

Quark theory with internal coordinates

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The free-particle Dirac wave function $\psi(x)$ is generalized to $\psi(x)\xi^a(z)$. Here, z denotes a set of three complex coordinates, called internal coordinates, in an abstract complex three-dimensional space, called internal space, a runs from 1 to 3, and $\xi^a(z)$ is assumed to contain a representation of the state of a quark triplet. The mass in the free-particle Dirac equation is replaced by a second-order operator $-\partial_a^2$ operating on $\xi^a(z)$. The so-modified Dirac equation is assumed to include a description of a free-quark triplet. Subsequently, symmetry-preserving interactions in space-time as well as in the internal space between two quark triplets are introduced. Two SU_3 -symmetry-breaking interactions, one transforming like the eighth component of an SU_3 octet vector and the other like the SU_3 charge operator, are also introduced. A similarly generalized Bethe-Salpeter equation in the ladder approximation is obtained. This equation is treated in greater detail in the following paper in which the Gell-Mann-Okubo formula for pseudoscalar mesons is derived with the coefficients determined by given relations. Then, spherical coordinates and corresponding spherical harmonics in the internal space are introduced. Finally, the equation for a one-quark system is briefly treated.

I. INTRODUCTION

Quantum mechanics, formulated about half a century ago, has indeed served us very well in the field of atomic physics. Quantum field theory, formulated shortly afterwards, together with the renormalization technique introduced about a quarter of a century ago has been spectacularly confirmed by experiments in the case of electromagnetic interactions, at the present time down to 10^{-15} cm, a distance considerably smaller than the length scale of a nucleus.

When applied to nuclear phenomena and strong interactions among the so-called elementary particles, quantum theory, here and hereafter limited to cases in which particle masses appear as given and predetermined constants, has proved to be far less satisfactory in accounting for experiments, both qualitatively and quantitatively. There are several reasons for this. In the first place, we do not have an accurate or even an adequate representation of the strong interaction in spite of a great wealth of experimental data, as we do in the case of the electromagnetic interactions. Thus, although we know the strong interactions phenomenologically, qualitatively, and semiquantitatively, there are features of strong interactions which cannot be naturally fitted into the quantum-theoretical description of today. Secondly, the strong-interaction strength is greater than unity and the perturbational approach, which is possible in theoretical quantum electrodynamics and is essential in bringing it into contact with experiments, fails. For the above reasons, the dispersion-relations approach of quantum theory to strong interactions was devised to complement

the field-theoretical approach. In spite of its partial success, the dispersion-relations approach is to be considered as an auxiliary theory in our search for a more adequate description of the strong interactions if our belief is that such a description is to take the form of a system of partial-differential or differential-integral equations. The last belief is based upon our experiences in all of the different basic branches of physics. Thirdly, particle masses are to be introduced as parameters into quantum theory with given and constant masses. This is practically manageable as long as the number of different masses is small. In an elementary-particle interaction, however, a great number of particles with different masses can be produced. To account for such interactions with quantum theory with given and constant masses, the large number of different masses gives the theory a rather clumsy appearance.

For these reasons at least, it is evident that quantum theory with given masses must be suitably modified or wholly replaced by another theory if we are to account for strong interactions more satisfactorily. The purpose of this paper is to present such a possible theory. In the search for such a more satisfactory theory, one may be guided by the above discussion and look for a system of partial-differential or differential-integral equations. The great success of quantum electrodynamics further suggests that at least some basic features of quantum field theory have fundamental importance and hold as well for interactions other than the electromagnetic ones. This is confirmed by the partial success of quantum field theory applied to strong interactions. Therefore, quantum

theory with constant masses will be used as a starting point and be suitably modified or generalized.

Another line of approach to account for our experiences in nuclear and strong-interaction phenomena began in the 1930's when the isospin formalism, associated with the mathematical group SU_2 , was proposed, which formalism turned out to be highly successful in classifying these phenomena. Generalization of the isospin concept to include strangeness or hypercharge and the corresponding generalization of SU_2 to SU_3 took place during the 1950's and early 1960's. Based on these generalizations, Gell-Mann¹ and Ne'eman² showed that essentially all the then-known hadrons could be fitted into an SU_3 classification scheme. Moreover, it was proposed that the masses of these particles were to be considered as eigenvalues of a mass operator operating on a set of functions representing irreducible SU_3 multiplets. In particular, the mass operator was assumed to consist of an SU_3 -singlet term and a term transforming like the eighth component of an SU_3 -octet vector.³ In this manner, the Gell-Mann-Okubo formula relating the masses of particles within a given SU_3 multiplet was deduced and it agreed well with experimental data under certain assumptions. A similar set of relations, concerning the electromagnetic splittings in SU_3 multiplets only, was also derived⁴ and likewise agreed well with experimental data. Along this line, Gell-Mann⁵ and Zweig⁶ showed that essentially all the then-known hadrons could be considered to consist of suitable combinations of a fundamental triplet, called quarks, much like the fundamental doublet, proton and neutron.

The SU_3 and quark formalism is so practical and powerful that it is natural that it has become of fundamental importance to strong-interaction physics. By its very nature, however, the formalism, apart from spin, does not contain any aspect of space-time mechanics of strong interactions. Since the experimental data are functions of space-time, the formalism alone is obviously insufficient in accounting for hadron interactions.

In the search for a more satisfactory theory for strong interactions, therefore, it appears natural to try to combine quantum theory with given and constant masses and the SU_3 and quark formalism in a suitable way. The above suggestion that quantum theory with constant masses is to provide a starting point and is to be suitably generalized can now be made more specific; the generalization consists of including the SU_3 and quark formalism or its equivalent in a suitable way. In particular, the suggestion leading to the Gell-Mann-Okubo formula, that masses are eigenvalues of a mass

operator, has been substantially followed. Further, use has been made of the hypothesis that quarks, formally represented by a triplet ξ^a , where a runs from 1 to 3, or included in such a formal representation for the time being, are the fundamental constituents from which all hadrons are built.

Since quarks have spin $\frac{1}{2}$, Dirac's equation for a free particle will be used as the starting point. The Dirac wave function is associated with a triplet ξ^a representing the quarks and the mass term in the Dirac equation is replaced by a suitable operator m_{op} operating on ξ^a . In this process, an abstract three-dimensional complex space, called internal space and denoted by M_3 , together with a set of three complex coordinates, called internal coordinates and denoted by z , are introduced. ξ^a then takes the meaning $\xi^a(z)$ and m_{op} becomes an operator operating on functions of z . These steps are carried out in Sec. II and the resulting equation is assumed to hold for a free-quark triplet or simply a free quark.

In Sec. III, a quark is assumed to interact with another quark with known state. Interactions in space-time as well as in the internal space are introduced. These interactions, however, preserve both the Lorentz and the U_3 invariance of the equations. In Sec. IV, interactions breaking the SU_3 invariance of the equations are introduced. In particular, an interaction term in the internal space transforming like the eighth component of an SU_3 octet vector is introduced following the line of Okubo.³

In Sec. V, the above generalizations are extended to the Bethe-Salpeter equation in the ladder approximation. The so-generalized Bethe-Salpeter equation for a quark-antiquark pair may account for mesons. This equation is treated in greater detail in the following paper⁷ in which the Gell-Mann-Okubo formula for pseudoscalar mesons is derived with the coefficients determined by given relations.

In Sec. VI, spherical coordinates and corresponding spherical harmonics in the internal space are introduced following the work of Bégg and Ruegg.⁸

In Sec. VII, the internal part of the free-quark equation is solved for the case of a single free quark. Solution is also given when an SU_3 -singlet-interaction function arising from a point source fixed in the internal space is included. A quantization rule for the $\xi^a(z)$ field is given.

