# Theory of multichannel potential scattering with permanently confined channels

R. F. Dashen*
*Institute for Advanced Study, Princeton, New Jersey 08540*

J. B. Healy†
*Department of Physics, Yale University, New Haven, Connecticut 06520*

I. J. Muzinich‡
*Brookhaven National Laboratory, Upton, New York 11973*
(Received 20 May 1976)

A detailed investigation of a class of nonrelativistic multichannel potential scattering models is presented. A subset of the channels contain confinement potentials that allow only a discrete spectrum with an accumulation at $+\infty$ (like the states of the quark model); the remaining channels contain the usual scattering states, which are allowed to communicate with the states of the permanently confined channel through an off-diagonal local potential. We formulate the problem in this paper in such a way that many of the usual properties of partial-wave scattering theory can be extended to this case. We generalize the Levinson theorem to include this class of models and attempt to use it to make a distinction between two types of resonances: those related to the fundamental states of the quark model and accidental resonances not associated with quark-model states.

## I. INTRODUCTION

In this paper we present a general discussion of the quantum theory of many-channel potential scattering in which scattering channels are allowed to interact with permanently confined channels. The confined channels can be thought of as being like a system of a quark and an antiquark, with infinite-range forces that permanently trap them; there are no scattering states in these channels. This is in contrast to the scattering channel, which can be thought of as a two-hadron system in which scattering states definitely exist.

In particular we consider a very simple model which contains much of the essential physics. Namely, we consider nonrelativistic quantum mechanics with the two-channel Hamiltonian

$$\mathcal{H}=\mathcal{H}_u+\mathcal{V}, \tag{1.1}$$

where

$$\mathcal{H}_u=\begin{pmatrix} H_c & 0 \\ 0 & H_s \end{pmatrix}, \quad \mathcal{V}=\begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}.$$

For simplicity $H_c=K+U$ and $H_s=K$, where $K$ is the kinetic energy ($K=-\vec{\nabla}^2$; $U$ and $V$ are the potentials of confinement and of communication respectively for two-particle systems in the c.m. frame). The quark structure of the hadrons is neglected and the potentials $V$ and $U$ are assumed to be local and depend only upon the relative separation of the two-body systems in each channel. A more realistic model would include many scattering channels with particle production and the quark structure of the hadrons. The neglect in the scattering channel of potentials of the same class as $V$ does not compromise our work in any way.[1] Also, the neglect of structure in the hadrons which are $q\bar{q}$ states is not crucial. We will communicate some progress in this direction elsewhere.

For low energies this class of models is somewhat similar to those of compound nuclear reaction theory,[2] but in the latter case all channels become scattering channels at a sufficiently high energy. To our knowledge there has been no systematic discussion of systems with permanently confined channels at all energies. We wish to treat the problem from a rigorous point of view and establish general results for a large class of potentials. We will find that while the scattering theory has many similarities to ordinary potential scattering, there are some novel and interesting features (see below). The motivations for our investigation are the following:

(1) We hope that a thorough study of such models will be useful to sharpen our ideas on hadron dynamics when the physics of confinement and scattering are present in different channels.

(2) It is also possible that nonrelativistic quantum mechanics could be directly relevant to quark confinement derived from a fundamental theory, such as Yang-Mills theory with exact color gauge symmetry. We have in mind that the confinement mechanism may well involve quarks of infinite mass for which nonrelativistic quantum mechanics might be directly relevant, if the energy of the scattering channel is not too high.

(3) In addition there may be some possible phenomenological applications to hadronic systems, such as charmonium[3] or baryon-antibaryon sys-

tems.

Very mathematical properties of the Hamiltonian in Eq. (1.1) that are usually taken for granted will be established for this model in a separate paper.[4] The mathematical results are the following:

(i) proof of the self-adjointness of $\mathcal{H}$ for $V \in L_2(E_3)$;

(ii) proof of the existence and completeness of the generalized wave operators for $V \in L_1(E_3) \cap L_2(E_3)$ ($E_3$ is the Euclidean three-space);

(iii) detailed spectral properties, namely that the spectrum of the Hamiltonian consists of two parts: a point spectrum at a discrete set of energies on the real axis, lower semibounded, with a possible accumulation point at $+\infty$, and an absolutely continuous spectrum on the real axis at energies above the scattering threshold; and

(iv) proof of the eigenfunction expansion associated with the Hamiltonian of Eq. (1.1) [the eigenfunctions consist of normalizable bound states and distorted plane waves (the scattering states)].

Such mathematical results are possible because of the detailed and elegant work on scattering theory discussed in the books by Kato and Simon.[5]

The theory for the partial-wave equation for spherically symmetric $U$ and $V$ will be presented in this paper. The starting point is to convert the time-independent Schrödinger equation implied by Eq. (1.1) to a Hilbert-Schmidt integral equation with the use of the Green's function for the unperturbed Hamiltonian. We avoid the use of Jost solutions, which do not, in general, exist for this problem. The analytic properties of the physical partial-wave solution, of the Fredholm determinants, and of the partial-wave $S$ matrices are relatively easy to establish.

The main novel and interesting features are (i) the existence of square-integrable eigenfunctions at a discrete set of energies above the scattering threshold (these bound states are degenerate with the continuum[6]; at these energies there are poles of the full resolvent kernel but not of the $S$ matrix and hence the unitarity of the $S$ matrix is not compromised) and (ii) a generalization of the Levinson theorem for the phase shifts. It is not surprising that the Levinson theorem is modified here, since its usual hypotheses are not satisfied in this case. However, when the level spacing between the states of the confined channel is bounded from below, $|E_n - E_{n+1}| > \Delta$ at large $E_n$, we obtain a generalization, namely the statement

$$\delta_l(0) - \delta_l(E_n+\epsilon) - (m-n)\pi = O((E_n+\epsilon)^{-\alpha}), \quad \alpha > 0 \tag{1.2}$$

for $0 < \epsilon < \Delta/2$. In Eq. (1.2) $m$ and $n$ are the numbers of bound states of $\mathcal{H}$ and $H_c$ respectively for energies less than $E_n + \epsilon$. One consequence of Eq. (1.2) is that the physical phase shift tends to $+\infty$ at $E_n = +\infty$ if $m_\infty$ is finite, and there are an infinite number of resonances where the phase shift increases by $\pi$. We then use Eq. (1.2) to make a distinction between the two types of resonances, those directly related to the states of $H_c$ and those arising from the coupling between channels.

The organization of our paper is the following: Section II contains a discussion of the Hilbert-Schmidt problem and the analytic properties of its solution. In Sec. III we derive expressions relating the partial-wave $S$ matrices to the Fredholm determinant; a proof of the generalization of the Levinson theorem for these models is also presented. Section IV contains a discussion of the resonance behavior associated with the bound states of $H_c$ and a mathematical example with a bound state in the continuum and no accumulation of bound states at $+\infty$. We also make some concluding remarks in this section. There are two appendixes dealing with some bounds for the Green's function relevant to these models.

## II. STATEMENT AND SOLUTION OF THE HILBERT-SCHMIDT PROBLEM

### A. Preliminaries

In the usual development of the theory of potential scattering,[1] use is made of the Jost solutions even in the case of many-channel scattering. From the Jost solutions, the physical solution is constructed and the partial-wave $S$ matrix is identified as the ratio of incoming to outgoing waves in the asymptotic behavior of the physical wave function. This is a useful construction since the Jost solutions are designed to approach incoming and outgoing waves in each channel, and relatively modest requirements on the potential guarantee the existence of such solutions. However, in our problem scattering-type boundary conditions cannot be maintained in both channels. Therefore, we avoid this procedure in the following and convert the Schrödinger equation to an integral equation that can be rigorously treated by general methods. Even though the confinement potential is unbounded, its Green's function is well behaved and its properties can be exploited through an integral equation.

The Schrödinger equation for our system in the time-independent formulation is

$$[-I\nabla_x^2 + \mathfrak{U}(\vec{x}) + \mathfrak{V}(\vec{x})]\Psi(\vec{x}) = E\,\Psi(\vec{x}), \tag{2.1}$$

where $\Psi$ is a vector

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix},$$

$I$ is the unit two-by-two matrix, and $\mathcal{U}=\frac{1}{2}(1+\sigma_3)U$ and $\mathcal{V}=V\sigma_1$ ($\sigma_i$ are the usual Pauli matrices). For spherically symmetric functions $U$ and $V$ we can simplify Eq. (2.1) by the standard partial-wave expansion

$$\Psi(\vec{x})=\sum_{lm}\psi_l(r)Y_{lm}(\hat{r}), \tag{2.2}$$

and obtain the ordinary differential equations

$$r\left\{I\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}\right)+\mathcal{U}(r)+\mathcal{V}(r)\right\}r\psi_l(r)=E\psi_l(r). \tag{2.3}$$

The operator on the left-hand side of Eq. (2.3) is well defined on piecewise differentiable functions of second order, the space $[C^2(0,\infty)]^2$.

Next we transform Eq. (2.3) to an integral equation; for $\mathcal{V}\psi_l\in\mathfrak{D}(R)$, the domain of the resolvent operator $\mathcal{R}=(IE-\mathcal{H}_u)^{-1}$:

$$\psi_l(r)=\psi_l^0(r)+\int_0^\infty r'^2\,dr'\frac{\mathcal{G}_u(r,r',E,l)}{rr'}\mathcal{V}(r')\psi_l(r'), \tag{2.4}$$

where $\psi_l^0(r)=(0,j_l(kr))$, $j_l(kr)$ is the spherical Bessel function ($k^2=E$), and $\mathcal{G}_u$ is the appropriate Green's kernel for the resolvent $\mathcal{R}$:

$$\mathcal{G}_u(r,r',k,l)=\begin{pmatrix}G_c(r,r',k,l) & 0\\ 0 & G_s^+(r,r',E,l)\end{pmatrix}. \tag{2.5}$$

$G_c$ and $G_s^+$ are the confined-channel and scattering-channel Green's functions which satisfy

$$\left(k^2+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-U\right)G_c(r,r',k,l)=\delta(r-r'), \tag{2.6a}$$

$$\left(k^2+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right)G_s^+(r,r',E,l)=\delta(r-r'). \tag{2.6b}$$

The confined-channel Hamiltonian $H_c$ has discrete spectrum $E=E_n\in\sigma(H_c)$; hence, the integral operator in Eq. (2.4) is not defined at $E=E_n\in\sigma(H_c)$. However, the precise location of the poles of $\mathcal{G}_u$ is due to our choice of separation of perturbed and unperturbed Hamiltonians. We can always choose a different separation of $\mathcal{H}$ into $\mathcal{H}_u'$ and $\mathcal{V}'$ such that $\mathcal{G}_u'\mathcal{V}'$ in Eq. (2.4) does not have a pole at a particular $E_n$. In our companion paper[4] we show how to deal with this question in detail.

