pi}\,\underline{Y}^{22}_{000}\underline{\lambda}^a_c\frac{9}{160}\left(F''''+\frac{10}{r}F'''-\frac{49}{r^2}F''-\frac{207}{r^3}F'+\frac{1152}{r^4}F\right)$$
$$+\frac{\pi}{4}\left(\frac{\pi}{2}\right)^{1/2}\left[\left(\frac{5}{3}\right)^{1/2}\underline{Y}^{11(8_1)}_{000}\underline{\lambda}^a_c+Y^{11}_{000}\frac{5}{3}\right]\frac{3}{40}\left(F''''+\frac{20}{r}F'''+\frac{81}{r^2}F''-\frac{177}{r^3}F'-\frac{768}{r^4}F\right)$$
$$-\frac{\pi}{4}\sqrt{3\pi}\,\underline{Y}^{00}_{000}\underline{\lambda}^a_c\frac{1}{32}\left(F''''+\frac{26}{r}F'''+\frac{207}{r^2}F''+\frac{561}{r^3}F'+\frac{384}{r^4}F\right). \tag{5.14}$$

VI. RADIAL EQUATIONS

With the aid of Secs. IV and V, (3.21), assigned to the pseudoscalar mesons, can be treated further. The interaction term M^{ad}_{bc} in (3.21), given by (3.14), has been assumed to consist of two SU_3-symmetry-preserving terms, one proportional to $\tau(r)$ and the other to $\omega(r)$, and two smaller SU_3-symmetry-breaking perturbation terms, one proportional to the hypercharge interaction term $G_m(r)$ and the other to the electromagnetic interaction term $G_{em}(r)$. Let

$$\xi^a_c(z)=(\xi_0)^a_c(z)+(\xi_1)^a_c(z), \tag{6.1}$$

$$K_0=K_{00}+K_{01}, \tag{6.2}$$

where ξ_0 and K_{00} correspond to the SU_3-symmetry-preserving parts and ξ_1 and K_{01} to the smaller SU_3-symmetry-breaking parts in (6.1) and (6.2). (3.21) can now be approximated by an SU_3-symmetry-preserving zeroth-order equation

$$\tfrac{1}{4}K_{00}{}^2(\xi_0)^a_c(z)\mp\tfrac{1}{2}K_{00}[\partial^a_b(\xi_0)^b_c(z)+\partial^b_c(\xi_0)^a_b(z)]+\partial^a_b(\xi_0)^b_d(z)\partial^d_c-\tau(r)(\xi_0)^a_c(z)-\omega(r)(\lambda_s)^a_b(\xi_0)^b_d(z)(\lambda_s)^d_c=0 \tag{6.3}$$

and an SU_3-symmetry-breaking first-order equation

$$\tfrac{1}{4}K_{00}{}^2(\xi_1)^a_c(z)\mp\tfrac{1}{2}K_{00}[\partial^a_b(\xi_1)^b_c(z)+\partial^b_c(\xi_1)^a_b(z)]+\partial^a_b(\xi_1)^b_d(z)\partial^d_c-\tau(r)(\xi_1)^a_c(z)-\omega(r)(\lambda_s)^a_b(\xi_1)^b_d(z)(\lambda_s)^d_c$$
$$=-\tfrac{1}{2}K_{00}K_{01}(\xi_0)^a_c(z)\pm\tfrac{1}{2}K_{01}[\partial^a_b(\xi_0)^b_c(z)+\partial^b_c(\xi_0)^a_b(z)]+G_m(r)[(\lambda_8)^a_b(\xi_0)^b_c(z)+(\xi_0)^a_b(z)(\lambda_8)^b_c]. \tag{6.4}$$

Here, (3.14) was used and the electromagnetic interaction term involving $G_{em}(r)$, considered to be of second order, has been dropped. In the case of the singlet or η' meson, $(\xi_0)^a_c(z)$ in (6.3) can be put equal to the expression given in (4.7). With the aid of (4.8) and (5.1)–(5.6), it can be seen that the angular parts of (6.3) cancel out and one obtains

$$\rho^3(g_s'''+\tfrac{2}{3}f_s''')+\rho^2(5g_s''+\tfrac{22}{3}f_s'')-\rho(5g_s'-\tfrac{26}{3}f_s')-\rho^3(2g_s'+\tfrac{4}{3}f_s')-(32+8\rho^2)f_s+\rho^4(\tfrac{1}{2}g_s-\tfrac{1}{6}f_s)$$
$$=2\tau\rho^4(g_s-\tfrac{1}{3}f_s)/K_{00}{}^2+12\omega\rho^4 g_s/K_{00}{}^2, \tag{6.5a}$$

$$\rho^4(g_s''''+\tfrac{2}{3}f_s'''')+\rho^3(10g_s'''+\tfrac{32}{3}f_s''')+\rho^2(15g_s''+38f_s'')-\rho^4(2g_s''+\tfrac{4}{3}f_s'')-\rho(15g_s'+6f_s')-\rho^3(10g_s'+\tfrac{44}{3}f_s')-(64+32\rho^2)f_s+3\rho^4 g_s$$
$$=12\rho^4(\tau+6\omega)g_s/K_{00}{}^2, \tag{6.5b}$$

in which the relation

$$(\lambda_s)^a_b(\xi_0)^b_d(z)(\lambda_s)^d_c = 2\delta^d_b(\xi_0)^b_d(z)\delta^a_c \quad (6.6)$$

has been used. The prime in this section and the following ones denotes $\partial/\partial\rho$ instead of $\partial/\partial r$ as in Sec. V. Further, the upper sign in (6.3) was used to obtain (6.5) and the substitution $\rho = r(2K_{00})^{1/2}$ has been made.