II. THE FREE-QUARK EQUATION

The free-particle Dirac equation reads

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \quad (2.1)$$

Since one may anticipate that the triplet ξ^a will be involved in a generalization of (2.1), (2.1) is re-written in the spinor form of van der Waerden⁹:

$$i\partial_{\dot{\sigma}}^{\nu}\eta^{\dot{\sigma}}(x) - m\chi^{\nu}(x) = 0, \quad i\partial_{\nu}^{\dot{\sigma}}\chi^{\nu}(x) + m\eta^{\dot{\sigma}}(x) = 0 \quad (2.2)$$

in order to facilitate a comparison between $\psi(x)$ and ξ^a . $\eta^{\dot{\sigma}}(x)$ and $\chi^{\nu}(x)$, on the one hand, and ξ^a on the other, are now on equal footing: $\dot{\sigma} = 1, 2$ or $\nu = 1, 2$ refers to the spin-up and spin-down states of a particle and $a = 1, 2$ refers to the isospin-up and isospin-down states of a particle neglecting electromagnetic interactions. $a = 3$ refers to a state having a nonvanishing strangeness. From van der Waerden⁹ and Laporte and Uhlenbeck¹⁰ we can obtain

$$\eta^{\dot{1}} = \psi_2 + \psi_4, \quad \eta^{\dot{2}} = -\psi_1 - \psi_3, \quad (2.3)$$

$$\chi^1 = \psi_1 - \psi_3, \quad \chi^2 = \psi_2 - \psi_4,$$

$$\begin{pmatrix} x_{1\dot{1}} & x_{2\dot{1}} \\ x_{1\dot{2}} & x_{2\dot{2}} \end{pmatrix} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \quad (2.4)$$

$$\begin{pmatrix} \partial_{1\dot{1}} & \partial_{2\dot{1}} \\ \partial_{1\dot{2}} & \partial_{2\dot{2}} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_0} & \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_0} \end{pmatrix},$$

$$\square = -\frac{1}{2}\partial_{\dot{\alpha}\beta}\partial^{\beta\dot{\alpha}}. \quad (2.5)$$

χ^{ν} transforms like a contravariant spinor or doublet function and is analogous to ξ^a which transforms as a contravariant triplet function. The covariant spinor χ_{ν} is defined by $\chi_{\nu} = \epsilon_{\nu\sigma}\chi^{\sigma}$, where

$$\epsilon_{\nu\sigma} = -\epsilon_{\sigma\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.6)$$

is a totally antisymmetric tensor. Similarly, $\eta^{\dot{\sigma}}$ transforms like the complex conjugate of a contravariant spinor and $\eta_{\dot{\sigma}}$ like that of a covariant spinor. $\eta^{\dot{\sigma}} = \epsilon^{\dot{\sigma}\nu}\eta_{\nu}$, $\eta_{\dot{\sigma}} = \epsilon_{\dot{\sigma}\nu}\eta^{\nu}$, $\epsilon^{\dot{\sigma}\nu}$ has the same form as $\epsilon^{\sigma\nu}$, and $\epsilon_{\dot{\sigma}\nu}$ the same form as $\epsilon_{\sigma\nu}$ in (2.6). The physical quantities $x_{\alpha\dot{\beta}}$ and physical operators $\partial_{\alpha\dot{\beta}}$ transform like a mixed tensor of second rank. Entities like x_{α} , $x_{\dot{\beta}}$, ∂_{α} , and $\partial_{\dot{\beta}}$ do not exist in the physical world.

The generalization of (2.2) to account for a free quark makes use of the fact that a triplet ξ^a has been assigned to represent quarks and of the earlier suggestion that m is the eigenvalue of an operator m_{op} operating on a set of an irreducible SU_3 multiplet. In such a generalization, ξ^a is obviously to be associated with χ^{ν} or $\eta^{\dot{\sigma}}$ in a multiplicative manner. It is natural to assume that ξ^a can represent an irreducible SU_3 multiplet, namely, an SU_3 triplet; m_{op} then obviously operates on ξ^a . The formal form of m_{op} is chosen according to the following discussion. In classical mechanics, both

the momenta p_{μ} , or in spinor form $p_{\nu}^{\dot{\sigma}}$, and the mass m are observables. In going over to quantum mechanics, the observable $p_{\nu}^{\dot{\sigma}}$ was replaced by the operator $i\partial_{\nu}^{\dot{\sigma}}$ which operates on a spinor function χ^{ν} introduced in this connection. The observable m remained unchanged. Presently, m is also to be replaced by an operator m_{op} operating on the triplet ξ^a introduced. Now ξ^a is on equal footing with χ^{ν} as discussed after (2.2). Therefore, m_{op} is assumed to take the formal mathematical form $\text{const} \times \partial_a^b$ by analogy with the form $i\partial_{\nu}^{\dot{\sigma}}$. Putting $\text{const} = -1$, (2.2) is thus formally generalized to

$$\begin{aligned} i\partial_{\dot{\sigma}}^{\nu}\eta^{\dot{\sigma}}(x)\xi^c + \partial_b^c\xi^b\chi^{\nu}(x) &= 0, \\ i\partial_{\nu}^{\dot{\sigma}}\chi^{\nu}(x)\xi^b - \partial_a^b\xi^a\eta^{\dot{\sigma}}(x) &= 0 \end{aligned} \quad (2.7)$$

which possess a formal symmetry between the spin or Greek indices and the triplet Latin indices.

The existence of $\eta^{\dot{\sigma}}$ transforming like the complex conjugate of χ^{ν} is associated with the existence of positive- and negative-energy components of $\psi(x)$. The latter existence depends upon the existence of energy which together with momentum form a Lorentzian space. If there were only positive-energy components, like those appearing in the nonrelativistic case in which the space is Euclidean, it would not be necessary to carry the dotted Greek indices. From our experience in working with the quark contravariant triplet function ξ^a , we have not encountered any aspect analogous to the above-mentioned positive- and negative-energy aspects. The antiquark triplet function can be represented by the covariant triplet ξ_a . Also, as is shown later in (2.15), the space to be associated with ξ^a is not Lorentzian but Euclidean. Therefore, the dots on the index b in (2.7) can be and are removed and (2.7) becomes

$$\begin{aligned} i\partial_{\dot{\sigma}}^{\nu}\eta^{\dot{\sigma}}(x)\xi^c + \partial_b^c\xi^b\chi^{\nu}(x) &= 0, \\ i\partial_{\nu}^{\dot{\sigma}}\chi^{\nu}(x)\xi^b - \partial_a^b\xi^a\eta^{\dot{\sigma}}(x) &= 0. \end{aligned} \quad (2.8)$$

$\partial_b^a\xi^b$ is now interpreted as

$$\partial_b^a\xi^b = \frac{\partial^2}{\partial z_a \partial z^b} \xi^b(z^1, z^2, z^3, z_1, z_2, z_3), \quad (2.9)$$

where $(z^1, z^2, z^3) \equiv z^a$ are three complex independent variables spanning an abstract complex three-dimensional space M_3 . Aspects of such a space have been discussed earlier by Bég and Ruegg⁸ and Tait.¹¹ $\xi^a(z^b, z_b)$ is assumed to transform as z^a or as a contravariant vector or triplet in M_3 .

The covariant space-time spinor χ_{ν} is defined in terms of the contravariant spinor χ^{ν} by means of the antisymmetric tensor $\epsilon_{\nu\sigma}$ in (2.6). The tensor corresponding to $\epsilon_{\nu\sigma}$ in M_3 is the totally antisymmetric tensor ϵ_{abc} where a, b , and c each run from 1 to 3. There is no ϵ_{ab} which can be used to

define a covariant vector $z_a = (z_1, z_2, z_3)$ in terms of z^b in M_3 . One is therefore free to define z_a as the Hermitian conjugate of z^a :

$$z_a = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = (z^a)^\dagger = (z^1, z^2, z^3)^\dagger = \begin{pmatrix} z^{1*} \\ z^{2*} \\ z^{3*} \end{pmatrix}. \quad (2.10)$$

$\xi_a(z^1, z^2, z^3, z_1, z_2, z_3)$, representing the antiquark triplet, is similarly defined as $(\xi^a)^\dagger$ and transforms like a covariant vector or z_a in M_3 .