The scattering Green's function has the customary representation satisfying outgoing-wave boundary conditions at $r=+\infty$ and vanishing as $O(r^{l+1})$ at $r=0$,

$$G_s^+(r,r',k,l)=-k(r_<j_l(kr_<))(rh_l^+(kr_>)), \tag{2.7a}$$

where $h_l^+(kr)$ is the spherical Hankel function ($h_l^+(kr)\sim\exp[i(kr-\frac{1}{2}l\pi)]/kr$ for $r\to\infty$). The confined-channel Green's function has a similar representation:

$$G_c(r,r',E,l)=-\frac{y_{0l}(r_<)y_{+l}(r_>)}{W}, \tag{2.7b}$$

where $y_{0l}(r)$ and $y_{+l}(r)$ are two linearly independent solutions to

$$\left(k^2+\frac{\partial^2}{\partial r^2}-\frac{l(l+1)}{r^2}-U\right)y=0$$

which satisfy the boundary conditions $y_{0l}(r)=O(r^{l+1})$ as $r\to 0$ for $U(r)$ less singular than $r^{-2}$ at $r=0$, and $\lim_{r\to\infty}y_{+l}(r)=0$; $W$ is the Wronskian $W=y_0y_+'-y_+y_0'$. Also in Eqs. (2.7a) and (2.7b)

$$r_{\gtrless}=\tfrac{1}{2}(r-r'\pm|r-r'|).$$

On the $L_2$ space of functions we expand an arbitrary function $f(r)\in L_2$ in terms of the eigenfunctions of $H_c$, $\psi_{nl}\in L_2$ as $f=\sum a_n r\psi_{nl}(r)$; hence, $G_c$ can be given a meaning in terms of an eigenfunction expansion,

$$(Gf)(r)=\int_0^\infty dr'G_c(r,r',E,l)f(r')=\sum_n\frac{\alpha_n r\psi_{nl}(r)}{E-E_n}, \tag{2.8}$$

which converges absolutely for $f\in L_2$.

For convenience we choose $H_c$ such that its spectrum lies above the scattering-channel threshold $E=0$, i.e., $E_n>0$ for all $n$. The modifications required for the case when some of the bound states of $H_c$ lie below threshold are trivial.

### B. Hilbert-Schmidt theory

The main result of this section is the following. Let the potentials $U$ and $V$ satisfy these conditions:

(i) $rV\in L_1(0,\infty)\cap L_2(0,\infty)$. (2.9a)

(ii) $V$ is of finite range, i.e., there exists an $\alpha_0>0$ such that for $0<\alpha<\alpha_0$ we have

$$e^{\alpha r}rV\in L_2(0,\infty). \tag{2.9b}$$

(iii) $V$ has at most a finite number of singularities.

(iv) $U$ is such that $G_c(r,r',E,l)$ satisfies

$$\frac{1}{r^2}\,||G_c(r,\cdot,E,l)||^2\equiv\frac{1}{r^2}\int dr'\,|G_c(r,r',E,l)|^2<C \tag{2.9c}$$

for $E \notin \sigma(H_c)$, where $C$ depends on $E$ but not on $r$. We will refer to this condition as the strong Carleman condition. In Appendix B we show that it is satisfied for the simple harmonic oscillator, and it is also easily seen to be satisfied by the infinite square well.

Let $B$ be the Banach space of continuously bounded functions with norm

$$\|\psi_l\|_B = \operatorname*{Sup}_{r,i} |\psi_l^{(i)}(r)| < m < \infty .$$

Then either there exists a unique solution in $B$ to the integral equation (2.4), or the homogeneous equation

$$\psi_l(r) = \int_0^\infty r'^2 dr' \frac{\mathcal{G}_u(r,r',E,l)}{rr'} \mathcal{V}\psi_l(r') \qquad (2.10)$$

has nontrivial solutions in $B$. The set of energies $\sigma(\mathcal{H}_l)$ for which there are nontrivial solutions to Eq. (2.10) is bounded below and discrete, with the only possible accumulation point at $E = +\infty$.

For $k^2 = E$ in the neighborhood of a particular eigenvalue $E_n$ of $H_c$, we can always shift the location of the pole in the confined-channel Green's function $G_c$ by choosing a different separation of the full Hamiltonian $\mathcal{H}$ into unperturbed Hamiltonian $\mathcal{H}_u' = \mathcal{H}_u - \delta V$ and perturbation $\mathcal{V}' = \mathcal{V} + \delta\mathcal{V}$, where $\delta\mathcal{V}$ is a diagonal potential in the confined channel. All of our arguments go through for the integral equation (2.4) with $\mathcal{G}_u\mathcal{V}$ replaced by $\mathcal{G}_u'\mathcal{V}'$, and the solutions to the new integral equation are identical with the solutions of the original integral equation for $k^2 = E$ not equal to the poles of either kernel.

To establish this result we convert Eq. (2.4) into a Hilbert-Schmidt integral equation for $E = k^2$ in a domain including the real axis.[7,8] In order to accomplish this we factor the potential as

$$\mathcal{V} = \mathcal{V}_1\mathcal{V}_2 = (V_1 I)(V_2\sigma_x) , \qquad (2.11)$$

with $V_1$ bounded and of finite range and $V_2$ of finite range, which is possible from Eqs. (2.9a) and (2.9b). Multiplying Eq. (2.4) from the left by $r\mathcal{V}_2$ we obtain

$$\Phi(r) = \Phi_0(r) + \int_0^\infty K(r,r')\Phi(r')dr' \qquad (2.12a)$$

with

$$K(r,r') = \mathcal{V}_2(r)\mathcal{G}_u(r,r',E,l)\mathcal{V}_1(r') , \qquad (2.12b)$$

where $\Phi = \mathcal{V}_2 r\psi$, $\Phi_0 = \mathcal{V}_2 r\psi_l^0$, and we drop the $l$ dependence for simplicity of notation.

Next we define a separable Hilbert space $\mathfrak{H} = L_2 \oplus L_2$ with the norm

$$\|\Phi\|_H{}^2 = \sum_{i=1}^{2} \|\Phi_i\|^2 < \infty$$

for

$$\Phi = (\Phi_1, \Phi_2) \in \mathfrak{H} ,$$

where

$$\|\Phi_i\|^2 = \int_0^\infty dr |\Phi_i(r)|^2 < \infty \qquad (2.13)$$

is the norm on $L_2(0,\infty)$.

It is easy to see that for every solution $\psi_l \in B$ to Eq. (2.4) [or Eq. (2.10)], $\Phi = r\mathcal{V}_2\psi_l$ is a solution to Eq. (2.12a) (or the corresponding homogeneous equation) and $\Phi \in \mathfrak{H}$:

$$\|\Phi\|_{\mathfrak{h}}{}^2 = \int_0^\infty |rV_2|^2(|\psi_l^{(1)}|^2 + |\psi_l^{(2)}|^2)dr \le \|\psi_l\|_B \|rV_2\|^2 < \infty . \qquad (2.14)$$

And for every solution $\Phi \in \mathfrak{H}$ to Eq. (2.12), the $\psi_l$ given by

$$\psi_l^{(1)}(r) = \int_0^\infty dr' \frac{G_c(r,r',E,l)}{r} V_1(r')\Phi_1(r') ,$$

$$\psi_l^{(2)}(r) = j_l(kr) + \int_0^\infty dr' \frac{G_s(r,r',k,l)}{r} V_2(r')\Phi_2(r')$$

satisfies Eq. (2.4) and $\psi_l \in B$:

$$|\psi_l^{(1)}| \le C\|\Phi_2\| \frac{\|G_c(r,\cdot,E)\|}{r} < \infty ,$$

$$|\psi_l^{(2)}| \le C + C\|V_1\| \, \|\Phi_2\| \operatorname{Sup} \frac{|G_s(r,r',k,l)|}{r} < \infty ,$$

where we have used the Schwarz inequality and the boundedness of $V_1$; and we have proved in Appendix A that

$$\left|\frac{G_s(r,r',k,l)}{r}\right| < C , \quad k^2 = E \text{ real}.$$

Having established the one-to-one correspondence between solutions of Eqs. (2.4) and (2.12a), we need only study Eq. (2.12a) in the Hilbert space $\mathfrak{H}$. The operator norm for a kernel $K(r,r') \in \mathfrak{H}\times\mathfrak{H}$ is defined by

$$\tau(KK^\dagger) \equiv \|K\|_{\mathfrak{h}\times\mathfrak{h}} = \sum_{j,i=1}^{2} \int_0^\infty dr \int_0^\infty dr' |K_{ij}(r,r')|^2 . \qquad (2.15)$$

And we can easily show for all $k \in D$, $D = \{k \,|\, \mathrm{Im}k > -\alpha_0/2, k^2 = E \notin \sigma(H_c)\}$, that

$$K = \mathcal{V}_2(r)\mathcal{G}_u(r,r',k,l)\mathcal{V}_1(r') \in \mathfrak{H}\times\mathfrak{H} .$$

$\tau(KK^\dagger)$ is a sum of two terms:

$$\tau(KK^\dagger)=\int_0^\infty dr\int_0^\infty dr'\,|V_2(r)|^2|V_1(r')|^2[\,|G_c(r,r',E)|^2+G_s(r,r',E)|^2]$$
$$=\tau^c+\tau^s. \tag{2.16}$$

For the scattering piece $\tau^s$ we have

$$\tau^s=\int_0^\infty dr\int_0^\infty dr'\,|V_1(r)|^2|r'V_2(r')|^2\left|\frac{G_s(r,r',E)}{r'}\right|^2$$
$$\le \operatorname*{Sup}_{r,r'}\left|\frac{G_s^+(r,r',E)e^{-(\alpha/2)r}e^{-(\alpha/2)r'}}{r'}\right|^2\|rV_2e^{(\alpha/2)r}\|^2\|V_1e^{(\alpha/2)r}\|^2<\infty, \tag{2.17a}$$

where we have used (see Appendix A or the text of De Alfaro and Regge[1])

$$\operatorname*{Sup}_{r,r'}\left|\frac{G_s^+(r,r',k)e^{-(\alpha/2)r}e^{-(\alpha/2)r}}{r'}\right|\le\text{const} \tag{2.17b}$$

for $\mathrm{Im}k\ge-\alpha/2$. The norm $\tau_c$ is bounded by

$$\tau^c=\int_0^\infty dr\int_0^\infty dr'\,|G_c(r,r',k^2)|^2|V_1(r)|^2|V_2(r')|^2$$
$$\le c^2\int_0^\infty dr'\frac{\|G_c(r',\cdot,k^2)\|}{r'^2}|r'V_2(r')|^2, \tag{2.18}$$

where we have used $|V_1(r)|<c$; and by the strong Carleman property of $G_c$ and the fact that $rV\in L_2$ we have

$$\tau_c\le C\left(\operatorname*{Sup}_{r\ge0}\frac{\|G_c(r,\cdot,E,l)\|}{r}\right)^2\|r'V_2\|^2<C \tag{2.19}$$

for $E\not\in\sigma(H_c)$, where $C$ depends only on $E$.