By analogy with the electromagnetic or the strong-interaction potential in space-time, which vanishes at large distances from the corresponding source, it will be assumed that

$$\tau(\rho\to\infty)\to 0, \quad \omega(\rho\to\infty)\to 0. \quad (6.7)$$

With this assumption, (6.5) yields

$$g_s \propto f_s \propto e^{\pm\rho}, e^{\pm i\rho} \quad (6.8)$$

when $\rho\to\infty$. The mass eigenvalue K_{00} in (6.5), if real, may be obtained by a kind of variational calculation. In order to avoid divergence in the integrals involved in such a calculation, the asymptotic form $e^{-\rho}$ is chosen among the four possibilities given in (6.8). Here, K_{00} is naturally interpreted as an approximate mass of the η' meson and it has been assumed for the sake of simplicity to be real and positive. If K_{00} is set real and negative, the lower sign in (6.3) also leads to (6.5) and (6.8) with $\rho = r\,|2K_{00}|^{1/2}$. With K_{00} real and positive, the lower sign in (6.3) under the above circumstances gives rise to

$$g_s \propto f_s \propto e^{(1\pm i)\rho/\sqrt{2}}, e^{(-1\pm i)\rho/\sqrt{2}},$$

when $\rho\to\infty$, and will therefore not be investigated further.

In the case of the octet meson with $Y=I=I_3=0$ or the η meson, the procedure is entirely analogous. $(\xi_0)^a_c(z)$ in (6.3) is now put equal to the expression given in (4.10). The derivatives given in (5.7)–(5.14) show that it is not necessary to include the $f_2^{(8_2)}$ and the two f_3 terms in (4.10). Putting

$$g_2(0,0,0,0,r)=g, \quad f_0(0,0,0,0,r)=-\sqrt{6}f,$$
$$f_2(0,0,0,0,r)=\tfrac{1}{2}\sqrt{15}\,h, \quad f_4(0,0,0,0,r)=\tfrac{1}{3}\sqrt{10}\,j, \quad (6.9)$$

and making use of (5.7)–(5.14), (4.11)–(4.13), and (4.17), (6.3) with the upper sign gives

$$\rho^2[6f''+3h''-2j'']+\rho(-6f'+15h'-30j')-36h-96j-3\rho^2(5h-2g)=12\tau\rho^2(5h-2g)/K_{00}^2-144\omega\rho^2 g/K_{00}^2, \quad (6.10a)$$

$$4[\rho^4(f''''+g''''+h''''+j'''')+\rho^3(4f'''+10g'''+10h'''+20j''')+\rho^2(-15f''-9g''-9h''+81j'')$$
$$+\rho(15f'-87g'-87h'-177j')+192(g+h-4j)]$$
$$-\rho^2[\rho^2(14f''+8g''+11h''+6j'')+\rho(-14f'+40g'+55h'+90j')-96g-132h+288j]+3\rho^4(5h+2g)$$
$$=12\tau\rho^4(5h+2g)/K_{00}^2+144\omega\rho^4 g/K_{00}^2 \quad (6.10b)$$

$$36[\rho^3(f'''+g'''+h'''+j''')+\rho^2(6g''+6h''+16j'')-12\rho(g'+h'-4j')+12(g+h-4j)]$$
$$-3\rho^2[\rho^2(6f''+3h''-2j'')+\rho(30f'+24g'+45h'-10j')+24g+84h+64j]+\rho^4(27f-5j)$$
$$=4\tau\rho^4(27f-5j)/K_{00}^2, \quad (6.10c)$$

$$36[\rho^4(f''''+g''''+h''''+j'''')+\rho^3(2f'''+8g'''+8h'''+18j''')+\rho^2(15f''+9g''+9h''+79j'')$$
$$+\rho(-15f'+87g'+87h'+177j')-192(g+h-4j)]$$
$$-12\rho^2[\rho^2(12f''+6g''+9h''+4j'')+\rho(60f'+18g'+75h'+20j')+96g+192h-128j]+8\rho^4(27f+5j)$$
$$=32\tau\rho^4(27f+5j)/K_{00}^2. \quad (6.10d)$$

Assuming (6.7), (6.10) yields a set of asymptotic relations entirely similar to (6.8) and the form $e^{-\rho}$ is chosen for similar reasons. Again, K_{00} has been assumed to be real and positive and the lower sign in (6.3) leads to consequences similar to the corresponding ones in the singlet or η' meson case.

In the following, it will be assumed that the mass-separation constants $m_P{}^2$ and $m_V{}^2$ in (2.2)–(2.5) both vanish. In this case, the interaction functions τ, ω, $G_m(r)$, and $G_{em}(r)$ all have the form shown in (7.8) of I, which, taking (6.7) into the consideration, is

$$\tau(r)=\frac{\mu_0}{128\pi^3}\frac{1}{r^2}+\frac{\mu_{10}}{r^4}, \quad (6.11)$$

where μ_{10} is an integration constant. Specifically, with $\rho = r(2K_{00})^{1/2}$,

$$\tau(\rho)=K_{00}{}^2(\rho^2\kappa_0+\gamma_0)/4\rho^4, \quad (6.12)$$

$$\omega(\rho)=K_{00}{}^2(\rho^2\kappa_n+\gamma_n)/4\rho^4, \quad (6.13)$$

$$G_{mP}(\rho)=G_{mV}(\rho)=K_{00}{}^2(\rho^2\kappa_8+\gamma_8)/4\rho^4, \quad (6.14)$$

$$G_{em}(\rho)=K_{00}{}^2(\rho^2\kappa_Q+\gamma_Q)/4\rho^4, \quad (6.15)$$

where

$$\kappa_0 = \mu_0/16\pi^3 K_{00}, \quad \gamma_0 = 16\mu_{10}, \tag{6.16}$$

$$\kappa_n = \mu_n/16\pi^3 K_{00}, \quad \gamma_n = 16\mu_{1n}, \tag{6.17}$$

$$\kappa_8 = \mu_8/16\pi^3 K_{00}, \quad \gamma_8 = 16\mu_{18}, \tag{6.18}$$

$$\kappa_Q = \mu_Q/16\pi^3 K_{00}, \quad \gamma_Q = 16\mu_{1Q}, \tag{6.19}$$

and μ_{1n}, μ_{18}, and μ_{1Q} are integration constants similar to μ_{10}. When (6.12) and (6.13) are substituted into (6.5) and (6.10), these equations become independent of their mass eigenvalue K_{00} except through κ_0 in (6.16) and κ_n in (6.17).