Consider a unitary transformation in M_3 : $z'_a = T_b^a z_b$. Its Hermitian conjugate is $z'^{\dagger a} = z^{\dagger a} = z^b T_b^a$. Unitarity requires that

$$z'^{\dagger a} z'_a = z^b T_b^a T_a^c z_c = z^b z_b, \quad (2.11)$$

so that $T_b^a T_a^c = \delta_b^c$. Using these relations, one can show that

$$\partial^{a'} \equiv \frac{\partial}{\partial z'^{\dagger a}} = \frac{\partial z_b}{\partial z'^{\dagger a}} \frac{\partial}{\partial z_b} = T_b^a \frac{\partial}{\partial z_b} \equiv T_b^a \partial^b \quad (2.12)$$

and thus transforms like a contravariant vector. Similarly

$$\partial'_a \equiv \frac{\partial}{\partial z^{\dagger a}} = \frac{\partial z_b}{\partial z^{\dagger a}} \frac{\partial}{\partial z_b} = T_a^b \frac{\partial}{\partial z_b} \equiv T_a^b \partial_b \quad (2.13)$$

and transforms like a covariant vector. The mass operator $-\partial_a^a \equiv -\partial^2 / \partial z_a \partial z^a$ transforms like a mixed tensor of second rank in M_3 and is analogous to the energy-momentum operator $i\partial_a^a$ in (2.8) which transforms like a mixed tensor of second rank in spinor space. The contracted mass operator $-\partial_a^a$ and contracted mass operator squared $\partial_a^a \partial_a^a$ transform like scalars in M_3 ; the latter is analogous to the d'Alembertian operator \square in (2.5). Writing

$$z^1 = y_1 + iy_2, \quad z^2 = y_3 + iy_4, \quad z^3 = y_5 + iy_6, \quad (2.14)$$

where the y 's are real quantities, the invariant quantity $z^b z_b$ in (2.11) becomes

$$z^b z_b = \sum_{i=1}^6 y_i^2, \quad (2.15)$$

showing that the space M_3 is Euclidean.

With the interpretation (2.9) and the discussions and definitions that follow, (2.8) becomes well defined. The spinor free-quark equations of (2.8) can be combined to produce a bispinor free-quark equation which is the corresponding generalization of (2.1), namely,

$$i\gamma^\mu \partial_\mu \psi(x) \xi^a(z) + \partial_a^a \psi(x) \xi^b(z) = 0. \quad (2.16)$$

Here, z denotes $(z^1, z^2, z^3, z_1, z_2, z_3)$ and will be referred to as internal coordinates. Similarly, M_3 can be called an internal space and $\xi^a(z)$ an internal quark function. One can now say that a principle guiding the quantum-mechanical equations (2.2) in which m is a constant to the general-

ized equations including internal coordinates, (2.8), is that the space-time part and the internal part of the generalized equations (2.8) are put on equal footing. Equation (2.16) can be separated to give a space-time part in the form of a Dirac equation

$$i\gamma^\mu \partial_\mu \psi(x) - m_{q0} \psi(x) = 0 \quad (2.17)$$

and an internal part in the following form

$$\partial_a^a \xi^b(z) + m_{q0} \xi^a(z) = 0, \quad (2.18)$$

where m_{q0} is a mass-separation constant between the space-time part and the internal part of the free-quark equation (2.16).

The transition from the Dirac equation for a free particle (2.1) to the free-quark equation (2.16) can be achieved by following the formal prescription: Multiply (2.1) by an internal quark function $\xi^b(z)$ from the right, replace m by $-\partial_a^a$, and multiply the term in the parentheses which does not involve m by δ_b^a . This prescription is similar to that taking classical mechanics to quantum mechanics with constant masses. Classically, we have, for a free particle

$$p_\mu p^\mu - m^2 = 0. \quad (2.19)$$

The prescription for going over to quantum mechanics is: Multiply (2.19) from the right by a wave function $\varphi(x)$ and replace p_μ by $i\partial/\partial x^\mu$ and p^μ by $i\partial/\partial x_\mu$. Carrying out these steps one obtains the Klein-Gordon equation for the same particle

$$(\square - m^2)\varphi(x) = 0. \quad (2.20)$$

If one considers the particle to be a pseudoscalar meson, one can further generalize (2.20) by following a prescription similar to that applied to the free-particle Dirac equation (2.1) mentioned between (2.18) and (2.19): Multiply (2.20) by an SU_3 -singlet internal function $\tau(z)$ from the right and replace m^2 by $\partial_a^c \partial_c^a$. Carrying out these two steps one obtains

$$(\square - \Delta_c) \varphi(x) \tau(z) = (-\frac{1}{2} \partial_{\alpha\beta} \partial^{\beta\alpha} - \partial_a^c \partial_c^a) \varphi(x) \tau(z) = 0, \quad (2.21)$$

where

$$\Delta_c \equiv \partial_a^c \partial_c^a = \partial_c^c \partial_a^a \quad (2.22)$$

and (2.5) has been used.

When going from the classical description (2.19) to the quantum-mechanical description (2.20), the observables p_μ were replaced by operators but the observable m was not. When going from the quantum-mechanical description (2.20) to the generalized description including internal coordinates (2.21), the observable m was also replaced by an operator. All the observable quantities in the clas-

sical relation (2.19), p_μ and m , have now been replaced by operators and in that sense have been put on equal footing. Equation (2.21) exhibits a symmetry between its space-time part and its internal part just like (2.16) does. Like (2.16), (2.21) can be separated to give

$$(\square - m_m^2)\varphi(x) = 0, \quad (2.23)$$

$$(\Delta_c - m_m^2)\tau(z) = 0 \quad (2.24)$$

which are analogous to (2.17) and (2.18), respectively. m_m^2 is the mass-separation constant between the space-time part and the internal part of the free-meson equation (2.21).

III. SYMMETRY-PRESERVING INTERACTIONS

Consider a fermion with mass m and wave function $\psi(x)$ interacting with another fermion with a known wave function $\chi(x)$ through the exchange of, say, a pseudoscalar particle with mass m_{P_0} . The appropriate equations are

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = i\gamma_5 U_P(x)\psi(x), \quad (3.1)$$

$$(\square - m_{P_0}^2)U_P(x) = \mu_P \bar{\chi}(x)\gamma_5 \chi(x), \quad (3.2)$$

where μ_P is a pseudoscalar interaction parameter. These two equations are generalizations of (2.1) and (2.20), respectively.