Thus the operator $K$ is a compact operator on $\mathfrak{H}$. And $K$ is easily seen to be meromorphic in $\mathrm{Im}k>-\alpha_0/2$ (see Ref. 4 for a proof). Thus we can apply the analytic Fredholm theorem, which we quote here in a form suitable to our purposes.

*Theorem*: Let $K(k)$ be a compact operator-valued function meromorphic in some connected domain $D$ of the complex $k$ plane, and let the residue of each pole of $K(k)$ be an integral kernel of finite rank. If $[I-K(k)]^{-1}$ exists at one point $k\in D$, then $[I-K(k)]^{-1}$ exists and is meromorphic in $D$. The poles of $[I-K(k)]^{-1}$ occur at the discrete set of points $[k_i]$ where the homogeneous equation

$$\Phi=K(\lambda)\Phi \tag{2.20}$$

has nontrivial solutions in $\mathfrak{H}$, and the residue of each pole is an integral kernel of finite rank. For the proof see the paper by Tiktopoulos.[9]

The detailed solution to Eq. (2.12a) can be found in any text on integral equations[10]:

$$\Phi(r)=\Phi_0(r)+\int_0^\infty R_\lambda(r,r',k)\Phi_0(r')dr', \tag{2.21}$$

where the resolvent $R_\lambda$ is given by

$$R_\lambda(r,r',k)=N_\lambda(r,r',k)D_\lambda^{-1}(k). \tag{2.22}$$

$\lambda$ is a coupling parameter which we have factored out of the potential, $\mathcal{V}\to\lambda\mathcal{V}$, to organize the determinantal expansions

$$D_\lambda(k)\equiv\det(I-\lambda K)$$
$$=\sum_{n=0}^\infty\lambda^n\delta_n(k)$$
$$=\det(I-\lambda\mathcal{G}_u\mathcal{V}) \tag{2.23a}$$

and

$$N_\lambda(r,r',k)=\sum_{n=1}^\infty\lambda^nN_n(r,r',k), \tag{2.23b}$$

where $\delta_n$ and $N_n$ are given by expressions involving $K$ and $\mathrm{Tr}K^n$ in the text by Smithies.[10] For our purposes we need only

$$N_1=K,$$
$$\delta_0=1,$$
$$\delta_1=0, \tag{2.24}$$
$$\delta_2=\mathrm{Tr}K^2=\sum_{ij}\int_0^\infty dr\int_0^\infty dr'K_{ij}(r,r',k)\times K_{ji}(r',r,k),$$

and the bounds

$$|\delta_n|\le\frac{[\tau(KK^+)]^{n/2}e^{n/2}}{n^{n/2}},$$
$$[\tau(N_nN_n^+)]^{1/2}\le\frac{[\tau(KK^+)]^{(n+1)/2}e^{(n+1)/2}}{n^{n/2}}. \tag{2.25}$$

The bounds in Eq. (2.25) ensure the uniform convergence of the series in Eqs. (2.23) for $k\in D$. Thus $D_\lambda(k)$ is meromorphic in $\mathrm{Im}k>-\alpha_0/2$ with poles at $k^2\in\sigma(H_c)$. And $D_\lambda(k)$ has zeros at the energies $k^2=E_i$ for which the homogeneous equation (2.20) has nontrivial solutions: The order of each zero is equal to the number of linearly independent solutions.[11]

We must still determine the order of the poles in

$D_\lambda(k)$ at $k^2 \in \sigma(H_c)$. Let us focus on one pole at $E = E_n$ and rewrite the unperturbed Green's function and the kernel $K$ in terms of a modified kernel plus a dyad:

$$\mathcal{G}_u = \mathcal{G}'_u + \frac{|n\rangle\langle n|}{E - E_n}$$

and (2.26)

$$K = K' + \frac{\mathcal{V}_2 |n\rangle\langle n| \mathcal{V}_1}{E - E_n},$$

where $K'$ and $\mathcal{G}'_u$ are defined through Eqs. (2.5), (2.7b), and (2.8) with one pole at $E = E_n$ missing, and $|n\rangle$ is an obvious Dirac notation for the state $\Phi_n(r)(1,0)$ of the confined channel. If $K \in \mathfrak{H} \times \mathfrak{H}$ then obviously $K' \in \mathfrak{H} \times \mathfrak{H}$ by the same technique as the proof that $K \in \mathfrak{H} \times \mathfrak{H}$. Using the identities

$$\begin{aligned}\det(1 - \lambda K) &= \det\left(1 - \lambda K' - \frac{\lambda \mathcal{V}_2 |n\rangle\langle n| \mathcal{V}_1}{E - E_n}\right)\\ &= \det(1 - \lambda K')\\ &\quad\times \det\left(1 - \frac{\lambda(1 - \lambda K')^{-1}\mathcal{V}_2 |n\rangle\langle n| \mathcal{V}_1}{E - E_n}\right)\end{aligned} \tag{2.27}$$

which follow from the Fredholm theory, and

$$\det(I + |f\rangle\langle g|) = 1 + \langle f, g\rangle \tag{2.28}$$

for $f, g \in L_2$, we obtain the following relation between determinants:

$$D_\lambda(k) = D'_\lambda(k) - \lambda \frac{\langle n|N'_\lambda(k)|n\rangle}{E - E_n}, \tag{2.29}$$

with

$$D'_\lambda(k) = \det(I - \lambda K'),$$

$$N'_\lambda(k) = \mathcal{V}_1[(I - \lambda K')^{-1} - I]\mathcal{V}_2 D'_\lambda(k).$$

The formal manipulations leading to Eq. (2.29) are justified since all of the operator products and expansions necessary to define the objects in Eqs. (2.27) and (2.29) exist and all factors are well-defined kernels in $\mathfrak{H} \times \mathfrak{H}$. Using the expansion corresponding to Eq. (2.23) for $N'_\lambda(k)$ we see that $\langle n|N'_\lambda(k)|n\rangle$ is regular at $k^2 = E_n$. And since by construction $K'$ has no pole at $k^2 = E_n$, $D'_\lambda(k)$ is also regular at that point. Thus $D_\lambda(k)$ has at worst simple poles at the eigenvalues $E_n$ of $H_c$, provided only that the eigenvectors of $H_c$ are not degenerate. It is easy to see that in general the maximum order of the pole in $D_\lambda(k)$ will be equal to the multiplicity of the eigenvalue $E_n$.

It is possible that a particular eigenvalue $E_n$ of $H_c$ will not show up as a pole in $D_\lambda(k)$ and that $D_\lambda(k)$ will be finite at this point. If a pole decouples in this way, then Eq. (2.20) must have a nontrivial solution $\Phi \in \mathfrak{H}$ at this point. To see this we must use a different separation of the complete Hamiltonian into $\mathcal{H}'_u = \mathcal{H} - \delta\mathcal{V}$ and $\mathcal{V}' = \mathcal{V} + \delta\mathcal{V}$ where $\delta\mathcal{V}$ has only a diagonal component $\delta\mathfrak{U}$ in the confined channel and is chosen such that $E_n$ is not contained in the spectrum $\{E'_n\}$ of $H'_c = H_c - \delta\mathfrak{U}$. Then, as discussed in Ref. 4, the resolvent kernel $\mathcal{G}'_u$ of $R' = (k^2 - \mathcal{H}'_u)^{-1}$ must satisfy the integral equation

$$\begin{aligned}\mathcal{G}'_u &= \mathcal{G}_u + \int \mathcal{G}_u \mathcal{V} \mathcal{G}'_u\\ &= \mathcal{G}_u + \int \mathcal{G}_u \mathcal{V}' \mathcal{G}_u - \int \mathcal{G}'_u \delta\mathcal{V} \mathcal{G}_u,\end{aligned} \tag{2.30}$$

from which we obtain the relation

$$\begin{aligned}D_\lambda(k) &= \det(I - \mathcal{G}_u \mathcal{V})\\ &= \det(I - \mathcal{G}'_u \mathcal{V}') \det(\mathcal{G}_u \mathcal{G}'^{-1}_u)\\ &= D'_\lambda(k) \prod_n \frac{E - E'_n}{E - E_n},\end{aligned} \tag{2.31}$$

where

$$D'_\lambda(k) = \det(E - \mathcal{G}'_u \mathcal{V}').$$

If a pole of $D_\lambda(k)$ at $k^2 = E_n$ decouples, then $D'_\lambda(k)$ must have a zero at $k^2 = E_n$; this zero of $D'_\lambda(k)$ corresponds to a solution to the homogeneous equation (2.10) with $\mathcal{G}_u \mathcal{V}$ replaced by $\mathcal{G}'_u \mathcal{V}'$.

### C. Relation to solutions of the Schrödinger equation $\mathcal{H}\Psi = E\Psi$

The previous subsection was limited to the discussion of the integral equations, (2.4) and (2.12a). In order to make full contact with the Schrödinger equation, (2.1), it is necessary to make some further assumptions on the potential $V$, namely, that $V$ is Hölder continuous except at a finite number of singularities and that $V(r) = O(r^{-5/2-\epsilon})$ $(\epsilon > 0)$ for $r \to \infty$. These assumptions enable us to prove[4] that the square-integrable solutions to the homogeneous integral equation (2.10) are also the square-integrable bound-state solutions to the Schrödinger equation $\mathcal{H}\psi = E\psi$. Furthermore, the set $\{E_i\} = \sigma(\mathcal{H})$ is restricted to the real axis, with a finite number of bound states below threshold and a discrete set above threshold (bound states embedded in the continuum) with a possible accumulation at $E = +\infty$.