VII. RECURRENCE RELATIONS FOR THE SINGLET MESON

The radial equations (6.5) will now be reduced to algebraic relations using Frobenius's method. Let

$$g_s(\rho) = e^{-\rho}\sum_{\nu=0}^{\infty} g_{s\nu}\rho^{\sigma+\nu}, \tag{7.1a}$$

$$f_s(\rho) = e^{-\rho}\sum_{\nu=0}^{\infty} f_{s\nu}\rho^{\sigma+\nu}, \tag{7.1b}$$

where σ is a constant. Inserting (7.1) into (6.5), multiplying it by e^{ρ}, and putting the coefficients of $\rho^{\sigma+\nu}$ equal to zero, one obtains the following recurrence relations:

$$\begin{aligned}(g_{s\nu}+\tfrac{2}{3}f_{s\nu})\mu(\mu-1)(\mu-2)&+(5g_{s\nu}+\tfrac{22}{3}f_{s\nu})\mu(\mu-1)-(5g_{s\nu}-\tfrac{26}{3}f_{s\nu})\mu-32f_{s\nu}\\ &-\tfrac{1}{2}\gamma_0(g_{s\nu}-\tfrac{1}{3}f_{s\nu})-3\gamma_n g_{s\nu}-3(g_{s\nu-1}+\tfrac{2}{3}f_{s\nu-1})(\mu-1)(\mu-2)-2(5g_{s\nu-1}+\tfrac{22}{3}f_{s\nu-1})(\mu-1)\\ &+5g_{s\nu-1}-\tfrac{26}{3}f_{s\nu-1}+(g_{s\nu-2}+\tfrac{2}{3}f_{s\nu-2})(\mu-2)+5g_{s\nu-2}-\tfrac{2}{3}f_{s\nu-2}-\tfrac{1}{2}\kappa_0(g_{s\nu-2}-\tfrac{1}{3}f_{s\nu-2})\\ &-3\kappa_n g_{s\nu-2}+g_{s\nu-3}+\tfrac{2}{3}f_{s\nu-3}+\tfrac{1}{2}(g_{s\nu-4}-\tfrac{1}{3}f_{s\nu-4})=0,\end{aligned} \tag{7.2a}$$

$$\begin{aligned}(g_{s\nu}+\tfrac{2}{3}f_{s\nu})\mu(\mu-1)(\mu-2)(\mu-3)&+(10g_{s\nu}+\tfrac{32}{3}f_{s\nu})\mu(\mu-1)(\mu-2)+(15g_{s\nu}+38f_{s\nu})\mu(\mu-1)\\ &-(15g_{s\nu}+6f_{s\nu})\mu-64f_{s\nu}-3(\gamma_0+6\gamma_n)g_{s\nu}-4(g_{s\nu-1}+\tfrac{2}{3}f_{s\nu-1})(\mu-1)(\mu-2)(\mu-3)\\ &-3(10g_{s\nu-1}+\tfrac{32}{3}f_{s\nu-1})(\mu-1)(\mu-2)-2(15g_{s\nu-1}+38f_{s\nu-1})(\mu-1)+15g_{s\nu-1}+6f_{s\nu-1}\\ &+4(g_{s\nu-2}+\tfrac{2}{3}f_{s\nu-2})(\mu-2)(\mu-3)+4(5g_{s\nu-2}+\tfrac{13}{3}f_{s\nu-2})(\mu-2)+15g_{s\nu-2}+6f_{s\nu-2}\\ &-3(\kappa_0+6\kappa_n)g_{s\nu-2}+4f_{s\nu-3}+2g_{s\nu-4}-\tfrac{2}{3}f_{s\nu-4}=0,\end{aligned} \tag{7.2b}$$

where $\mu=\sigma+\nu$, ν runs from 0 to ∞, and $g_{s\alpha}$ and $f_{s\alpha}$ both vanish when $\alpha<0$. In the limit as $\nu\to\infty$, (7.2) yields

$$\nu g_{s\nu} = 2g_{s\nu-1}, \quad \nu f_{s\nu} = 2f_{s\nu-1} \tag{7.3}$$

which, together with (7.1), yields

$$g_s(\rho),\ f_s(\rho) \propto \rho^{\sigma} e^{\rho}. \tag{7.4}$$

These relations yield diverging g_s and f_s when $\rho\to\infty$, contrary to the asymptotic form $e^{-\rho}$ discussed in connection with (6.8). Thus, the series $g_{s\nu}$ and $f_{s\nu}$ must terminate so that the $e^{-\rho}$ factor in (7.1) dominates when $\rho\to\infty$. The situation is analogous to that of the energy levels in a hydrogen atom in which a kind of terminated hypergeometric series is involved. It will be assumed that the series in (7.1) terminate when $\nu\geq\lambda$, so that

$$g_{s\lambda+\lambda'} = f_{s\lambda+\lambda'} = 0, \quad \lambda'\geq 1. \tag{7.5}$$

A set of necessary conditions is provided by (7.2) with $\nu=\lambda+4$, $\lambda+3$, $\lambda+2$, and $\lambda+1$. Making use of (7.5), one obtains

$$g_{s\lambda} = \tfrac{1}{3}f_{s\lambda}, \tag{7.6}$$

$$g_{s\lambda-1} = \tfrac{1}{3}f_{s\lambda-1} - 2f_{s\lambda}, \tag{7.7}$$

$$g_{s\lambda-2} = \tfrac{1}{3}f_{s\lambda-2} - 2f_{s\lambda-1} - 2(\mu'-1-\kappa_n)f_{s\lambda}, \tag{7.8}$$

$$\kappa_0 + 2\kappa_n = (2\mu'+3)(2\mu'+5), \tag{7.9}$$

$$\begin{aligned}g_{\lambda-3} = \tfrac{1}{3}f_{\lambda-3} &- 2f_{\lambda-2} - 2(\mu'-2-\kappa_n)f_{\lambda-1}\\ &+ 2(3\mu'^2+19\mu'+13-\kappa_0-8\kappa_n)f_{\lambda},\end{aligned} \tag{7.10}$$

$$4(2\mu'+3)f_{\lambda-1} = -(4\mu'^3+18\mu'^2-40\mu'-87)f_{\lambda}, \tag{7.11}$$