The generalization of these equations to include internal coordinates and functions begins by following the prescriptions given between (2.18) and (2.19) for (3.1) and those between (2.20) and (2.21) for (3.2). Multiplying (3.1) by $\xi^b(z)$ from the right, replacing m by $-\partial_b^a$, and multiplying the terms not containing m by δ_b^a , one has

$$i\gamma^\mu \partial_\mu \psi(x)\xi^a(z) + \partial_b^a \psi(x)\xi^b(z) = i\gamma_5 U_P(x)\psi(x)\xi^a(z). \quad (3.3)$$

Multiplying (3.2) by $\tau(z)$ from the right and replacing $m_{P_0}^2$ by Δ_c , one obtains

$$(\square - \Delta_c)U_P(x)\tau(z) = \mu_P \bar{\chi}(x)\gamma_5 \chi(x)\tau(z). \quad (3.4)$$

The left-hand side of (3.3), same as that of (2.16), and the left-hand side of (3.4), similar to (2.21), both possess symmetry between space-time and internal parts. The right-hand sides of these equations, one representing an interaction and the other a source function in space-time, do not possess such a symmetry. The next step in the generalization consists of providing such a symmetry by completing the right-hand sides of (3.3) and (3.4) in such a way that these equations become

$$i\gamma^\mu \partial_\mu \psi(x)\xi^a(z) + \partial_b^a \psi(x)\xi^b(z) = i\gamma_5 U_P(x)\psi(x)\xi^a(z) + \tau(z)\psi(x)\xi^a(z), \quad (3.5)$$

$$(\square - \Delta_c)U_P(x)\tau(z) = \mu_P \bar{\chi}(x)\gamma_5 \chi(x)\tau(z) - \mu_0 \xi_c(z)\xi^c(z)U_P(x), \quad (3.6)$$

where μ_0 is an SU_3 -singlet interaction parameter. The space-time wave function $\chi(x)$, associated with one of the fermions, has been generalized to the quark-triplet wave function or simply quark function $\chi(x)\xi^a(z)$ just as $\psi(x)$ was generalized to $\psi(x)\xi^a(z)$ and as the pseudoscalar interaction function $U_P(x)$ was generalized to the pseudoscalar SU_3 -singlet-interaction function $U_P(x)\tau(z)$. As with $\chi(x)$, $\xi^a(z)$ is assumed to be known. By requiring that (3.3) and (3.4) be made symmetric with respect to their space-time and internal parts, interactions between the two fermions in the internal space are naturally introduced here and are indicated by the last and introduced terms in (3.5) and (3.6). Such internal interactions play a dominant role in the present theory. Equation (3.5) can be separated to give

$$i\gamma^\mu \partial_\mu \psi(x) - m_q \psi(x) = i\gamma_5 U_P(x)\psi(x), \quad (3.7)$$

$$\partial_b^a \xi^b(z) + m_q \xi^a(z) = \tau(z)\xi^a(z) \quad (3.8)$$

and (3.6) can be separated to give

$$(\square - m_p^2)U_P(x) = \mu_P \bar{\chi}(x)\gamma_5 \chi(x), \quad (3.9)$$

$$(\Delta_c - m_p^2)\tau(z) = \mu_0 \xi_c(z)\xi^c(z). \quad (3.10)$$

Again m_q appears as the separation constant between the space-time and internal parts of (3.5) and m_p^2 as that of (3.6). The mass-separation constants are analogous to the angular-momentum-separation constant which arises when the time-independent Schrödinger equation with a central potential is separated into a radial part and an angular part or to the energy-separation constant which arises when a time-dependent Schrödinger equation is separated into a time-dependent part and a time-independent part.

Another possibility to achieve a symmetry between the space-time parts and the internal parts of the right-hand sides of (3.3) and (3.4) is to generalize them to the following product forms:

$$i\gamma_5 U_P(x)\psi(x)\xi^a(z) \rightarrow i\gamma_5 U_P(x)\psi(x)\tau(z)\xi^a(z), \quad (3.11)$$

$$\mu_P \bar{\chi}(x)\gamma_5 \chi(x)\tau(z) \rightarrow \mu_P \mu_0 \bar{\chi}(x)\gamma_5 \chi(x)\xi_c(z)\xi^c(z). \quad (3.12)$$

The equations obtained by combining (3.3) with (3.11) and (3.4) with (3.12) are, however, generally not separable. The convenient mass-separation constants generally do not exist and these equations may be difficult to solve. As an example to further illustrate this aspect, let us consider the generalization of the following one-dimensional Schrödinger equation

$$\frac{\partial^2}{\partial x_1^2} \psi_1(x_1) + E\psi_1(x_1) = V_1(x_1)\psi_1(x_1) \quad (3.13)$$

to a two-dimensional one. First, multiply (3.13)

by $\psi_2(x_2)$ from the right and replace $\partial^2/\partial x_1^2$ by $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ to obtain

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\psi_1(x_1)\psi_2(x_2) + E\psi_1(x_1)\psi_2(x_2) = V_1(x_1)\psi_1(x_1)\psi_2(x_2). \quad (3.14)$$

If we let x_1 represent the space-time coordinates x and x_2 the internal coordinates z , the transition from (3.13) to (3.14) is similar to the transition from (3.1) and (3.2) to (3.3) and (3.4). Now the interaction term on the right-hand side of (3.14) is not symmetric with respect to x_1 and x_2 . One possibility is to generalize $V_1(x_1)$ to the symmetric-product form $V_1(x_1)V_2(x_2)$ which leads to a Schrödinger equation generally not separable into x_1 and x_2 parts. If $V_1(x_1) = x_1^2$ and $V_2(x_2) = x_2^2$, then the interaction potential is $V_1(x_1)V_2(x_2) = x_1^2 x_2^2$, a form of little interest in atomic physics. Such a generalization to a product form is analogous to the generalizations to the product forms given in (3.11) and (3.12). Another possibility is to generalize $V_1(x_1)$ to the symmetric-sum form $V_1(x_1) + V_2(x_2)$ which leads to a Schrödinger equation separable in x_1 and x_2 . In particular, for $V_1(x_1) + V_2(x_2) = x_1^2 + x_2^2$, two harmonic-oscillator equations, one in x_1 and the other in x_2 , are obtained. This last generalization to a sum form is analogous to the generalization of (3.3) and (3.4) to (3.5) and (3.6), and this sum type of generalization will be adopted.

So far in this section, the interaction function for the two-quark system has been assumed to be a pseudoscalar-singlet function $U_P(x)\tau(z)$ generated by one of the quark-triplet functions $\chi(x)\xi^a(z)$. In principle, however, the singlet function $\tau(z)$ can also be associated with, instead of the pseudoscalar function $U_P(x)$, a scalar, vector, pseudovector, or tensor function in space-time. In addition to the singlet function $\tau(z)$, there is a nonet function $\omega_b^a(z)$ which can be associated with $U_P(x)$ or any one of the other four mentioned interaction functions in space-time to form an interaction function on the same footing as $U_P(x)\tau(z)$. For instance, if the interaction function is a vector nonet function, denoted by $U_\mu(x)\omega_b^a(z)$, (3.5) and (3.6) become

$$i\gamma^\mu \partial_\mu \psi(x)\xi^a(z) + \partial_b^a \psi(x)\xi^b(z) = \gamma^\mu U_\mu(x)\psi(x)\xi^a(z) + \omega_b^a(z)\psi(x)\xi^b(z), \quad (3.15)$$

$$(\square - \Delta_c)U_\mu(x)\omega_b^a(z) = \mu_\nu \bar{\chi}(x)\gamma_\mu \chi(x)\omega_b^a(z) - \mu_n \zeta_b(z)\xi^a(z)U_\mu(x), \quad (3.16)$$

respectively, where μ_ν is a vector interaction parameter and μ_n a nonet interaction parameter. Upon separation, these two equations become

$$i\gamma^\mu \partial_\mu \psi(x) - m_{qV}\psi(x) = \gamma^\mu U_\mu(x)\psi(x), \quad (3.17)$$

$$\partial_b^a \xi^b(z) + m_{qV}\xi^a(z) = \omega_b^a(z)\xi^b(z), \quad (3.18)$$

$$(\square - m_V^2)U_\mu(x) = \mu_\nu \bar{\chi}(x)\gamma_\mu \chi(x), \quad (3.19)$$

$$(\Delta_c - m_V^2)\omega_b^a(z) = \mu_n \zeta_b(z)\xi^a(z), \quad (3.20)$$

where m_{qV} and m_V^2 are two new mass-separation constants.

In general terms, each of the two internal-interaction functions, the singlet-interaction function $\tau(z)$ and the nonet-interaction function $\omega_b^a(z)$, can be associated with each of the five interaction functions in space-time, namely, the scalar, pseudoscalar, vector, pseudovector, and tensor interaction functions. Therefore, one has in principle $2 \times 5 = 10$ different interaction functions to choose from. Furthermore, the general interaction function for the two-quark system can in principle consist of a linear combination of the mentioned 10 different interaction functions.