The confined-channel element $\psi_l^{(1)}(r)$ of any solution $\psi_l(r)$ to Eq. (2.10) is guaranteed to be in $L_2$ because of the strong Carleman condition on $G_c$. For $E < 0$ the scattering-channel Green's function $G_s$ is also a Carleman-type operator, so the negative-energy solutions to Eq. (2.10) are obviously square integrable. From the homogeneous integral equation we see that a necessary condition for the square integrability of a solution to this equation for $E_i > 0$ is

$$\langle \psi_{0l}|\mathcal{V}|\psi_l\rangle = \int_0^\infty r^2\, dr\, \psi_{0l}(r)\mathcal{V}(r)\psi_l(r) = 0, \tag{2.32}$$

since[4]

$$\psi_l^{(2)}(r) = \frac{e^{ikr}}{r}\langle\psi_{0l}|\mathcal{V}|\psi_l\rangle + O(r^{-3/2-\epsilon}) \qquad (2.33)$$

and $\langle\psi_{0l}|\mathcal{V}|\psi_l\rangle$ is the coefficient of the leading term in the expansion of the wave function at large $r$. In Ref. 4, Eq. (2.32) was shown to be satisfied provided $E>0$. The condition Eq. (2.32) decouples the bound state from the scattering amplitude and gives a zero width for such bound states embedded in the continuum.

A zero-energy solution to the homogeneous integral equation requires special consideration. In that case, the confined-channel element of $\psi_l(r)$ is still in $L_2$ and the scattering-channel element is

$$\psi_l^{(2)}(r) = -r^{-l-1}\int_0^\infty r'^{l}V(r')\psi_l^{(1)}(r')dr' + O(r^{-3/2-\epsilon}) \qquad (2.34)$$

for large $r$, where we have used the fact that the free Green's function reduces to $G_s(r, r', 0, l) = -r_>^{-l}r_<^{l+1}$ for $k=0$. We see that $\psi_l$ is always square integrable for $l \geq 1$, and for the $s$ wave $\psi_{l=0}$ will be square integrable provided $\int_0^\infty r'V(r')\psi_{l=0}^{(1)}(r')dr' = 0$. The latter possibility is excluded in ordinary potential scattering by the positivity of the derivatives of the Jost function, but there is no analog of this argument to prevent it from occurring in our models. On the other hand, if $\int_0^\infty r'V(r')\psi_{l=0}^{(1)}(r')dr' \neq 0$ then the zero-energy $s$-wave solution to Eq. (2.30) is not square integrable and is not a bound-state wave function.

Thus the situation is essentially the same as in ordinary potential scattering. The only solutions to the homogeneous integral equation (2.10) which are not normalizable bound-state solutions to the Schrödinger equation are those zero-energy $s$-wave solutions which are not square integrable. And just as in ordinary potential scattering, the nature of the zero-energy solution (if one is present) to Eq. (2.10) will manifest itself in the order of the zero of $D_\lambda(k)$ at $k^2=0$. To see this we employ a method which was originally introduced by Weinberg.[12] We consider the integral equation

$$\eta(E)\psi(r,E) = \int r'^2\,dr'\,\frac{\mathcal{G}_u(r,r',E,l)}{rr'}\,\mathcal{V}(r')\psi(r',E) = (E-\mathcal{H}_u)^{-1}\mathcal{V}\psi\,. \qquad (2.35)$$

For each fixed $E$ there will be a discrete set of values $\eta_i(E)$ for which Eq. (2.38) has nontrivial solutions. Following Weinberg we will refer to these $\eta_i(E)$ as the complex eigenvalue trajectories. The $\eta_i(E)$ are real analytic functions of $E$, and for any fixed $E$ the Fredholm determinant can be written in the form[12]

$$D_\lambda(k) = \prod_i [1-\eta_i(k^2)]\,. \qquad (2.36)$$

If $D(k)$ has a zero at $k^2=0$, then one (or more) of the eigenvalue trajectories must pass through $\eta_i(0)=1$; to each $\eta_i(E)$ which satisfies this condition there corresponds a linearly independent solution to Eq. (2.10) for $E=0$. So to examine the order of the zero of $D_\lambda(k)$ at $k^2=0$ we need only examine the behavior of the eigenvalue trajectories near $E=0$.

If the solution $\psi(r,E)=\langle r|\psi_i(E)\rangle$ to Eq. (2.35) with $\eta(E)=\eta_i(E)$ is square integrable, then applying the Schrödinger operator $(\mathcal{H}_u - E)$ to Eq. (2.35),

$$\langle\psi_i(E)|[(\mathcal{H}_u-E)\eta_i(E)+\mathcal{V}]|\psi_i(E)\rangle = 0\,, \qquad (2.37)$$

and differentiating with respect to $E$ we obtain

$$0 = \frac{d}{dE}\langle\psi_i(E)|[(\mathcal{H}_u-E)\eta_i(E)+\mathcal{V}]|\psi_i(e)\rangle = \left\langle\psi_i(E)\left|\frac{d}{dE}[(\mathcal{H}_u-E)\eta_i(E)]\right|\psi_i(E)\right\rangle \qquad (2.38\text{a})$$

or

$$\frac{1}{\eta_i(E)}\frac{d\eta_i(E)}{dE} = \frac{\langle\psi_i(E)|\psi_i(E)\rangle}{\langle\psi_i(E)|(\mathcal{H}_u-E)|\psi_i(E)\rangle}\,. \qquad (2.38\text{b})$$

Evaluating this at $E=0$ and using the positivity of $\mathcal{H}_u$ we have

$$\eta_i'(0) \equiv \left.\frac{d\eta_i(E)}{dE}\right|_{E=0} = \frac{\langle\psi_i(0)|\psi_i(0)\rangle}{\langle\psi_i(0)|\mathcal{H}_u|\psi_i(0)\rangle} > 0\,.$$

If $\eta_i(0)=1$, then

$$\eta_i(E) \approx 1 + \eta_i'(0)E$$

for small $E$, and $D_\lambda(k)$ has a double pole at $k^2=0$. To each square-integrable zero-energy solution to Eq. (2.10) corresponds a double pole of $D_\lambda(k)$ at $k^2=0$.

On the other hand, if there is an $s$-wave solution to Eq. (2.10) which is not square integrable, then we obtain from Eq. (2.35) the relation

$$\eta_i(E) = \frac{\langle\psi_i(E)|\mathcal{V}(E-\mathcal{H}_u)^{-1}\mathcal{V}|\psi_i(E)\rangle}{\langle\psi_i(E)|\mathcal{V}|\psi_i(E)\rangle}\,;$$

evaluating the imaginary part of the right-hand side explicitly for small real $E>0$ we have

$$\eta_i(E) \approx 1 + ik\,\frac{|\int_0^\infty r\,dr\,V(r)\psi_i^{(1)}(r,E)|^2}{\langle\psi_i(E)|\mathcal{V}|\psi_i(E)} + O(E)\,.$$

Thus in this case $D_\lambda(k)$ has a simple zero at $k=0$.

We summarize: At every energy $k^2=E_n$ for which the Schrödinger equation (2.3) has a normalizable bound-state solution the Fredholm determinant $D_\lambda(k)$ for that partial wave has a zero. For $E_i \neq 0$, the order of the zero is equal to the multiplicity of the bound state. For $E_i = 0$ there is a zero of order 2 for each bound state, and for every non-normalizable $s$-wave solution to the homogeneous integral equation (2.10) the $D_\lambda(k)$ for the $s$ wave has a simple zero at $k^2 = 0$.

## III. THE SCATTERING MATRIX AND LEVINSON'S THEOREM

Here we derive the standard relation between the partial-wave $S$ matrix and the Fredholm determinant and then use this relation and the analyticity properties of $D_\lambda(k)$ to establish a generalization of Levinson's theorem.

The scattering amplitude can be deduced by simply finding the asymptotic behavior of the wave function in Eq. (2.4). First we separate $\psi_l$ into $\psi_l^0$ and the scattered wave $\psi_l^{sc}$ in the standard way:

$$\psi_l = \psi_l^0 + \psi_l^{sc}. \tag{3.1}$$

The asymptotic behavior of the scattered wave which follows from Eqs. (2.4) and (2.7a) is

$$[\psi_l^{sc}(r)]_2 \underset{r\to\infty}{\sim} -\frac{1}{r}\exp(ikr)\int_0^\infty r'^2\psi_l^0(r')\mathcal{V}(r')\psi_l(r') + O(r^{-3/2-\delta}). \tag{3.2}$$

The on-shell scattering amplitude is simply

$$\begin{aligned} T(k) &= -\int_0^\infty r'^2 dr'\psi_l^0(r')\mathcal{V}(r')\psi_l(r') \\ &= -\langle\psi_l^0|\mathcal{V}|\psi_l\rangle, \end{aligned} \tag{3.3}$$

where we recall that $\psi_l^0 = (0, j_l(kr))$. Using Eqs. (2.21) and (2.22) with $\Phi = r\mathcal{V}_2\psi_l$ and $\Phi_0 = r\mathcal{V}_2\psi_l^0$ we can write Eq. (3.3) as

$$T(k) = -\langle\psi_l^0|\mathcal{V}_1 N_\lambda \mathcal{V}_2|\psi_l^0\rangle(D_\lambda(k))^{-1}. \tag{3.4}$$

Then, with the usual relation between the $S$-matrix element and the $T$-matrix element,

$$S(k) = 1 + 2ikT(k), \tag{3.5}$$

we arrive at the expression for the partial-wave $S$ matrices:

$$S_l(k) = 1 - 2ik\langle\psi_l^0|\mathcal{V}_1 N_\lambda \mathcal{V}_2|\psi_l^0\rangle(D_\lambda^+(k))^{-1}, \tag{3.6}$$

where we use a + superscript to remind the reader that the scattering Green's function which enters $\mathcal{G}_u$ in $D_\lambda^+(k) = \det(1 - \lambda\mathcal{V}_2\mathcal{G}_u^+\mathcal{V}_1)$ has the customary $+i\epsilon$ boundary condition.