where

$$\mu' = \sigma + \lambda. \tag{7.12}$$

Putting $\nu=0$ in (7.2), the indicial equation

$$\begin{aligned}\sigma(\sigma-2)(\sigma+2)(\sigma+4)(\gamma_0+4\gamma_n)&\\ -16(\sigma-2)(\sigma+4)(\gamma_0+6\gamma_n)&-\gamma_0(\gamma_0+6\gamma_n)=0\end{aligned} \tag{7.13}$$

and the relation between g_{s0} and f_{s0},

$$[\sigma(\sigma-2)(\sigma+4)-\tfrac{1}{2}\gamma_0-3\gamma_n]g_{s0} = -\tfrac{2}{3}[(\sigma-2)(\sigma+4)(\sigma+6)+\tfrac{1}{4}\gamma_0]f_{s0}, \tag{7.14}$$

are obtained. If $\gamma_n=0$ and $\gamma_0\neq 0$, (7.13) and (7.14) become

$$\gamma_0=\sigma^4+4\sigma^3-20\sigma^2-48\sigma+128, \tag{7.15}$$

$$3(\sigma-4)g_{s0}=(\sigma+2)f_{s0}. \tag{7.16}$$

In order for the integrals mentioned after (6.8) to converge at $\rho=0$, it is required that $\sigma>-3$.

A procedure to solve (7.2) is to choose a value for γ_n/γ_0 and one for κ_n/κ_0 which, to begin with, can both be put to zero. Next, a pair of values σ and λ are suitably chosen so that γ_0, κ_0, γ_n, and κ_n are known. By putting $f_{s0}=1$, g_{s0} is known through (7.14). (7.2) can now be used to generate f_{s1}, g_{s1}, f_{s2}, g_{s2}, ..., $f_{s\lambda}$, and $g_{s\lambda}$. Then, the boundary conditions (7.6)–(7.8), (7.10), and (7.11) are checked. If these are satisfied, a solution with the associated eigenvalue κ_0 is found. If not, another pair of values for σ and λ are chosen and the whole process repeated.

Obviously, one may instead of checking with the boundary conditions (7.6)–(7.8), (7.10), and (7.11) extend the calculation to $f_{s\lambda+4}$ and $g_{s\lambda+4}$ and see if all of the last four f and four g coefficients vanish. It is noted, however, that if a solution is found and κ_0 known, the mass K_{00} is still not determined since the interaction parameter μ_0 in (6.16) is unknown. To determine μ_0, and also the usefulness of the present theory with the assumptions made, other members of the pseudoscalar-meson nonet and possibly the first-order equation (6.4) need be treated first and the results be compared with the observed masses.

A preliminary computer calculation of (7.2), with $\gamma_n=\kappa_n=0$, indicated that the $f_{s\nu}$ and $g_{s\nu}$ series could not be made to terminate; (7.2) does not appear to possess terminated series as solutions. However, the eight coefficients $f_{s\lambda+1}, \ldots, f_{s\lambda+4}, g_{s\lambda+1}, \ldots, g_{s\lambda+4}$ can each be made to vanish if $\lambda\geq 5$ and σ is allowed to vary slightly around values near 0.5 for each of these eight coefficients. Thus, for $\lambda=5$, $\sigma=0.4813\pm 1.3\times 10^{-3}$, $\lambda=8$, $\sigma=0.4975925\pm 10^{-6}$, and $\lambda=30$, $\sigma=0.5129\cdots$. If these values are taken as an indication of some interest, one may consider the following. With $\lambda=5$ and $\sigma\approx 0.5$, (7.12), (7.9), (7.15), and (6.16) give $\kappa_0\approx 224$ and $\gamma_0\approx 100$. Since these numbers are greater than the usual strong-interaction coupling parameter, ~15, the singlet internal strong interaction (6.11) thus appears to be stronger than the usual strong interaction in space-time. This may therefore lend some support to the assumption, stated between (3.19) and (3.20), that the G's, the space-time interaction functions, are of ϵ order or higher and can be neglected here.

The scalar-meson equation (3.20) can be treated in an entirely analogous manner. A zero-order equation corresponding to (6.3) has been obtained. Using the SU_3 singlet zero-order internal function (4.7), two coupled equations corresponding to (6.5) are obtained. With (6.7), (6.8) also holds in this case. Two recurrence relations similar to (7.2) are also obtained. With $\nu\to\infty$, however, one finds that either (7.3) or

$$\nu g_{s\nu}=(1\pm i)g_{s\nu-1}, \quad \nu f_{s\nu}=(1\pm i)f_{s\nu-1}$$

holds. Assuming that the series termination condition (7.5) holds in this case, it was found that a boundary condition from one of the recurrence relations contradicts a similar one from the other. Thus, no terminated series exists as solution to the scalar counterpart of the pseudoscalar (7.2).

The absence of terminated series as solutions to the two cases of singlet mesons treated above was found assuming that the zero-order mass K_{00} was real. The particles to be associated with these cases, namely, the pseudoscalar η' and the scalar η_{0+}, are unstable and therefore have complex masses. They are akin to the vector mesons in this respect and differ from the pseudoscalar octet mesons which are stable and have real masses in the present context. The present equations with complex K_{00} have not been treated.

VIII. RECURRENCE RELATIONS FOR AN OCTET MESON

Equation (6.10) can be treated in essentially the same way (6.5) was. The functions g, f, h, and j in (6.10) are put in a form similar to that of (7.1) with $\sigma\to\sigma_0$. The recurrence relations consist of four equations of the type given in (7.2) but are not reproduced here. In a similar manner, f_α, g_α, h_α, and j_α vanish if $\alpha<0$. In the limit as $\nu\to\infty$, one finds a set of relations similar to (7.3). Therefore, the series f_ν, g_ν, h_ν, and j_ν must also terminate. Assuming a set of series termination conditions similar to (7.5) with $\lambda\to\lambda_0$, a set of 13 boundary conditions similar to (7.6)–(7.11) are obtained for $\kappa_n=0$. One of these conditions reads

$$\kappa_0=(2\mu'+3)(2\mu'+5), \tag{8.1}$$

which is similar to (7.9). In (8.1), $\mu'=\sigma_0+\lambda_0$ replaces (7.12).