IV. SYMMETRY-BREAKING INTERACTIONS

Equations (3.5) and (3.6) are written in such a form which shows that they are invariant under Lorentz transformations and unitarity transformations in the internal space. Further, since the two interacting quarks, represented by $\psi(x)\xi^a(z)$ and $\chi(x)\xi^a(z)$, respectively, are by themselves identical, (3.5) and (3.6) also hold when $\psi(x)$ is replaced by $\chi(x)$ and vice versa and simultaneously $\xi^a(z)$ is replaced by $\zeta^a(z)$ and vice versa. Thus, the mass-separation constants in (3.7)–(3.10) can become invariants under unitary transformations in the internal space. The above discussion obviously also holds for (3.15), (3.16), and (3.17)–(3.20) and for the case with a more general interaction function.

Such a conclusion appears to disagree with the rather successful proposal that the masses of the members of an SU_3 multiplet consist of a term invariant under SU_3 transformations and a term transforming like the eighth component of an SU_3 octet. Further, there are smaller electromagnetic corrections to these masses. Taking these considerations into account, (3.5) is further generalized to include two SU_3 -symmetry-breaking interaction terms:

$$i\gamma^\mu \partial_\mu \psi(x)\xi^a(z) + \partial_b^a \psi(x)\xi^b(z) = i\gamma_5 U_P(x)\psi(x)\xi^a(z) + [\tau(z) + G_m(z)(\lambda_8)_b^a + G_{em}(z)Q_b^a]\psi(x)\xi^b(z), \quad (4.1)$$

where

$$2Q = \lambda_3 + \lambda_8/\sqrt{3} \quad (4.2)$$

and the λ 's are two of the Gell-Mann matrices. $G_m(z)$ is supposed to represent a semistrong interaction and $G_{em}(z)$ an electromagnetic interaction

in the internal space. Since the G_m and G_{em} terms are assumed to be corrections to the singlet term $\tau(z)$ in (4.1), they are assumed to obey equations similar to (3.10):

$$(\Delta_c - m_8^2)G_m(z) = \mu_8 \xi_a(z) \xi^a(z), \quad (4.3)$$

$$(\Delta_c - m_Q^2)G_{em}(z) = \mu_Q \xi_a(z) \xi^a(z). \quad (4.4)$$

Here, μ_8 is a semistrong interaction parameter and μ_Q an electromagnetic interaction parameter in the internal space. Guided by the Gell-Mann-Okubo formula,³ m_8 is assumed to be equal to m_P and the associated $G_m(z)$ is denoted by $G_{mP}(z)$. Equations (3.6), (3.8), and (3.10) now become

$$\begin{aligned} (\square - \Delta_c)U_P(x)(\tau_{P8})_b^a(z) &= \mu_P \bar{\chi}(x) \gamma_5 \chi(x) (\tau_{P8})_b^a(z) \\ &\quad - [\mu_0 + \mu_8 (\lambda_8)_b^a] \xi_c(z) \xi^c(z) U_P(x), \end{aligned} \quad (4.5)$$

$$\partial_b^a \xi^b(z) + m_Q \xi^a(z) = (\tau_{P8})_b^a(z) \xi^b(z), \quad (4.6)$$

$$(\Delta_c - m_P^2)(\tau_{P8})_b^a(z) = [\mu_0 + \mu_8 (\lambda_8)_b^a] \xi_c(z) \xi^c(z), \quad (4.7)$$

respectively, where

$$(\tau_{P8})_b^a(z) = \tau(z) + G_{mP}(z) (\lambda_8)_b^a. \quad (4.8)$$

The internal electromagnetic interaction $G_{em}(z)$ is naturally associated with the electromagnetic interaction in space-time. Therefore, a corresponding space-time term for the electromagnetic interaction, $\gamma^\mu A_\mu(x) \xi^a(z)$, is added to the right-hand side of (4.1), where

$$\square A_\mu(x) = \mu_\alpha \bar{\chi}(x) \gamma_\mu \chi(x) \quad (4.9)$$

and μ_α is proportional to the fine-structure constant $\approx \frac{1}{137}$. Since the photon mass is zero, m_Q in (4.4) is therefore also put to zero so that (4.4) becomes

$$\Delta_c G_{em}(z) = \mu_Q \xi_a(z) \xi^a(z). \quad (4.10)$$

$$i\gamma^\mu \partial_\mu \psi(x) \xi^a(z) + \partial_b^a \psi(x) \xi^b(x) = [i\gamma_5 U_P(x) + \gamma^\mu U_\mu(x) + \gamma^\mu A_\mu(x)] \psi(x) \xi^a(z) + [(\tau_{P8})_b^a(z) + (\omega_{V8})_b^a(z) + G_{em}(z) Q_b^a] \psi(x) \xi^b(z). \quad (4.15)$$

Just like (3.6) was generalized to (4.5), (3.16) is similarly generalized to

$$(\square - \Delta_c)U_\mu(x)(\omega_{V8})_b^a(z) = \mu_V \bar{\chi}(x) \gamma_\mu \chi(x) (\omega_{V8})_b^a(z) - [\mu_n \xi_b(z) \xi^a(z) + \mu_8 \xi^c(z) \xi_c(z) (\lambda_8)_b^a] U_\mu(x). \quad (4.16)$$

(4.15) together with (4.5) and (4.16) form a set of equations describing the influence of a quark triplet with the supposedly known wave function $\chi(x) \xi^a(z)$ upon another with the wave function $\psi(x) \xi^a(z)$. (4.15) and (4.16) can, like (4.5), be separated. As remarked at the end of Sec. III, it is in principle possible to suitably include sca-

lar, pseudovector, and tensor interaction functions in (4.15) in addition to the pseudoscalar and vector interaction functions present in it.

$$(\omega_{V8})_b^a(z) = \omega_b^a(z) + G_{mV}(z) (\lambda_8)_b^a, \quad (4.11)$$

When the interaction function is $U_P(x)(\tau_{P8})_b^a(z)$, $U_P(x)$ is coupled to $(\tau_{P8})_b^a(z)$ via the mass-separation constant m_P^2 . If the interaction function is $U_\mu(x)(\omega_{V8})_b^a(z)$, where

$$U_\mu(x) \text{ is coupled to } (\omega_{V8})_b^a(z) \text{ via the mass-separation constant } m_V^2. \text{ Here, } G_m(z) \rightarrow G_{mV}(z) \text{ and } m_8 \rightarrow m_V \text{ in (4.3). In the case of the electromagnetic interaction, however, such a mass-separation constant vanishes. The internal electromagnetic interaction function } Q_b^a G_{em}(z) \text{ is decoupled from the electromagnetic interaction function in space-time, the photon function } A_\mu(x). \text{ This may be associated with the fact that the classical equation (2.19) can be generalized to include } A_\mu(x), \text{ but not any other type of interaction, at least up to now, so that}$$

$$[p_\mu - A_\mu(x)][p^\mu - A^\mu(x)] - m^2 = 0. \quad (4.12)$$

Following the prescriptions between (2.19) and (2.21), we obtain

$$\{[i\partial_\mu - A_\mu(x)][i\partial^\mu - A^\mu(x)] - \Delta_c\} \varphi(x) \tau(z) = 0, \quad (4.13)$$

which replaces (2.21). The remark between (2.22) and (2.23) that all the observable quantities in the classical equation (2.19) were replaced by operators in (2.21) also holds for the transition from (4.12) to (4.13) since $A_\mu(x)$ is not an observable quantity. Separation of (4.13) gives (2.24) and

$$\{[i\partial_\mu - A_\mu(x)][i\partial^\mu - A^\mu(x)] - m_m^2\} \varphi(x) = 0 \quad (4.14)$$

which replaces (2.23).