To relate the numerator on the right-hand side of Eq. (3.6) directly to the Fredholm determinant, we evaluate $D_\lambda^-(k) = \det(1 - \lambda\mathcal{V}_2\mathcal{G}_u^-\mathcal{V}_1)$, with the $-i\epsilon$ boundary condition used for the open-channel Green's function:

$$\begin{aligned} D_\lambda^-(k) &= \det(I - \lambda\mathcal{V}_2\mathcal{G}_u^-(E)\mathcal{V}_1) \\ &= \det(I - \lambda\mathcal{V}_2(\mathcal{G}_u^+ - [\mathcal{G}_u])\mathcal{V}_1) \\ &= \det(I - \lambda\mathcal{V}_2\mathcal{G}_u^+\mathcal{V}_1) \\ &\quad\times\{I + (I - \lambda\mathcal{V}_2\mathcal{G}_u^+\mathcal{V}_1)^{-1}\mathcal{V}_2[\mathcal{G}_u]\mathcal{V}_1\} \\ &= D_\lambda^+(k)\det(I + (I - \lambda\mathcal{V}_2\mathcal{G}_u^+\mathcal{V}_1)^{-1}\mathcal{V}_2[\mathcal{G}_u]\mathcal{V}_1). \end{aligned} \tag{3.7}$$

Here the discontinuity of the unperturbed Green's function

$$\begin{aligned} [\mathcal{G}_u] &\equiv \mathcal{G}_u^+(E) - \mathcal{G}_u^-(E) \\ &= -2ik|\psi_l^0\rangle\langle\psi_l^0| \end{aligned}$$

is a separable kernel or dyadic, and so the last factor on the right-hand side of Eq. (3.7) can be evaluated using the identity (2.28):

$$D_\lambda^-(k) = D_\lambda^+(k)[1 - 2ik\langle\psi_l^0|\mathcal{V}_2 N_\lambda \mathcal{V}_1|\psi_l^0\rangle(D_\lambda^+(k))^{-1}]. \tag{3.8}$$

The series expansions (2.23) for the determinants $D_\lambda^+(k)$ and $D_\lambda^-(k)$ converge uniformly in the domains $\text{Im}k > -\alpha_0/2$ and $\text{Im}k < \alpha_0/2$, respectively; thus the formal manipulations leading to Eq. (3.8) are justified.

Comparing Eqs. (3.6) and (3.8) we see that

$$S_l(k) = D_\lambda^-(k)/D_\lambda^+(k). \tag{3.9}$$

While this is a well-known standard result for ordinary potential scattering, we have derived it here directly from the Hilbert-Schmidt theory.[13] Even though the precise form of $D_\lambda(k)$ depends on the particular separation of the Hamiltonian $\mathcal{H}$ into $\mathcal{H}_u$ and $\mathcal{V}$ in Eq. (2.4), the $S$ matrix is unique; and the (infinite number of) poles of $D_\lambda(k)$ at the spectrum of the particular $H_c$, while inevitable, do not effect the unitarity of the $S$ matrix since the same poles occur in $D_\lambda^-(k)$ and $D_\lambda^+(k)$.

Although the analysis presented here was carried out for the case of only one scattering channel, it is clear that the method can be extended to any finite number of scattering channels. In particular, Eq. (3.9) then takes the form

$$\det S_l = D_\lambda^-(k)/D_\lambda^+(k), \tag{3.10}$$

and one can calculate all elements of the partial-wave $S$ matrix from the Fredholm determinant as a function of the independent channel momenta. The results have the same form as in ordinary multichannel potential scattering.[1]

It is obvious that Eq. (3.9) embodies the unitarity of the $S$ matrix, which is generally satisfied for ordinary potential scattering, and was established for this particular class of models in Ref. 4. Since $D_\lambda(k)$ is a real analytic function of $E = k^2$ and satisfies

$$D_\lambda^-(k) = (D_\lambda^+(k))^* \tag{3.11}$$

by virtue of Eq. (3.12), we can represent the $S$ matrix in terms of the phase

$$\eta(k) = -\arg D_\lambda^+(k) \tag{3.12}$$

or

$$\begin{aligned} S_l(k) &= D_\lambda^-(k)/D_\lambda^+(k) \\ &= (D_\lambda^+(k))^*/D_\lambda^+(k) \\ &= \exp[2i\eta(k)] \end{aligned} \tag{3.13a}$$

with the symmetry property

$$D_\lambda^+(-k) = D_\lambda^-(k), \quad k \text{ real}. \tag{3.13b}$$

The discontinuity of $\ln D_\lambda(k)$ across the cut in the energy plane is given by

$$\begin{aligned} [\ln D_\lambda(k)] &\equiv \lim_{\epsilon\to 0}[\ln D_\lambda(k+i\epsilon) - \ln D_\lambda(-k-i\epsilon)] \\ &= \ln D_\lambda^+(k) - \ln D_\lambda^-(k) = -2i\eta(k). \end{aligned} \tag{3.14}$$

The phase, $\eta(k)$, is determined by Eqs. (3.13a) modulo $\pi$. Since $D(k)$ has poles at $k^2 = E_n \in \sigma(H_c)$ and zeros at the bound states imbedded in the continuum $k^2 = E_i \geq 0$, the phase $\eta(k)$ has discontinuities

$$\lim_{\epsilon\to 0}[\eta((E_i)^{1/2}+\epsilon) - \eta((E_i)^{1/2}-\epsilon)] = \pi$$

and

$$\lim_{\epsilon\to 0}[\eta((E_n)^{1/2}+\epsilon) - \eta((E_n)^{1/2}-\epsilon)] = \pi .$$

We must define the physical phase shift $\delta_l(k)$, with the discontinuities removed, by

$$\delta_l(0) = \eta(0),$$

$$\begin{aligned} \delta_l(k) = \eta(k) &- \sum_{E_n<E} \lim_{\epsilon\to 0}[\eta((E_n)^{1/2}+\epsilon) - \eta((E_n)^{1/2}-\epsilon)] \\ &- \sum_{E_i<E} \lim_{\epsilon\to 0}[\eta((E_i)^{1/2}+\epsilon) - \eta((E_i)^{1/2}-\epsilon)] . \end{aligned} \tag{3.15}$$

It is this phase shift that will actually be important in the discussion of the Levinson theorem. Of course, if a continuum bound state occurs at an eigenvalue of $H_c$, then the corresponding discontinuities will cancel, but to count states properly we must include these terms in Eq. (3.15) anyway.

We now have all of the ingredients for the proof of the Levinson theorem except the crucial asymptotic behavior of $D_\lambda(k)$. In order to establish that $D(k) \to 1$ for $|k| \to \infty$ with $\operatorname{Im} k \geq 0$ and $E = k^2 \notin \sigma(H_c)$ it is sufficient to have

$$0 \leq \tau < O(|k|^{-\alpha}), \quad \alpha > 0 \tag{3.16}$$

in all directions in the half plane $\operatorname{Im} k > 0$ and on sequences of points $S_\epsilon$ for real $E = k^2$ that avoid the poles of the confined-channel Green's function. Such sequences are defined by

$$S_\epsilon = \{E \,|\, |E_n - E| > \epsilon > 0 \text{ for all } n\} \tag{3.17}$$

for any fixed $\epsilon > 0$ such that $2\epsilon < \Delta = \min_{n>n_0} |E_{n+1} - E_n|$ for some finite $n_0$. Then it follows from Eqs. (2.23) and (2.25) that

$$0 \leq |D(k) - 1| < O(|k|^{-\alpha}), \quad \alpha > 0 \tag{3.18}$$

for $|k| \to \infty$ in all complex directions and on the sequences $S_\epsilon$. In Eq. (3.18) we have retained the first nontrivial term ($n=2$) in the expansion of $D_\lambda(k)$, Eq. (2.23). We have checked Eq. (3.16) in detail for the harmonic-oscillator confining potential in Appendix B, and it can be easily established for the infinite square well.

There will be no real sequences $S_\epsilon$ on which Eq. (3.18) is satisfied unless the level spacing is lower semibounded, $|E_{n+1} - E_n| > \Delta$, for large $n$. This actually places a requirement on the confinement potential, namely that $E_n$ must increase at least as fast as $n$ for large $n$. The harmonic-oscillator potential is the borderline case. For a potential increasing as $r^p$ as $r \to \infty$, one can show that $E_n = O(n^{2p/(p+2)})$ as $n \to \infty$, by a simple WKB argument. Thus $p < 2$ (and hence the linear potential in nonrelativistic quantum mechanics) is excluded.

To establish the Levinson theorem, we use Eqs. (3.15), (3.16), and (3.18) and consider the Cauchy integral of $(d/dk)\ln D(k)$ over a contour $C$ in the upper half plane,

$$-m_{E<0} + \frac{1}{2\pi i}\int_C d\ln D_\lambda(k) = 0 . \tag{3.19}$$

The contour $C$ is from $-E_N - \epsilon$ to $E_N + \epsilon$ ($E_N$ is a large eigenvalue of $H_c$), avoiding the poles of $D_\lambda$, zeros of $D_\lambda$, and the origin by small semicircles in the upper half $k$ plane; the contour is closed by a semicircle of radius $R$ large enough to include all $m_{E<0}$ of the zeros of $D_\lambda(k)$ below threshold.

Using the symmetry property (3.13b) we can write out Eq. (3.19) as

$$\begin{aligned} \frac{1}{2i\pi}\Bigg\{ &-2i\eta((E_N)^{1/2}+\epsilon) + 2i\sum_{E_n<E_N}[\eta((E_n)^{1/2}+\epsilon) - \eta((E_n)^{1/2}-\epsilon) + \pi] \\ &+ 2i\sum_{E_i<E_N}\eta((E_i)^{1/2}+\epsilon) - \eta((E_i)^{1/2}-\epsilon) - \pi) + 2i\eta(0) + \int_R dk\,\frac{d}{dk}\ln D(k)\Bigg\} = n_{E<0} . \end{aligned} \tag{3.20}$$

From Eq. (3.14) and the asymptotic behavior (3.18) we obtain the relation for the physical phase shift

$$\delta_l(0) - [\delta_l((E_N)^{1/2} + \epsilon) - \pi n_{E_N+\epsilon} + \pi m_{E_N+\epsilon}] + O(E_N^{-\alpha/2}) = \pi m_{E<0}\,, \quad (3.21)$$

where $n_{E_n+\epsilon}$ and $m_{E_N+\epsilon}$ are, respectively, the numbers of poles and zeros of $D_\lambda(k)$, counted according to their multiplicities, between threshold and $E_{N+\epsilon}$. As in ordinary potential scattering, if there is a non-normalizable $s$-wave solution to the homogeneous integral equation, then this counts as half a bound state in Eq. (3.21) for $l=0$. If the number of positive energy bound states of $\mathcal{H}$, $m_\infty$, is finite then we obtain in the limit of large $E_N$

$$\delta_l(0) - \lim_{E_N\to\infty} [\delta((E_N)^{1/2} + \epsilon) - \pi n_{E_N+\epsilon}] = \pi m\,, \quad (3.22)$$

where $m = m_\infty + m_{E<0}$ is the total number of bound states of the complete Hamiltonian $\mathfrak{H}$. We have not been able to prove that $m_\infty$ is finite in general, although an example in the next section does have this property.