If $\kappa_n\neq 0$, one of the 13 boundary conditions cannot be satisfied because two boundary conditions contradict each other, similar to that in the case of the scalar singlet meson discussed near the end of the last section. This may lend some support to the assumption of $\kappa_n=\gamma_n=0$ for the pseudoscalar singlet case treated near the end of the last section.

Putting $\nu=0$ and $\gamma_n=0$, one obtains the indicial equation

$$\gamma_0=\sigma_0^4+4\sigma_0^3-44\sigma_0^2-96\sigma_0+608. \tag{8.2}$$

If σ_0 is to be real, γ_0 must $\geqslant 32$. This γ_0 value is in order-of-magnitude agreement with the value $\gamma_0\approx 100$ mentioned after (7.16) and again indicates that the internal interaction (6.11) is strong. Assuming $f_0=1$, g_0, h_0, and j_0 can be obtained from the recurrence relations with $\nu=0$. The rest of the treatment can be entirely similar to that described after (7.16). If $\gamma_0=0$, a set of relations is $\sigma_0=2$ and $g_0=j_0=0$ with the relation between f_0 and h_0 not determined. This and a study of (7.13)–(7.14) with $\gamma_0=0$ together with (7.15)–(7.16) seem to indicate that γ_n or possibly γ_n and κ_n or the whole nonet interaction $\omega(\rho)$ in (6.13) can be dropped here.

The octet meson of this section, the η meson, has real mass and the associated zero-order mass K_{00} has been assumed to be real and positive.

IX. GELL-MANN–OKUBO FORMULA

The zero-order equation (6.3) has been applied to the η' and η mesons and may also be applied to the other seven members of the pseudoscalar nonet mesons. Knowing the zero-order wave function $(\xi_0)_c^a(z)$ and K_{00}, the first-order equation (6.4) can readily be solved. For this purpose, the matrix

$$\lambda_0=\left(\frac{2}{3}\right)^{1/2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \tag{9.1}$$

is introduced. Let the subscripts S, T, U, and V each run from 1 to 8. These can then be associated with the SU_3 regular representation and correspond to s, t, u, and v with each of which running from 0 to 8. λ_t then denotes (λ_0,λ_T) where λ_T are the usual eight Gell-Mann matrices. With the usual relations for λ_T we can show that

$$(\lambda_s)_b^a(\lambda_t)_c^b\delta_a^c=2\delta_{st}, \tag{9.2}$$

$$[\lambda_s,\lambda_t]=2if_{stu}\lambda_u, \tag{9.3}$$

$$\{\lambda_s,\lambda_t\}=\tfrac{4}{3}\delta_{st}+2id_{stu}\lambda_u. \tag{9.4}$$

Here, f_{stu} is totally antisymmetric and d_{STU} totally symmetric. Further,

$$f_{0tu}=0, \tag{9.5}$$

$$d_{0TU}=d_{T0U}=(\tfrac{2}{3})^{1/2}\delta_{TU}, \tag{9.6}$$

$$d_{00t}=d_{su0}=0. \tag{9.7}$$

The wave functions of (6.3) and (6.4) are now expanded according to (4.5) and written in the following form:

$$(\xi_0)_c^a(z)=\sum_N A_{0N}(\xi_N)_c^a(z)$$
$$\equiv\sum_N A_{0N}\xi_{Ns}(z)(\lambda_s)_c^a, \tag{9.8}$$

$$(\xi_1)_c^a(z)=\sum_N A_{1N}(\xi_N)_c^a(z)$$
$$\equiv\sum_N A_{1N}\xi_{Nt}(z)(\lambda_t)_c^a, \tag{9.9}$$

where N denotes (p,q,Y,I,I_3), the set of internal quantum numbers describing the state of a multiplet member, and A_{0N} and A_{1N} are expansion coefficients. Similarly, it is defined that

$$\partial_c^a\equiv(\lambda_s)_c^a\partial_s. \tag{9.10}$$

For the zero-order internal meson wave function $(\xi_0)_c^a(z)$, only nine terms in (9.8) with $N=M$, referring to the nonet-meson eigenfunctions with $p=q=0$, given in (4.7), and with $p=q=1$ of the type given in (4.10), will be of substantial interest here. The corresponding eigenvalue is denoted by K_{00M}. Equation (6.4) now can be written as

$$[\tfrac{1}{4}K_{00M}{}^2(\lambda_s)_c^a-\tfrac{1}{2}K_{00M}\{\lambda_s,\lambda_t\}_c^a\partial_t+(\lambda_t)_b^a(\lambda_s)_d^b(\lambda_u)_c^d\partial_t\partial_u$$
$$-\tau(r)(\lambda_s)_c^a-\omega(r)(\lambda_t)_b^a(\lambda_s)_d^b(\lambda_t)_c^d]\sum_N A_{1N}\xi_{Ns}(z)$$
$$=[-\tfrac{1}{2}K_{01}(K_{00M}(\lambda_s)_c^a-\{\lambda_s,\lambda_t\}_c^a\partial_t)$$
$$+G_m(r)\{\lambda_8,\lambda_s\}_c^a]A_{0M}\xi_{Ms}(z), \tag{9.11}$$

where only the upper signs in (6.4) were used. This is consistent with the related discussion between (6.8) and (6.9).

With Sec. V, it can be shown that for the singlet member $M=(0,0,0,0,0)$ and one of the octet members $M=(1,1,0,0,0)$ the relations

$$\{\lambda_s,\lambda_t\}_c^a\partial_t\xi_{Ms}(z)=\zeta_{Ms}(z)(\lambda_s)_c^a, \tag{9.12}$$

$$(\lambda_t)_b^a(\lambda_s)_d^b(\lambda_u)_c^d\partial_t\partial_u\xi_{Ms}(z)=\eta_{Ms}(z)(\lambda_s)_c^a, \tag{9.13}$$

hold where ζ_{Ms}, η_{Ms}, and ξ_{Ms} have the same angular functions but different radial functions. Equations (9.12) and (9.13) are assumed to hold for the other seven of the octet members as well. Further, using (6.6) and (9.1), one obtains

$$\omega(r)(\lambda_t)_b^a(\lambda_s)_d^b(\lambda_t)_c^d\xi_{Ms}(z)=6\omega(r)\xi_{M0}(z)(\lambda_0)_c^a. \tag{9.14}$$