Taking the SU_3 symmetry-breaking pseudoscalar singlet and vector-nonet interaction functions and the photon function, shown between (4.8) and (4.9), into account, (4.1) is further generalized to

lar, pseudovector, and tensor interaction functions in (4.15) in addition to the pseudoscalar and vector interaction functions present in it.

It may be noted that the vector field $U_\mu(x)$ associated with the internal nonet interaction function $(\omega_{V8})_b^a(z)$ in (4.15) and (4.16) is not of a generalized Yang-Mills¹² type of field. In (4.15), the

orientations of the z^a axes in the internal space remain unchanged for different space-time points.

V. GENERALIZED BETHE-SALPETER EQUATION

In Sec. III and IV, the interaction between two quarks was described assuming that the wave function of one of the quarks, $\chi(x)\xi^a(z)$, was known. This is generally not true. In quantum-field theory, the interaction between two fermions is described by the Bethe-Salpeter equation.^{13, 14} In this investigation only the so-called ladder approximation of this equation, namely,

$$G_{PV}(x_I, x_{II}) = G_{PV}(|x_I - x_{II}|) = i^2 \gamma_{5I} \gamma_{5II} G_{Pm}(|x_I - x_{II}|) + g_{\mu\nu} \gamma_I^\mu \gamma_{II}^\nu [G_{Vm}(|x_I - x_{II}|) + G_{ph}(|x_I - x_{II}|)], \quad (5.2)$$

where $G_{Pm}(|x_I - x_{II}|)$ is proportional to a relativistic generalization of the Yukawa potential with a mass m_P and an interaction parameter μ_P , $G_{Vm}(|x_I - x_{II}|)$ is proportional to a similar potential with a mass m_V and an interaction parameter μ_V , $G_{ph}(|x_I - x_{II}|)$ is proportional to $\alpha_f |x_I - x_{II}|^{-2}$, where $\alpha_f \approx \frac{1}{137}$, and $g_{\mu\nu} = \delta_{\mu\nu}$ for $\nu = 0$ and $g_{\mu\nu} = -\delta_{\mu\nu}$ otherwise.

The generalization of (5.1) to describe the interaction of two quarks begins by following a prescription similar to that intervening (3.2) and

$$\begin{aligned} (i\gamma_I^\mu \partial_{\mu I} \delta_b^a + \partial_{bI}^a)(i\gamma_{II}^\mu \partial_{\mu II} \delta_d^c + \partial_{dII}^c) \Psi_{2q}(x_I, x_{II}) \Xi^{bd}(z_I, z_{II}) &= G_{PV}(|x_I - x_{II}|) \Psi_{2q}(x_I, x_{II}) \Xi^{ac}(z_I, z_{II}) \\ &+ \{T'(|z_I - z_{II}|) \delta_b^a \delta_d^c + \omega'(|z_I - z_{II}|) (\lambda_s)_b^a (\lambda_s)_d^c \\ &+ [G'_{mP}(|z_I - z_{II}|) + G'_{mV}(|z_I - z_{II}|)] (\lambda_s)_b^a \delta_d^c + \delta_b^a (\lambda_s)_d^c \\ &+ G'_{em}(|z_I - z_{II}|) (Q_b^a \delta_d^c + \delta_b^a Q_d^c)\} \Psi_{2q}(x_I, x_{II}) \Xi^{bd}(z_I, z_{II}). \end{aligned} \quad (5.3)$$

Here, z_I and z_{II} denote the positions of the two quark triplets or quarks in the internal space. The internal interaction functions are assumed to depend upon $|z_I - z_{II}|$ only in analogy with the case of the space-time interaction functions given in (5.2) and satisfy

$$(\Delta_c - m_P^2) T'(|z_I - z_{II}|) = \mu_0 \delta(z_I - z_{II}), \quad (5.4)$$

$$(\Delta_c - m_V^2) \omega'(|z_I - z_{II}|) = \mu_n \delta(z_I - z_{II}), \quad (5.5)$$

$$(\Delta_c - m_P^2) G'_{mP}(|z_I - z_{II}|) = \mu_8 \delta(z_I - z_{II}), \quad (5.6)$$

$$(\Delta_c - m_V^2) G'_{mV}(|z_I - z_{II}|) = \mu_8 \delta(z_I - z_{II}), \quad (5.7)$$

$$\Delta_c G'_{em}(|z_I - z_{II}|) = \mu_Q \delta(z_I - z_{II}). \quad (5.8)$$

Here, Δ_c is to be interpreted as operating on $z_I - z_{II}$ instead of on z as was the case in the previous sections. Obviously, (5.4) has its origin in (3.10), (5.5) in (3.20), (5.6) in (4.3) with $m_8 \rightarrow m_P$, (5.7) in (4.3) with $m_8 \rightarrow m_V$, and (5.8) in (4.10). The two λ_8 terms and the two Q terms in (5.3) instead of the corresponding ones in (4.15) are introduced to assure symmetry of (5.3) with respect to the inter-

$$\begin{aligned} (i\gamma_I^\mu \partial_{\mu I} - m_I)(i\gamma_{II}^\mu \partial_{\mu II} - m_{II}) \Psi_{2q}(x_I, x_{II}) \\ = G_{PV}(x_I, x_{II}) \Psi_{2q}(x_I, x_{II}), \end{aligned} \quad (5.1)$$

will be considered. Here, I and II refer to particles I and II, respectively. $\Psi_{2q}(x_I, x_{II})$ is a 16-component wave function in space-time and $G_{PV}(x_I, x_{II})$ is an interaction function corresponding to a one-particle exchange between the two interacting fermions.

If a pseudoscalar particle, a vector particle, and a photon are exchanged, as was indicated in (4.15), G_{PV} takes the form

(3.3). Equation (5.1) is multiplied by an internal two-quark function $\Xi^{bd}(z_I, z_{II})$ from the right, m_I is replaced by $-\partial_{bI}^a$, m_{II} by $-\partial_{dII}^c$, $\partial_{\mu I}$ is multiplied by δ_b^a , $\partial_{\mu II}$ by δ_d^c , and G_{PV} by $\delta_b^a \delta_d^c$. The right-hand side of the resulting equation is, as that of (3.3), not symmetric with respect to the space-time and the internal interaction functions. The equation is therefore completed in a similar way as (3.3) was completed to become (4.15). The result is

change of the two quarks. Further, λ_s represents the Gell-Mann matrices with s running from 0 to 8. λ_0 is defined here as

$$\lambda_0 = \left(\frac{2}{3}\right)^{1/2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.9)$$

Each of the indices a , b , c , and d runs from 1 to 3.

If one of the two quarks is replaced by an anti-quark, (5.3) is to be modified in the following way. The space-time part of (5.3) is modified by multiplying (5.3) by $\gamma_I \gamma_3$ from the right¹⁵ and by replacing $\Psi_{2q} \gamma_I \gamma_3$ by a quark-antiquark space-time wave function $\Psi_{q\bar{q}}(x_I, x_{II})$. The internal part of (5.3) is modified by changing the contravariant indices c and d to covariant ones and the covariant index d to a contravariant one. In this way, the generalized Bethe-Salpeter equation in the ladder approximation for a quark-antiquark system reads

$$\begin{aligned}
 (i\gamma^\mu \partial_\mu \delta_b^a + \partial_{b1}^a) \Psi_{qa}(x_I, x_{II}) \Xi_d^b(z_I, z_{II}) (i\gamma^\mu \partial_\mu \delta_c^d + \partial_{cII}^d) = i^2 G_{Pm}(|x_I - x_{II}|) \gamma_5 \Psi_{qa}(x_I, x_{II}) \gamma_5 \Xi_c^a(z_I, z_{II}) \\
 + [G_{ph}(|x_I - x_{II}|) + G_{Vm}(|x_I - x_{II}|)] \gamma_\mu \Psi_{qa}(x_I, x_{II}) \gamma^\mu \Xi_c^a(z_I, z_{II}) \\
 + \{\tau'(|z_I - z_{II}|) \delta_b^a \delta_c^d + \omega'(|z_I - z_{II}|) (\lambda_b)_b^a (\lambda_c)_c^d\} \\
 + [G'_{mP}(|z_I - z_{II}|) + G'_{mV}(|z_I - z_{II}|)] [(\lambda_b)_b^a \delta_c^d + \delta_b^a (\lambda_c)_c^d] \\
 + G'_{em}(|z_I - z_{II}|) (Q_b^a \delta_c^d + \delta_b^a Q_c^d) \Psi_{qa}(x_I, x_{II}) \Xi_d^b(z_I, z_{II}).
 \end{aligned}
 \tag{5.10}$$

As remarked at the ends of Secs. III and IV, (5.10) can be generalized to include scalar, pseudovector, and tensor interactions in space-time. (5.10) has been treated in the following paper.⁷

VI. TRANSFORMATION TO SPHERICAL COORDINATES

It is useful to introduce spherical coordinates and associated spherical harmonics in the internal space that are analogous to the ordinary spherical coordinates and spherical harmonics in three-dimensional real space.