Equations (3.21) and (3.22) are the generalization of the Levinson theorem for this class of models. Although there have been some earlier statements of the Levinson theorem when hidden channels are present, namely within the context of the Low[14] equation and the Lee model, to our knowledge ours is the first systematic discussion within the context of confinement potentials.

It should come as no surprise that the number of poles of the confined-channel Hamiltonian $H_c$ plays a role in the final behavior of the phase shift. The physical reason lies in Jauch's[15] interpretation of the Levinson theorem as a statement of the equality of the total number of states for interacting and noninteracting particles. The difference between the phase shifts at $E=0$ and $E=\infty$ is just the difference between the dimensionalities of the spaces of scattering states for interacting and noninteracting particles. To correctly balance states in the generalized Levinson theorem we must also include the bound states of the confined-channel Hamiltonian $H_c$ with the free-particle states and include the bound states of the full Hamiltonian $\mathcal{H}$ with the interacting states. This is exactly what Eqs. (3.21) and (3.22) do.

Of course, the confining potential can never be regarded as a small perturbation relative to the kinetic energy operator, so one cannot define a phase shift relative to the free-particle state in both channels. For "noninteracting" particles we must always choose free particles in the scattering channel and confined particles in the confined channel. In this connection it would be very satisfying if one could prove our generalized Levinson theorem from the existence and completeness of the generalized wave operators, as has been suggested in other situations by Jauch[15] and Polkinghorne.[14]

## IV. RESONANCES AND BOUND STATES IN THE CONTINUUM

This section will be more heuristic than the preceding sections and will contain our concluding remarks. Some of the results quoted in this section will be based upon examples as opposed to general mathematical methods.

### A. Weak coupling and high-energy limit

By weak coupling, we simply mean that $\lambda\tau$ is a small parameter in the expansion in Eqs. (2.23) and (2.24). We then can expand the Fredholm determinant and keep the first nontrivial term in $\lambda\tau$:

$$\begin{aligned} D_\lambda(k) &\approx 1 - \frac{\lambda^2}{2}\sigma_2 \\ &= 1 - \lambda^2 \int_0^\infty dr \int_0^\infty dr' V(r) G_c(r,r') V(r') G_s(r',r) \\ &= 1 - \lambda^2 \sum_n \frac{1}{E-E_n} \int_0^\infty \frac{k^2 dk}{(2\pi)} \frac{|\langle nl|\mathcal{V}|\psi_l^0\rangle|^2}{E-k^2+i\epsilon}\,; \end{aligned} \quad (4.1)$$

the last step follows after use of the eigenfunction expansion (2.8) and the spectral representation for the scattering Green's function (2.7a). In Eq. (4.1) $|nl\rangle$ and $|\psi_l^0\rangle$ correspond to the unperturbed eigenfunctions for the confined channel and the scattering channel $\psi_{nl}$ (1, 0) and $j_l(kr)(0,1)$, respectively.

The approximation in Eq. (5.1) is valid for $\lambda\tau$ small, either for high energy, $E\in S_\epsilon$ of Eq. (3.17), or for weak coupling, or for both. If we retain the approximation even when $E$ is near a particular $E_n$, i.e., $|E-E_n| \ll |E-E_m|$, $E_m \neq E_n$, even though $\tau$ is not small we obtain

$$D_\lambda(k) \approx (E-E_n)^{-1}\Bigg(E - E_n - \lambda^2\, \mathrm{P}\int_0^\infty \frac{k^2 dk |\langle nl|\mathcal{V}|\psi_l^0\rangle|^2}{E-k^2} + i\pi\lambda^2 \int_0^\infty \frac{k^2 dk}{2\pi} |\langle nl|\mathcal{V}|\psi_l^0\rangle|^2 \times \delta(E-k^2)\Bigg)\,, \quad (4.2)$$

where P means the principal-value integral. While we have not justified the use of Eq. (4.2) in the neighborhood of $E_n$, we know from the discussion of the Levinson theorem that $D\to 1$ at high energy except in shrinking neighborhoods of $\sigma(H_c)$, where $D$ has simple poles. Thus we know that the physical phase shift increases by $\pi$ as we pass each $E_n$, indicating resonances of shrinking widths. We expect that the resonance is characterized by a

Breit-Wigner formula with position

$$E_{\text{res}} \approx E_n + \lambda^2 \,\mathrm{P}\int_0^\infty \frac{k^2 dk}{2\pi}\,\frac{|\langle nl|\mathcal{V}|\psi_l^0\rangle|^2}{E-k^2} \tag{4.3}$$

and width

$$\Gamma_{\text{res}} \approx \lambda^2 (E_n)^{1/2} |\langle nl|\mathcal{V}|\psi_l^0\rangle|^2 . \tag{4.4}$$

We now estimate the size of the widths at asymptotically large energy in terms of the behavior of the potential at the origin. For all partial waves we have

$$\langle nl|\mathcal{V}|\psi_l^0\rangle = \int_0^\infty r^2 dr\; \psi_{nl}(r) V(r) j_l(kr) . \tag{4.5}$$

Next we split the integral in two pieces:

$$\langle nl|\mathcal{V}|\psi_l^0\rangle = \int_0^R r^2 dr\; \psi_{nl}(r) V(r) j_l(kr) + \int_R^\infty r^2 dr\; \psi_{nl}(r) V(r) j_l(kr) , \tag{4.6}$$

where $R = O((1/k)^{1-\epsilon})$ at large $k$. In the second member of Eq. (4.6) we use the asymptotic behavior

$$j_l(x) \underset{x\to\infty}{\sim} \frac{1}{x}\sin\left(x-(l+1)\frac{\pi}{2}\right),$$

and the facts that $\psi_{nl}$ is bounded and $rV \in L_1$, to prove that

$$\int_R^\infty r^2 dr\; \psi_{nl}(r) V(r) j_l(kr) \leq \frac{1}{k}\int_0^\infty r\,dr\; V(r) \leq \frac{C}{k} . \tag{4.7a}$$

For the first term in Eq. (4.6), we use the bound for $j_l(kr)$ in Appendix A, $j_l(kr) \leq (1/r)C$ for $\mathrm{Im}k = 0$. Next, we assume that $r\,V(r) = O(r^{-\epsilon})$ for $0 < r < R$, to obtain

$$\int_0^R r^2 dr\; \psi_{nl}(r) V(r) j_l(kr) < k^{-(1-\epsilon)^2} . \tag{4.7b}$$

Putting all of this together we obtain

$$\Gamma_{\text{res}} = O(k^{1-2(1-\epsilon)^2}) \tag{4.8}$$

for large $k$; the width decreases for $\epsilon \lesssim 0.3$. Therefore the partial-wave scattering amplitude for this class of models consists of an infinite number of resonances with decreasing widths for reasonable off-diagonal potentials, i.e., $r\,V(r) = O(r^{-\epsilon})$ for $r\to 0$, $\epsilon \lesssim 0.3$.

### B. Bound states in the continuum

In this subsection we present an explicit example of a coupled-channel model of the type discussed in the previous sections which can be solved exactly. This example contains a bound state embedded in the continuum with no accumulation of such states at $+\infty$.

The example is simply an infinite square well for the confinement potential and a finite square-well off-diagonal perturbation. For $s$ waves we have in the notation of Eq. (2.1) ($U$ is the confinement potential), letting $\phi = r\psi$,

$$\left(-I\frac{d^2}{dr^2} + \sigma_1\theta(a-r)V + \tfrac{1}{2}(1+\sigma_3)U\right)\phi(r) = E\,\phi(r) \tag{4.9}$$

where $a < 1$. If we introduce the vector notation $\phi = (\phi_1, \phi_2)$ we obtain from Eq. (4.9) the following coupled equations in the various regions:

$$\left.\begin{aligned} &-\frac{d^2}{dr^2}\phi_1(r) + V\phi_2(r) = E\,\phi_1(r) ,\\ &-\frac{d^2}{dr^2}\phi_2(r) + V\phi_1(r) = E\,\phi_2(r) ,\end{aligned}\right\}\quad r \leq a \tag{4.10}$$

$$\begin{aligned} &-\frac{d^2}{dr^2}\phi_1(r) = E\,\phi_1(r) , \quad 1 \geq r \geq a\\ &-\frac{d^2}{dr^2}\phi_2(r) = E\,\phi_2(r) , \quad r > a\\ &\phi_1 = 0 , \quad r > 1\end{aligned} \tag{4.11}$$

for $U = 0$ $(r < 1)$, $U = \infty$ $(r \geq 1)$.

The solutions are straightforward and are given by

$$\left.\begin{aligned} &\phi_1(r) = A_+ \sin k_+ r + A_- \sin k_- r ,\\ &\phi_2(r) = A_+ \sin k_+ r - A_- \sin k_- r ,\end{aligned}\right\}\quad r \leq a \tag{4.12}$$

and

$$\begin{aligned} &\phi_1(r) = A\sin k(r-1), \quad 1 \geq r > a\\ &\phi_2(r) = Be^{ikr} + Ce^{-ikr} , \quad r > a\end{aligned} \tag{4.13}$$

where

$$k_\pm = (k^2 \mp V)^{1/2} ,$$

and we have used the physical boundary conditions at $r = 0$ and $r = \infty$.

Matching boundary conditions at $r = a$ yields the equations

$$\begin{aligned} &A_+ \sin k_+ a + A_- \sin k_- a = A\sin k(a-1) ,\\ &A_+ k_+ \cos k_+ a + A_- k_- \cos k_- a = kA\cos k(a-1) ,\end{aligned} \tag{4.14}$$

$$\begin{aligned} &A_+ \sin k_+ a - A_- \sin k_- a = Be^{ika} + Ce^{-ika} ,\\ &A_+ k_+ \cos k_+ a - A_- k_- \cos k_- a = ik(Be^{ika} - Ce^{-ika}) .\end{aligned} \tag{4.15}$$

Equations (4.4) and (4.15) are sufficient to determine the physical wave function up to one constant that can be fixed by an incident flux normalization for scattering states or by $\langle \Phi, \Phi \rangle = 1$ for bound states.