(9.11) can now be rewritten as

$$(\lambda_s)^a_c \sum_N (A_{1N}/A_{0M})[\tfrac{1}{4}K_{00M}{}^2 \xi_{Ns}(z) - \tfrac{1}{2}K_{00M}\zeta_{Ns}(z) + \eta_{Ns}(z) - \tau(r)\xi_{Ns}(z) - 6\omega(r)\xi_{N0}(z)\delta_{s0}]$$
$$= -\tfrac{1}{2}K_{01}(\lambda_s)^a_c[K_{00M}\xi_{Ms}(z) - \zeta_{Ms}(z)] + (\lambda_s)^a_c[(4/\sqrt{6})\delta_{s0}\xi_{M8}(z) + 2d_{8ts}\,\xi_{Mt}(z)]G_m(r) \tag{9.15}$$

where (9.4) has been consulted. In an entirely similar fashion, (6.3) takes the form

$$(\lambda_s)^a_c[\tfrac{1}{4}K_{00M}{}^2\xi_{Ms}(z) - \tfrac{1}{2}K_{00M}\zeta_{Ms}(z) + \eta_{Ms}(z) - \tau(r)\xi_{Ms}(z) - 6\omega(r)\xi_{M0}(z)\delta_{s0}] = 0\,. \tag{9.16}$$

One can write

$$(\lambda_s)^a_c\,\xi_{Ms}(z) = (\lambda_0)^a_c\,\xi_{M0}(z) + (\lambda_S)^a_c\,\xi_{MS}(z) \tag{9.17}$$

where $\xi_{MS}(z)$ constitutes an irreducible SU_3 octet. Using (9.17), a comparison between (9.8) and (4.5) gives

$$\lambda_0\xi_{M0}(z) = g_{p+q}(p-q, Y, I, I_3, r)Y^{pq}_{YII_3}\,, \tag{9.18}$$

$$(\lambda_S)^a_c\,\xi_{MS}(z) = \underline{X}^{pq}_{YII_3}(z)\underline{\lambda}^a_c\ . \tag{9.19}$$

Putting $M = N$ in (9.16), the last three terms on the left sides of (9.15) and (9.16) can be eliminated. The resulting equation is now multiplied from the left by $\int d^6z(\lambda_0)^c_a\,\xi_{L0}(z)$, its Hermitian conjugate by $\int d^6z(\lambda_0)^c_a\,\xi^*_{L0}(z)$, and both integrated equations are added together. One obtains

$$\tfrac{1}{4}(K_{00M}{}^2 - K_{00L}{}^2)\int d^6z(A_{1L}/A_{0M} + A^*_{1L}/A^*_{0M})\xi^*_{L0}\xi_{L0} - \tfrac{1}{2}(K_{00M} - K_{00L})\int d^6z(A_{1L}\,\xi^*_{L0}\zeta_{L0}/A_{0M} + A^*_{1L}\,\xi_{L0}\zeta^*_{L0}/A^*_{0M})$$
$$= -K_{01}\delta_{LM}\left[2K_{00M}\int d^6z\,\xi^*_{M0}\xi_{M0} - \int d^6z(\xi^*_{M0}\zeta_{M0} + \xi_{M0}\zeta^*_{M0})\right] + (8/\sqrt{6})\int d^6z(\xi^*_{L0}\xi_{M8} + \xi_{L0}\xi^*_{M8})G_m(r)\,, \tag{9.20}$$

where d^6z is given by (6.5) of I and (9.7) was consulted.

The whole procedure is repeated with the multiplicative function $(\lambda_0)^c_a\,\xi^*_{L0}(z)$ replaced by $(\lambda_T)^c_a\,\xi^*_{LT}(z)$ and $(\lambda_0)^c_a\,\xi_{L0}(z)$ by $(\lambda_T)^c_a\,\xi_{LT}(z)$. Use is made of (9.19), (4.6), and the orthogonality relation (4.18) which is now assumed to hold, in addition to $M = (0,0,0,0,0)$ and $M = (1,1,0,0,0)$, also for the other seven of the octet members with $p = q = 1$. Here, it may be noted that the two $\underline{\lambda}$'s in (4.18) are eliminated via (9.2). The resulting equation corresponding to (9.20) reads

$$\tfrac{1}{4}(K_{00M}{}^2 - K_{00L}{}^2)\int d^6z(A_{1L}/A_{0M} + A^*_{1L}/A^*_{0M})\xi^*_{LT}\xi_{LT} - \tfrac{1}{2}(K_{00M} - K_{00L})\int d^6z(A_{1L}\,\xi^*_{LT}\zeta_{LT}/A_{0M} + A^*_{1L}\,\xi_{LT}\zeta^*_{LT}/A^*_{0M})$$
$$= -K_{01}\delta_{LM}\left[2K_{00M}\int d^6z\,\xi^*_{MT}\xi_{MT} - \int d^6z(\xi^*_{MT}\zeta_{MT} + \xi_{MT}\zeta^*_{MT})\right]$$
$$+ \int d^6z[\,4d_{8UT}(\xi^*_{LT}\xi_{MU} + \xi_{LU}\xi^*_{MT}) + (8/\sqrt{6})(\xi^*_{L8}\xi_{M0} + \xi_{L8}\xi^*_{M0})G_m(r)]\,, \tag{9.21}$$

where (9.6) was consulted. Following the known procedure[3, 10] one has

$$\begin{aligned} d_{8UT}\xi_{MU} = (D_8)_{TU}\xi_{MU} &= \tfrac{2}{3}d_{8VW}(F_VF_W)_{TU} \\ &= (1/\sqrt{3})[-\tfrac{1}{3}F_VF_V + (F_1{}^2 + F_2{}^2 + F_3{}^2) - \tfrac{1}{3}F_8{}^2]_{TU}\xi_{MU} \\ &= (1/\sqrt{3})[-1 + I_M(I_M+1) - \tfrac{1}{4}Y_M{}^2]\,\xi_{MT} \end{aligned} \tag{9.22}$$