Following Bég and Ruegg⁸ closely, the following transformation is made:

$$\begin{aligned}
 z^1 &= r \sin\vartheta \cos\xi e^{i\varphi_1}, \\
 z^2 &= r \sin\vartheta \sin\xi e^{i\varphi_2}, \\
 z^3 &= r \cos\vartheta e^{i\varphi_3},
 \end{aligned}
 \tag{6.1}$$

where

$$0 \leq \vartheta \leq \pi/2, \quad 0 \leq \xi \leq \pi/2, \quad 0 \leq \varphi_1, \varphi_2, \varphi_3 \leq 2\pi.
 \tag{6.2}$$

Taking the complex conjugate of (6.1), one obtains

$$\begin{aligned}
 z_1 &= r \sin\vartheta \cos\xi e^{-i\varphi_1}, \\
 z_2 &= r \sin\vartheta \sin\xi e^{-i\varphi_2}, \\
 z_3 &= r \cos\vartheta e^{-i\varphi_3}.
 \end{aligned}
 \tag{6.3}$$

By means of (6.1) and (6.3), one can write

$$\begin{aligned}
 \partial_1 &\equiv \frac{\partial}{\partial z^1} = \frac{1}{2} e^{-i\varphi_1} \left(\sin\vartheta \cos\xi \frac{\partial}{\partial r} + \cos\vartheta \cos\xi \frac{1}{r} \frac{\partial}{\partial \vartheta} - \frac{\sin\xi}{r \sin\vartheta} \frac{\partial}{\partial \xi} - \frac{i}{r \sin\vartheta \cos\xi} \frac{\partial}{\partial \varphi_1} \right), \\
 \partial_2 &\equiv \frac{\partial}{\partial z^2} = \frac{1}{2} e^{-i\varphi_2} \left(\sin\vartheta \sin\xi \frac{\partial}{\partial r} + \cos\vartheta \sin\xi \frac{1}{r} \frac{\partial}{\partial \vartheta} + \frac{\cos\xi}{r \sin\vartheta} \frac{\partial}{\partial \xi} - \frac{i}{r \sin\vartheta \sin\xi} \frac{\partial}{\partial \varphi_2} \right), \\
 \partial_3 &\equiv \frac{\partial}{\partial z^3} = \frac{1}{2} e^{-i\varphi_3} \left(\cos\vartheta \frac{\partial}{\partial r} - \sin\vartheta \frac{1}{r} \frac{\partial}{\partial \vartheta} - \frac{i}{r \cos\vartheta} \frac{\partial}{\partial \varphi_3} \right), \\
 \partial^1 &= \partial_1^*, \quad \partial^2 = \partial_2^*, \quad \partial^3 = \partial_3^*.
 \end{aligned}
 \tag{6.4}$$

A volume element in the internal space M_3 can be written as

$$d^6z \equiv dz_1 dz_2 dz_3 dz_1^* dz_2^* dz_3^* = |J_6| dr d\vartheta d\xi d\varphi_1 d\varphi_2 d\varphi_3,
 \tag{6.5}$$

where the absolute value of the Jacobian J_6 is given by

$$|J_6| = 8r^5 \cos\vartheta \sin^3\vartheta \cos\xi \sin\xi.
 \tag{6.6}$$

Bég and Ruegg⁸ defined a set of harmonic functions for the group SU_3 as follows:

$$Y_n^{m_1 m_2 m_3 I}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = \frac{1}{\sin\vartheta} e^{i(m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3)} d_{(m_3+2I+1)/2, (m_3-2I-1)/2}^{(n+1)/2} (2\vartheta) d_{(m_1+m_2)/2, (m_1-m_2)/2}^I(2\xi).
 \tag{6.7}$$

They also pointed out that these functions (6.7) form a complete orthogonal set in the intervals specified in (6.2) using $|J_6|/8r^5$ as the density function.

Here, the set of harmonic functions is normalized:

$$Y_n^{m_1 m_2 m_3 I}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = e^{i\delta} \left[\frac{(2I+1)(n+2)}{2\pi^3} \right]^{1/2} Y_n^{m_1 m_2 m_3 I},
 \tag{6.8}$$

where δ is a phase angle not yet determined and may depend upon m_1, m_2, m_3, I , and n . Using the density function $|J_6|/8r^5$ and observing (6.2), one can verify the following orthonormality relation:

$$\int Y_n^{m_1' m_2' m_3' I'} Y_n^{m_1 m_2 m_3 I} \frac{1}{8r^5 dr} d^6z = \delta_{m_1' m_1} \delta_{m_2' m_2} \delta_{m_3' m_3} \delta_{n' n} \delta_{I' I}.
 \tag{6.9}$$

A function of z_a and z^a , with $a = 1, 2$, and 3 , can be expanded as follows:

$$F(z_a, z^a) = f(r, \vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = \sum_{n=0}^{\infty} \sum_{2I=0}^n \sum_{m_3=-(n-2I)}^{n-2I} \sum_{m_2=-2I}^{2I} \sum_{m_1=-(2I-|m_2|)}^{2I-|m_2|} f_{nI}^{m_1 m_2 m_3}(\vartheta) Y_{nI}^{m_1 m_2 m_3}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3). \quad (6.10)$$

One can also show the following completeness relation:

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{2I=0}^n \sum_{m_3=-(n-2I)}^{n-2I} \sum_{m_2=-2I}^{2I} \sum_{m_1=-(2I-|m_2|)}^{2I-|m_2|} Y_{nI}^{m_1 m_2 m_3*}(\vartheta', \xi', \varphi_1', \varphi_2', \varphi_3') Y_{nI}^{m_1 m_2 m_3}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) \\ = \frac{8r^5}{|J_6|} \delta(\vartheta' - \vartheta) \delta(\xi' - \xi) \delta(\varphi_1' - \varphi_1) \delta(\varphi_2' - \varphi_2) \delta(\varphi_3' - \varphi_3). \end{aligned} \quad (6.11)$$

In accordance with Bég and Ruegg,⁸ the following identifications with quark theory are made:

$$\begin{aligned} p+q=n, \quad p-q=m_1+m_2+m_3, \\ Y=\frac{1}{3}(-2m_1+m_2+m_3), \quad I_3=\frac{1}{2}(m_1-m_2). \end{aligned} \quad (6.12)$$

From this equation one obtains

$$m_1=\frac{1}{3}(p-q)+\frac{1}{2}Y+I_3, \quad m_2=\frac{1}{3}(p-q)+\frac{1}{2}Y-I_3, \quad m_3=\frac{1}{3}(p-q)-Y. \quad (6.13)$$