For any normalizable states degenerate with the continuum $E > 0$, we must have $B = C = 0$. Equations (4.14) and (4.15) immediately gives us the necessary and sufficient conditions:

$$k_+ \cot k_+ a = k_- \cot k_- a = k \cot k(a-1) . \tag{4.16}$$

Using the definitions of $k_\pm$ and $k$, we looked for solutions with $0 \leqslant E \leqslant V$ and $a \approx 0.5$. In particular, after rearrangement of $k_+ \cot k_+ a$ for $E \leqslant V$ we found that a solution of

$$(E+V)^{1/2} \cot(E+V)^{1/2} a = -(E)^{1/2} \cot(E)^{1/2}(1-a) = (V-E)^{1/2} \coth(V-E)^{1/2} a : \tag{4.17}$$

is given by $V = 44.5$, $E = 23$, and $a = 0.51$ in arbitrary units. The example is simple and is presented primarily to show that such states exist. The bound-state wave function is given by Eqs. (4.12), (4.13), (4.14), (4.15), and (4.16) with $B = C = 0$, and a remaining overall constant is obtained by normalizing the state. The wave function is confined inside the region $r < 1$, as are the unperturbed states of the confined potential, but for other confinement potentials we do not expect such bound states to vanish like the eigenstates of the unperturbed confinement potential.

We also have examined the solution of Eqs. (4.14) and (4.15) for the scattering states with the normalization $C = 1$, which is compatible with the integral equation (2.4). We quote here the result for $r > 1$ which is relevant to scattering:

$$\phi_2(r) = -e^{-ikr} + S e^{ikr} , \tag{4.18}$$

where $S$ is the $s$-wave scattering matrix and is given by

$$S = f(-k)/f(k) , \tag{4.19}$$

and $f(k)$ is given by the formula

$$f(k) = e^{ik}\left[-2ki + \frac{d}{dr}\ln[u_+(r,k)u_-(r,k)]\right]_{r=1} , \tag{4.20}$$

where $u_\pm(r,k)$ are the regular solutions to the single-channel Schrödinger equations with potentials $\pm V$. Equation (4.20) is actually very general and gives the exact $S$ matrix for any off-diagonal potential $V$ and the infinite square-well confinement potential. For the particular example of the finite square-well perturbation the solutions $u_\pm$ are given by the linear combinations $u_\pm = (\psi_1 \pm \psi_2)$, Eqs. (4.12). In general, since the Schrödinger equation can be diagonalized in the region $r \leq 1$ the relevant solutions are defined by the Volterra equations[1]:

$$u_\pm(k,r) = \frac{\sin kr}{k} \pm \int_0^r \frac{\sin k(r-r')V(r')}{k} u_\pm(k,r') . \tag{4.21}$$

The bound states in the continuum for this problem are formally very similar to those found in ordinary multichannel potential scattering some years ago.[6] In the usual multichannel potential scattering with different thresholds, the poles corresponding to the bound states in the continuum which occur between thresholds are contained in the analytic continuation of $S$-matrix elements below the higher threshold. This is not the case here, since there do not exist $S$-matrix elements in both channels. The reader is referred to the excellent text by Newton[1] for a general treatment of the bound states in the continuum and the associated resonance theory.

Finally we point out that in our example there is no accumulation of bound states at $+\infty$. For asymptotically large $E$ it is impossible to satisfy Eq. (4.16), since the leading terms in the expansion of $k_\pm$ are $k_\pm = k \mp V/2k$. For arbitrary $l$, $l > 0$, this is also true since the Bessel functions can be replaced by their respective asymptotic expansions and the same argument will apply.

### C. Discussion

In principle the Levinson theorem (3.22) differentiates between two types of resonances, namely "fundamental" resonances, which are related to the bound states of $H_c$, and "accidental" resonances (we are tempted to call these "hadron molecular states"), which are not. If we gradually turn off the off-diagonal coupling (i.e., take $\lambda \to 0$) then the fundamental resonances approach the positions of the bound states of $H_c$, and their widths shrink to zero; on the other hand the widths of the accidental resonances increase, and in the limit of zero coupling they disappear altogether.

More importantly from a physical point of view, each fundamental resonance makes a net contribution of $\pi$ to the asymptotic behavior of the partial-wave phase shift in which it occurs but the net contribution of each accidental resonance is zero. What we mean by this is the following: If the phase shift is set to a finite value at threshold, then as the energy is increased the physical partial-wave phase shift continuously increases by $\pi$ every time $E$ passes a pole of the confined-channel Hamiltonian, and there is a resonance of the type mentioned earlier in this section. At high energy the phase shift escalates very rapidly through an odd multiple of $\pi/2$ as $E$ passes each $E_n$, and the resonances decrease in width as they approach the locations of the eigenvalues of $H_c$. In the limit $E\to\infty$ the phases shift approaches $+\infty$. Near accidental resonances the phase shift also increases rapidly by $\pi$, but as the energy continues to increase the phase shift then gradually falls back down by the same amount. This is in accord with Jauch's interpretation[15] of the Levinson theorem as a statement of the equality of the total numbers of states for interacting and noninteracting particles.

If the resonances are widely separated relative to their widths, then this last point may give us an opportunity to distinguish phenomenologically between the two types of resonances. As an example the analysis of Dashen and Kane[16] shows that the deuteron should be considered an accidental state, not related to the fundamental quark-model states, since when the relevant $p$-$n$ phase shift is averaged over an energy range of a few hundred MeV around the deuteron the net contribution is zero. This is satisfying since everyone agrees that the deuteron is a composite state. On the other hand, as Dashen and Kane have pointed out, the major well-studied hadronic resonances, including the $\rho$, $\Delta$, etc., do produce net contributions to the relevant phase shifts and are correctly regarded as fundamental.

Another system for which our analysis might be relevant is the $\psi$-$J$ system. In models of the type we have discussed both accidental and fundamental resonances can be very narrow, and so one should be aware that the assignment of every state of the $\psi$-$J$ system to a quark-model classification, as in the charmonium scheme, is not the only possibility. Likewise, the apparent absence or gross displacement of a particular quark-model state should not necessarily be regarded as a failure for the quark model since even in the nonrelativistic regime the state could be radically altered by the complicated hadron dynamics.

Of course, at high energies one would expect to see only regularly spaced narrow fundamental resonances if models of the type we have described are relevant in that regime. However, at high energies inelastic channels, which we have neglected, become important; in any case we are under no illusion that a phase-shift analysis can really be carried out in the presence of increasing numbers of inelastic channels at high energy. Therefore we do not think it is meaningful to perform any detailed numerical calculations here.

There are several directions for future work on the problem of scattering theory with confined channels. It would be very interesting to investigate the analytic structure of the scattering amplitude in detail for such systems, and in particular to establish the validity or nonvalidity of dispersion relations. Also one could try to formulate the inverse scattering problem for these systems. And of course one could certainly enlarge the class of potentials $U$ and $V$ for which the results we have already established are valid. In this connection we recall that we have not dealt with the linear confinement potential, which seems to be a favorite candidate for the quark-antiquark binding potential. However, we wish to emphasize that one should expect the general properties of the nonrelativistic harmonic oscillator, not the linear potential, to carry over to the relativistic linear potential. The crucial issue is the growth of $E_n$ for large $n$. The linear potential in relativistic quantum mechanics actually corresponds (for large momenta) to a wave equation linear in both coordinates and momenta, with an energy spectrum satisfying $E_n = O(n)$ as $n\to\infty$, just as for the nonrelativistic harmonic oscillator. For the nonrelativistic linear potential, $E_n = O(n^{2/3})$ and the Levinson theorem we have derived does not even make sense.

On a more grandiose scale one would like to derive the confinement force and the finite-range forces between observable hadrons from local quantum field theory. There is some work in this direction from two-dimensional Yang-Mills theories,[17] from the Schwinger[18] model, and from lattice gauge theories.[19] Perhaps in some approximation our model could be obtained from such a theory.

*Note added in proof.* Let $D_\lambda^+(E)$ have a zero of order $p$ at $E=E_i$, and let $P_i$ be the projection onto the $m$-dimensional eigenspace of $\mathcal{H}$ at $E_i$. Then by using the integral equation for the Green's function it is easy to see that

$$\begin{aligned}p &= \lim_{E\to E_i}(E-E_i)\frac{d}{dE}\ln D_\lambda^+(E)\\ &= -\lim_{E\to E_i}(E-E_i)\,\mathrm{Tr}\left[(1-\mathcal{G}_u\mathcal{V})^{-1}\frac{d\mathcal{G}_u}{dE}\mathcal{V}\right]\\ &= \lim_{E\to E_i}(E-E_i)\,\mathrm{Tr}[\mathcal{G}\mathcal{G}_0\mathcal{V}]\\ &= \mathrm{Tr}[P_i\mathcal{G}_0\mathcal{V}]\\ &= \mathrm{Tr}P_i\\ &= m\,.\end{aligned}$$

For more details see Ref. 11.

After the completion of this work we received a report from C. Dullemond and E. Van Beveren [University at Nijmegen report (unpublished)] who treat essentially the same problem.

## ACKNOWLEDGMENTS

All three of the authors would like to thank Professor Tullio Regge for many discussions about potential scattering. Two of the authors (I.J.M. and J.B.H.) would like to thank Professor Carl Kaysen for hospitality at the Institute for Advanced Study. One of the authors (I.J.M.) would also like to thank Professor B. W. Lee for hospitality at Fermilab, where parts of this work were completed. We would also like to thank Professor R. G. Newton for a useful correspondence.


## APPENDIX A

In this appendix we establish the bounds on $G_s$ used in Sec. II and the Carleman property of the harmonic-oscillator Green's function.

The bound for the scattering Green's function is found in the text of De Alfaro and Regge[1]; we merely quote the results here. From Eq. (2.7a) and the bounds for Bessel functions,

$$|r j_l(kr)|\, e^{-|\mathrm{Im} k| r} < \left(\frac{C|k|r}{1+|k|r}\right)^{l+1}, \tag{A1}$$

$$|r h_l^+(kr)|\, e^{+(\mathrm{Im} k) r} < \left(\frac{C|k|r}{1+|k|r}\right)^{-l} \tag{A2}$$

we obtain

$$|G_s(r,r',k,l)| < Cr(r/r')^l\left(\frac{1+|k|r'}{1+|k|r}\right)^l \frac{1}{1+|k|r} < C\left(\frac{r}{1+|k|r}\right) \text{ for } \mathrm{Im}k \geq 0,$$

$$|G_s(r,r',k,l)| < C\left(\frac{r}{1+|k|r}\right) e^{|\mathrm{Im}k| r} \text{ for } \mathrm{Im}k < 0, \tag{A3}$$

for $r<r'$ and all physical $l$.