where $(D_V)_{TU} \equiv d_{VTU}$, $(F_V)_{TU} \equiv -if_{VTU}$, I_M denotes the isospin of the multiplet member M, and Y_M the corresponding hypercharge. The last step in (9.22) was possible as ξ_{MU} represents an irreducible SU_3 octet. Putting $L = M$, subtracting (9.20) from (9.21), and making use of (9.22), one obtains

$$K_{01M}\left[K_{00M}\int d^6z\,2(\xi^*_{MT}\xi_{MT} - \xi^*_{M0}\xi_{M0}) - \int d^6z(\xi^*_{MT}\zeta_{MT} + \xi_{MT}\zeta^*_{MT} - \xi^*_{M0}\zeta_{M0} - \xi_{M0}\zeta^*_{M0})\right]$$
$$= (8/\sqrt{3})(-1 + I_M(I_M+1) - \tfrac{1}{4}Y_M{}^2)\int d^6z\,\xi^*_{MT}\xi_{MT}G_m(r)\,, \tag{9.23}$$

where K_{01} was replaced by K_{01M}, which indicates that the mass correction K_{01M} here refers to the multiplet member M.

The mass of the pseudoscalar meson K_0 with internal quantum numbers denoted by M can similarly be denoted by K_{0M}. Up to first order, one can write

$$K_{0M} = K_{00M} + K_{01M} \,. \tag{9.24}$$

Similarly,

$$K_{0M}{}^2 = K_{00M}{}^2 + 2K_{00M}K_{01M} \,, \tag{9.25}$$

which together with (9.23) reproduces the Gell-Mann–Okubo formula[3] for pseudoscalar mesons. The term linear in Y in the Gell-Mann–Okubo formula is both absent and unnecessary in (9.23). If such a term is to be included, a term proportional to f_{8ts} needs to be inserted in (9.15) and this requires that a term including the antisymmetric expression

$$(\lambda_8)^a_b\delta^d_c - \delta^a_b(\lambda_8)^d_c \tag{9.26}$$

be inserted in (2.1). This expression destroys, however, the symmetry of (2.1) and was therefore not included.

X. MIXING, ELECTROMAGNETIC INTERACTIONS, AND CHARMED MESONS

It is noted that (9.23) is not complete in the sense that, if degeneracy exists in the zero-order states, possible first-order degeneracy may exist and has not been removed. Assuming that a degeneracy between two zero-order states described by L and M exists, i.e., $K_{00M} = K_{00L}$, (9.20) and (9.21) require that

$$P_{0LM} = \int d^6z(\xi^*_{L0}\xi_{M8} + \xi_{L0}\xi^*_{M8})G_m(r) = 0 \,, \tag{10.1a}$$

$$P_{8LM} = \int d^6z(\xi^*_{L8}\xi_{M0} + \xi_{L8}\xi^*_{M0})G_m(r) = 0 \,. \tag{10.1b}$$

If these expressions do not vanish, ξ_L and ξ_M are to be replaced by $\xi_{L'}$ and $\xi_{M'}$, which are mixed states of ξ_L and ξ_M. Specifically, one can put

$$\xi_{L'0} = \xi_{L0} + B_{M0}\xi_{M0}, \quad \xi_{L'T} = \xi_{LT} + B_{MT}\xi_{MT} \,, \tag{10.2a}$$

$$\xi_{M'0} = \xi_{M0} + B_{L0}\xi_{L0}, \quad \xi_{M'T} = \xi_{MT} + B_{LT}\xi_{LT} \,, \tag{10.2b}$$

where the B constants are determined by the requirement that the matrix elements of K_{01} between the L' and M' states vanish:

$$\int d^6z(\xi^*_{L'0}\xi_{M'8} + \xi_{L'0}\xi^*_{M'8})G_m(r) = 0 \,, \tag{10.3}$$

$$\int d^6z(\xi^*_{L'8}\xi_{M'0} + \xi_{L'8}\xi^*_{M'0})G_m(r) = 0 \,, \tag{10.4}$$

$$\int d^6z\, d_{8UT}(\xi^*_{L'T}\xi_{M'U} + \xi_{L'T}\xi^*_{M'U})G_m(r) = 0 \,. \tag{10.5}$$

Because d_{STU} is totally symmetric, only three relations, (10.3)–(10.5), are obtained to determine the four B's and, therefore, a relation between two of the B's can be chosen. Calculation of (10.5) can be simplified using (9.22). Equation (9.23) now becomes

$$K_{01L'}\left[K_{00M}\int d^6z 2(\xi^*_{L'T}\xi_{L'T} - \xi^*_{L'0}\xi_{L'0}) - \int d^6z(\xi^*_{L'T}\zeta_{L'T} + \xi_{L'T}\zeta^*_{L'T} - \xi^*_{L'0}\zeta_{L'0} - \xi_{L'0}\zeta^*_{L'0})\right]$$
$$= (8/\sqrt{3})\left[\left(-1 + I_L(I_L+1) - \tfrac{1}{4}Y_L{}^2\right)\int d^6z\,\xi^*_{LT}\xi_{LT}G_m(r) + |B_{MT}|^2\left(-1 + I_M(I_M+1) - \tfrac{1}{4}Y_M{}^2\right)\int d^6z\,\xi^*_{MT}\xi_{MT}G_m(r)\right] \,. \tag{10.6}$$

The equation for $K_{01M'}$ is given by (10.6) with $L' \to M'$, $L \to M$, and $M \to L$.

To illustrate such mixing or removal of degeneracy in first order, let L refer to the singlet function (4.7) and M to the octet function with $Y = I = 0$ (4.10). Equation (10.1), with the help of (4.7)–(4.13), (4.17), and (6.5) and (6.6) of I, now gives

$$P_{0LM} \propto \int_0^\infty dr\, r^5 g_0(0,0,0,0,r) f_0(0,0,0,0,r) G_m(r) \,, \tag{10.7}$$

$$P_{8LM} \propto \int_0^\infty dr\, r^5 g_2(0,0,0,0,r) f_2(0,0,0,0,r) G_m(r) \,. \tag{10.8}$$

The angular parts of these integrals did not vanish and these radial integrals may not vanish; the vanishing of other similar integrals depend usually upon the vanishing of their anuglar parts. Assuming that at least one of (10.7) and (10.8) is not zero and that a degeneracy between L and M exists or $K_{00L} = K_{00M}$, then such a degeneracy can be removed by suitably mixing the singlet and the $Y = I = 0$ octet states according to (10.2)–(10.5). The mass correction to the L' state, $K_{01L'}$, is given by (10.6) in which I_L, Y_L, I_M, and Y_M are all zero.