(6.8) and (6.7) can now be rewritten as follows:

$$\begin{aligned} Y_{YII_3}^{pq}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) &= Y_{nI}^{m_1 m_2 m_3} \\ &= e^{i\delta} \left(\frac{(2I+1)(p+q+2)}{2\pi^3} \right)^{1/2} \frac{1}{\sin\vartheta} d_{(p-q-3Y+6I+3)/6, (p-q-3Y-6I-3)/6}^{(p+q+1)/2} (2\vartheta) d_{(p-q)/3+Y/2, I_3}^I(2\xi) \\ &\quad \times \exp\{i[\frac{1}{3}(p-q)+\frac{1}{2}Y+I_3]\varphi_1 + i[\frac{1}{3}(p-q)+\frac{1}{2}Y-I_3]\varphi_2 + i[\frac{1}{3}(p-q)-Y]\varphi_3\}. \end{aligned} \quad (6.14)$$

Similarly, (6.10) can be rewritten as:

$$\begin{aligned} F(z_a, z^a) &= f(r, \vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) \\ &= \sum_{p+q=0}^{\infty} \sum_{2I=0}^{p+q} \sum_{(p-q)/3-Y=-(p+q-2I)}^{p+q-2I} \sum_{(p-q)/3+Y/2=-I}^I \sum_{I_3=-I}^I f(p, q, Y, I, I_3, r) Y_{YII_3}^{pq}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3). \end{aligned} \quad (6.15)$$

The operator ∂_a^a can be written in the following way:

$$\partial_a^a \equiv \partial_1^1 + \partial_2^2 + \partial_3^3 = \frac{1}{r^5} \frac{\partial}{\partial r} r^5 \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_5, \quad (6.16)$$

where Δ_5 is a second-order operator operating on the angles $\vartheta, \xi, \varphi_1, \varphi_2,$ and φ_3 only. Bég and Ruegg⁸ have shown that $Y_{YII_3}^{pq}$ is an eigenfunction of Δ_5 :

$$\Delta_5 Y_{YII_3}^{pq} = -(p+q)(p+q+4) Y_{YII_3}^{pq}. \quad (6.17)$$

VII. ONE-QUARK SYSTEM

Returning to the case of free quarks, (2.18) can be solved by assuming that

$$\xi^a(z) = \text{const} \times k^a \exp[i(k_b z^b + k^b z_b)], \quad (7.1)$$

so that

$$m_{q0} = k_b k^b = k_1 k^1 + k_2 k^2 + k_3 k^3. \quad (7.2)$$

If a box type of boundary conditions in M_3 is assumed, discrete k_a and k^b values are obtained. The solution (7.1) does not, however, refer to a single quark. The internal wave function of a single quark is obtained by considering the following expansion of $\xi^a(z)$ for a one-quark system in

spherical harmonics:

$$\begin{aligned} \xi^a(z) &= (\xi^1(z), \xi^2(z), \xi^3(z)) \\ &= \sum_{p, q, Y, I, I_3} \begin{pmatrix} Q_1(p, q, Y, I, I_3, r) Y_{Y+1/3, I+1/2, I_3+1/2}^{p+1, q} \\ Q_2(p, q, Y, I, I_3, r) Y_{Y+1/3, I+1/2, I_3-1/2}^{p+1, q} \\ Q_3(p, q, Y, I, I_3, r) Y_{Y-2/3, I, I_3}^{p+1, q} \end{pmatrix}^T, \end{aligned} \quad (7.3)$$

where T denotes transpose. Keeping the lowest-order term, associated with $p=q=Y=I=I_3=0$, only and putting $Q_1=Q_2=Q_3=Q$, (7.3) becomes

$$\xi^a(z) = Q(0, 0, 0, 0, 0, r) \frac{1}{\pi} \left(\frac{\delta}{\pi} \right)^{1/2} \begin{pmatrix} \sin\vartheta \cos\xi e^{i\varphi_1} \\ \sin\vartheta \sin\xi e^{i\varphi_2} \\ \cos\vartheta e^{i\varphi_3} \end{pmatrix}^T, \quad (7.4)$$

which can be considered as the internal wave function of a single free quark. $\xi^a(z)$ in (7.3) generally represents states more complex than a single quark; for instance, the $p=q=1$ term may represent a single quark together with a quark-antiquark pair. In obtaining (7.4), (6.14) was used in which δ was put equal to 0 for ξ^2 and ξ^3 but to π for ξ^1 . This

procedure enables (7.4) to take a form conforming to (6.1) and leads to the cancellation of the angular parts in the derivation of the next equation. Inserting (7.4) into (2.18) and applying (6.4), the angular parts cancel out and one obtains

$$Q'' + \frac{5}{r} Q' - \frac{5}{r^2} Q + 4m_{q0} Q = 0, \quad (7.5)$$

where $Q' \equiv \partial Q / \partial r$. The solution of (7.5) is

$$Q = \text{const} \times \frac{1}{r^2} J_3(2\sqrt{m_{q0}} r), \quad (7.6)$$

where J denotes Bessel's function. m_{q0} can be interpreted as an indeterminate mass of a single free quark.

If a singlet interaction term, $\tau(z)$ in (3.10), is included, (2.18) becomes (3.8). For simplicity, it will be assumed that the source function in (3.10) can be replaced by a point source function $\mu_0 \delta(z)$. This assumption is analogous to the assumption in space-time that the source of an interaction function, corresponding to $\tau(z)$, is a point source located at the origin of a three-dimensional real coordinate system in space. Further it will be assumed that the mass-separation constant m_p^2 vanishes and that $z \rightarrow |z| = r$. (3.10) then becomes

$$\Delta_c \tau(r) = \mu_0 \delta(z). \quad (7.7)$$

Making use of (2.22) and (6.16), (7.7) yields

$$\tau(r) = \frac{\mu_0}{128\pi^3} \frac{1}{r^2} + \mu_{10} \frac{1}{r^4} + \mu_2 r^2 + \mu_3, \quad (7.8)$$

where μ_{10} , μ_2 , and μ_3 are integration constants. In this connection one may note that the equation

$$\partial_a^2 \tau_0(r) = \nu_0' \delta(z) \quad (7.9)$$

has the solution

$$\tau_0(r) = -\frac{\nu_0'}{32\pi^3} \frac{1}{r^4} + \nu_1', \quad (7.10)$$

where ν_1' is an integration constant.

By analogy with $G_{\text{ph}}(|x_I - x_{II}|)$, discussed after (5.2), one may assume that $\tau(r \rightarrow \infty) \neq \infty$ so that $\mu_2 = 0$.

Equation (7.4) is now substituted into (3.8) and, again, the angular parts cancel out leaving a radial equation like (7.5) with m_{q0} replaced by $m_q - \tau(r)$. Application of Frobenius's method to this radial equation using (7.8) yields no solution when $\mu_{10} \neq 0$. If, however, $\mu_{10} = 0$, one finds that

$$Q = \text{const} \times \frac{1}{r^2} J_{\pm(9+4mu)^{1/2}/2}(2(m_q - \mu_3)^{1/2} r), \quad (7.11)$$

where $mu = \mu_0 / 128\pi^3$. Equation (7.11) is a modified form of (7.6) with m_q not determined. Thus, a single quark cannot be confined by a central potential in the internal space of the type (7.8) or (7.10) in the sense that these potentials do not confine the radial internal function Q into a certain region in the internal space and make it vanish exponentially outside that region.

Finally, it is suggested¹⁶ that, in a second quantization, $\xi^a(z)$ obeys anticommutation relations

$$\{\xi^a(z), \xi^b(z')\} = \{\xi_a(z), \xi_b(z')\} = 0, \quad (7.12)$$

$$\{\xi^a(z), \xi_b(z')\} = \delta_3^a \delta(z - z')$$

but commutes with $\psi(x)$. Thus, the quark wave function $\psi(x)\xi^a(z)$ is always even, leading to the correct relation between spin and statistics of the baryon decuplet.

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