Next we establish the Carleman property of the partial-wave Green's function given the analogous property for the full Green's function. In our companion work[4] we establish the strong Carleman condition for the three-dimensional harmonic-oscillator potential. In the interest of economy we quote the result here:

$$\|G_c(\vec{x},\cdot,k^2)\|^2 \equiv \int d^3\vec{y}\,|G_c(\vec{x},\vec{y},k^2)|^2 = \sum_n \frac{|\Psi_n(\vec{x})|^2}{|k^2-E_n|^2} < C, \tag{A4}$$

where $\psi_n(\vec{x})$ is the full three-dimensional harmonic-oscillator eigenfunction.

It is clear that we can introduce a partial-wave analysis of the eigenfunctions for the spherically symmetric three-dimensional oscillator and obtain

$$\|G_c(x,\cdot,k^2)\|^2 = \sum_{nl} \frac{|\psi_{nl}(r)|^2(2l+1)}{|k^2-E_n|^2}, \tag{A5}$$

where $r=|\vec{x}|$ and

$$\psi_{nl}(r) = \{(2n+1)^{-1}[\tfrac{1}{2}(n-l-1)]![\Gamma(\tfrac{1}{2}(n+l+1))]^{-3}\}^{1/2} r^l e^{r^2/2} L^{l+1/2}_{(n-l-1)/2}(r^2),$$

for $0 \leq l \leq n-1$ and $1 \leq n \leq \infty$; $L^\alpha_\beta$ are Laguerre polynomials. No Legendre functions enter Eq. (A5) since both arguments of the product of the Green's function with its adjoint are the same. Identifying the sum over $n$ in Eq. (A5) with the norm, $\|G_c(r,\cdot,k^2,l)\|$, for the partial-wave Green's function, we have

$$\left(\frac{\|G_c(r,\cdot,k^2,l)\|}{r}\right)^2 = \sum_{n=l+1}^{\infty} \frac{|\psi_{nl}(r)|^2}{|k^2-E_n|^2} < C < \infty, \tag{A6}$$

where $C$ depends only on $E=k^2$ and is finite for $k^2 \notin \sigma(H_c)$.

## APPENDIX B

In this appendix we prove that for the harmonic-oscillator confining Hamiltonian

$$\tau \leq O(|E|^{-1/2+\epsilon}), \text{ with } \epsilon < \tfrac{1}{2} \text{ for } s \text{ waves} \tag{B1}$$

as $|k| \to \infty$ in all complex directions and on the real sequences $S_\epsilon$ of Eq. (3.17) which avoid the spectrum of the harmonic-oscillator Hamiltonian, $H_c$. First we write the Hilbert-Schmidt norm $\tau(KK^\dagger)=\tau^s+\tau^c$ as in Eq. (2.16). We carry out the procedure for $s$ waves explicitly. For the scattering part we have

$$\tau^s=\int_0^\infty dr\,|V_1(r)|^2\int_0^r |r'V_2(r')|^2\left|\frac{\sin kr' e^{ikr}}{r'k}\right| + \int_0^\infty dr\,|r\,V_2(r)|^2\int_0^r |V_1(r')|^2\left|\frac{\sin kr' e^{ikr}}{kr}\right|. \tag{B2}$$

Noticing that $|e^{ikr}|\le|e^{ikr'}|$ and $|e^{ikr}/r|\le|e^{ikr'}/r'|$ for all $r\ge r'$ and $\mathrm{Im}k\ge 0$, we have the bound

$$\tau^s\le\|V_1\|^2\int_0^\infty dr'\,|r'V_2(r')|^2\left|\frac{\sin kr' e^{ikr'}}{kr'}\right|^2 + \|rV_2\|^2\int_0^\infty dr'\,|V_1(r')|^2\left|\frac{\sin kr' e^{ikr'}}{kr'}\right|^2. \tag{B3}$$

Now divide up the integrals into pieces with $0\le r<R$ and $r>R$, and choose $R$ such that $rV_2=O(r^{-\epsilon})$, $\epsilon>\frac{1}{2}$, in that region, which we can do since $V$ has only a finite number of singularities. Then using

$$\left|\frac{\sin kr' e^{ikr'}}{kr'}\right|^2\le \mathrm{Min}(1,(kr')^{-2})$$

we obtain

$$\tau^s\le C_1\|V_1\|\int_0^R dr'\,r'^{-2\epsilon}\left|\frac{\sin kr' e^{ikr'}}{kr'}\right|^2 + C_2\|rV_2\|\int_0^R dr'\left|\frac{\sin kr' e^{ikr'}}{kr'}\right|^2 + \frac{2}{|kR|^2}\|rV_2\|\,\|V_1\|. \tag{B4}$$

After changing the variable of integration in the first two integrals to $y=r'k$, we obtain

$$\tau^s\le C_1\|V_1\|(k)^{-1+2\epsilon}+\frac{2}{|kR|^2}\|rV_2\|\,\|V_1\|+C_2\|rV_2\|k^{-1}\le C|E|^{-1/2+\epsilon}. \tag{B5}$$

Next we bound the Hilbert-Schmidt norm for the confined-channel kernel:

$$\tau^c=\int_0^\infty dr\,|V_1(r)|^2\int_0^\infty dr'\,|V_2(r')|^2|G_c(r,r',k^2)|^2. \tag{B6}$$

For positive real $E$ we split the integral into three pieces:

$$\begin{aligned}\tau^c=&\int_0^R dr\,|V_1(r)|^2\int_0^R dr'\,|r'V_2(r')|^2\left|\frac{G_c(r,r',k^2)}{r'}\right|^2 + \int_0^R dr\,|V_1(r)|^2\int_R^\infty dr'\,|V_2(r')|^2\left|\frac{G_c(r,r',k^2)}{r'}\right|^2\\ &+\int_R^\infty dr\,|V_1(r)|^2\int_0^\infty dr'\,|r'V_2(r')|^2\left|\frac{G_c(r,r',k^2)}{r'}\right|^2\\ \equiv&\,\tau_1^c+\tau_2^c+\tau_3^c.\end{aligned} \tag{B7}$$

The first term $\tau_1^c$ can be dealt with by using the following asymptotic behavior[20] for real $E\gg r,r'$:

$$|G_c(r,r',E)|\underset{E\to\infty}{\le}\left|\frac{\sin(2E)^{1/2}r'/(2E)^{1/2}}{\cos\pi E}\right|. \tag{B8}$$

The analysis proceeds in the same way as the scattering Green's function. Choosing $R=|E|^K$, $K<\frac{1}{2}$, we obtain

$$\tau_1^c\le\frac{O(E)^{-1/2+\epsilon}}{(\cos\pi E)^2},\quad \epsilon>\tfrac{1}{2} \tag{B9}$$

which decreases like $O(E^{-1/2+\epsilon})$ as long as $E=k^2\in S_\epsilon$ of Eq. (3.17).

For the second term, $\tau_2^c$, we use the finite-range nature of $V_2$ and the fact that $V_1$ is bounded to write

$$\tau_2^c\le Ce^{-\alpha_2 R}\int_R^\infty |r'V_2(r')e^{\alpha_2 r'}|^2\|G_c(r',\cdot,k^2)\|^2. \tag{B10}$$

For the third term, we obtain in a similar fashion

$$\tau_3^c\le Ce^{-\alpha_1 R}\int_0^\infty |r'V_2(r')|^2\|G_c(r',\cdot,k^2)\|^2 \tag{B11}$$

by using the fact that we can choose $V_1$ such that $V_1e^{\alpha_1 R}$ is bounded.

Next we use the eigenfunction expansion for $\|G_c(r',\cdot,k^2)\|^2$ and the bound for $s$-wave eigenfunctions

$|\psi_k(r')|^2 < Cn^{1/2}$ (see the article of Schwid[20]) to establish

$$\|G_c(r', \cdot, k^2)\|^2 \le C \sum_{n=0}^{\infty} \frac{n^{1/2}}{|E-E_n|^2} \le \operatorname{Sup}_{n\ge 0}\left|\frac{E_n}{E-E_n}\right| \sum_{n=0}^{\infty} \frac{n^{1/2}}{E_n} < C \operatorname{Sup}\left|\frac{E_n}{E-E_n}\right| . \tag{B12}$$

For the harmonic oscillator $E_n = (n+\frac{1}{2})\hbar\omega_c$ and the series in Eq. (B11) for $C$ certainly converges. We have a large margin because of the exponential factors in Eqs. (B9) and (B10). In particular, $\operatorname{Sup}|E_n/(E-E_n)| < E/\epsilon$ on the sequences $S_\epsilon$ which avoid the eigenvalues of $H_c$ at $E_n$. Therefore (we summarize), on the sequences $S_\epsilon$ we have

$$\tau_2^c \le Ce^{-\alpha_2 R}\|r' e^{\alpha_2 r'} V_2\| E < O(Ee^{-\alpha_2 E^K}) , \tag{B13a}$$

$$\tau_3^c \le Ce^{-\alpha_1 R}\|r' V_2\| E < O(Ee^{-\alpha_1 E^K}) , \tag{B13b}$$

for $R = |E|^K$ with $0 < K < \frac{1}{2}$.

Next we return to Eq. (B6) for complex $E$; we proceed exactly as in the way we treated $\tau_2^c$ and $\tau_3^c$. We use the fact that $V_1(r)$ is bounded and $rV \in L_2$ to obtain

$$\tau^c \le C \int_0^\infty dr' \|G_c(r', \cdot, k^2)\|^2 |r' V_2(r')|^2 \le C \|r' V_2\|^2 \sum_{n=0}^{\infty} \frac{n^{1/2}}{|E-E_n|^2} = Cf(E) . \tag{B14}$$

For $\arg E \ne 0$, $f(E) = O(|E|^{-1/2})$, which can be seen by relating $f(E)$ to the generalized Riemann $\zeta$ functions. Thus we have

$$0 \le \tau^c \le O(|E|^{-1/2+\epsilon}), \quad \epsilon < \tfrac{1}{2} \tag{B15}$$

for $|k| \to \infty$ in all complex directions and on the sequences $S_\epsilon$, and this completes the proof of Eq. (B1) for $s$ waves. For higher partial waves and for the infinite square well the analysis proceeds in essentially the same way. For the infinite square well $E_n = O(n^2)$ for large $n$ and there is a larger margin of convergence than for the harmonic-oscillator case.

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*Research sponsored in part by ERDA under Grant No. E(11-1-2220).

†Research sponsored in part by ERDA under Contract No. E(11-1)-3075.

‡Work supported in part by ERDA.

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