The results of this and the last sections were derived by starting from the generalized version of the ladder approximation of the Bethe-Salpeter equation, (2.1).

Thus, higher-order terms not included in the ladder approximation have been dropped. Further, it was assumed around (3.18) and (3.19) that the space time functions in (3.16) and (3.17) could be ordered with reference to powers of a small parameter ϵ. If the results of Secs. IX and X, essentially first-order results based upon the first-order part of (3.21), can be of use, the mentioned higher-order terms and the space-time-ϵ-order terms must be of second order or it must be possible to separate the relevant effects of these terms from the mentioned first-order effects. These requirements are pushed one step further if the second-order part of (3.21) involving the internal electromagnetic interactions is considered. Such a second order equation is obtained by adding a second-order term $(\xi_2)^a_c(z)$ to (6.1) and another second-order term K_{02} to (6.2). Substituting the so-modified (6.1) and (6.2) into (3.21) and keeping the second-order electromagnetic interaction term involving $G_{\rm cm}(r)$, the second-order part of (3.21) reads

$$\begin{aligned}
&\tfrac{1}{4}K_{00}{}^2(\xi_2)^a_c(z)-\tfrac{1}{2}K_{00}[\partial^a_b(\xi_2)^b_c(z)+\partial^b_c(\xi_2)^a_b(z)+\partial^a_b(\xi_2)^b_d(z)\partial^d_c-\tau(r)(\xi_2)^a_c(z)-\omega(r)(\lambda_s)^a_b(\xi_2)^b_d(z)(\lambda_s)^d_c]\\
&\qquad+\tfrac{1}{2}K_{00}K_{01}(\xi_1)^a_c(z)-\tfrac{1}{2}K_{01}[\partial^a_b(\xi_1)^b_c(z)+\partial^b_c(\xi_1)^a_b(z)]-G_m(r)[(\lambda_8)^a_b(\xi_1)^b_c(z)+(\xi_1)^a_b(z)(\lambda_8)^b_c]\\
&\qquad=\tfrac{1}{4}(2K_{00}K_{02}+K_{01}{}^2)(\xi_0)^a_c(z)+\tfrac{1}{2}K_{02}[\partial^a_b(\xi_0)^b_c(z)+\partial^b_c(\xi_0)^a_b(z)]+G_{\rm cm}(r)[Q^a_b(\xi_0)^b_c(z)+(\xi_0)^a_b(z)Q^b_c]\,,
\end{aligned}\tag{10.9}$$

where the upper sign was again adopted and $2Q = \lambda_3 + \lambda_8/\sqrt{3}$ according to (4.2) of I. The second-order equation (10.9) can be solved to yield K_{02} in a way analogous to that followed for the first-order equation. The coefficients A_{1N} in (9.9) and K_{01} are obtained from (9.15), (9.16), and (9.23). $(\xi_2)^a_c(z)$ can be expanded in a form similar to (9.9) with coefficients A_{2N}. With the help of the zero-order and first-order results, K_{02} can be evaluated. In doing so, it may be desirable from certain points of view to make a unitary transformation in the internal space so that the separation constants Y, I, and I_3 are transformed to Q, U, and U_3; U-spin formalism is naturally more suited for describing electromagnetic interactions. If degeneracy exists among the zero-order states as well as among the first-order states, it can be removed in a fashion similar to that given in this section, giving rise to second-order mixing of states.

During the past year, two particles, ψ(3.1 GeV)[11, 12] and ψ(3.7 GeV)[13], have been found. It has been suggested that each of these particles is a charmed meson consisting of a charmed antiquark and a charmed quark.[14] If this is the case and if such charmed quarks are to be included in the present formalism, a fourth complex coordinate z^4, in addition to $z^a = (z^1, z^2, z^3)$ in (2.8) spanning M_3, is to be introduced. In such a case, one may try to match the internal coordinates $z^a = (z^1, z^2, z^3)$ with the spatial coordinates $\vec{x} = (x_1, x_2, x_3)$ and, therefore, z^4 with the time coordinate $x_0 = t$. In such a matching, one may be led to assume that under coordinate transformations in the enlarged internal space spanned by z^1, z^2, z^3, and z^4 the quantity

$$z_1 z^1 + z_2 z^2 + z_3 z^3 - z_4 z^4 = r^2 - z_4 z^4 \tag{10.10}$$

is left invariant in analogy with the fact that $\vec{x}^2 - t^2$ is invariant under Lorentz transformations.

ACKNOWLEDGMENT

To A. Pagnamenta, I would like to extend my thanks for his supply of references on the subject of the Bethe-Salpeter equation.

[1] F. C. Hoh, preceding paper, Phys. Rev. D 14, 2790 (1976).

[2] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).

[3] S. Okubo, Prog. Theor. Phys. 27, 949 (1962).

[4] M. Gourdin, Nuovo Cimento 7, 338 (1958).

[5] W. Kummer, Nuovo Cimento 31, 219 (1964).

[6] G. C. Wick, Phys. Rev. 96, 1124 (1954).

[7] M. A. B. Bég and H. Ruegg, J. Math. Phys. 6, 677 (1965).

[8] J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963).

[9] P. Carruthers, *Introduction to Unitary Symmetry* (Interscience, New York, 1966).

[10] D. B. Lichtenberg, *Unitary Symmetry and Elementary Particles* (Academic, New York, 1970).

[11] J. J. Aubert *et al.*, Phys. Rev. Lett. 33, 1404 (1974).

[12] J.-E. Augustin *et al.*, Phys. Rev. Lett. 33, 1406 (1974).

[13] G. S. Abrams *et al.*, Phys. Rev. Lett. 33, 1453 (1974).

[14] J. D. Bjorken and S. L. Glashow, Phys. Lett. 11, 255 (1964).