Collective model of the hadrons*

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We describe a model of the baryons and mesons as trilocal and bilocal collective excitations of a self-consistent ground state. The fundamental theory we adopt (in analogy with the BCS Hamiltonian) is quantum chromodynamics (QCD). The hadrons are flavored excitations of a color-singlet vacuum condensate of $\bar{q}q$ pairs. Our approach is completely conventional, fully relativistic, and incorporates both a γ_5-noninvariant vacuum [partially conserved axial-vector current (PCAC)] and quark confinement. We derive the Gor'kov or gap excitation equations for mesons from QCD. These Gor'kov equations provide the connection between QCD and the phenomenological theory of the hadrons. We also discuss the solutions to the gap equation and the gap excitation equation for confined quarks. It is noted that for an infrared-singular gauge field propagator the exact gap equation becomes a differential equation for the quark propagator. These equations are solved analytically and have the property of PCAC and confinement in the infrared limit. Qualitatively, the same features are found as in two-dimensional QCD. The Gor'kov equations for the bound states are obtained, and the ground-state meson mass spectrum is computed. Potentially, this approach can provide for the determination of the fundamental features of hadron phenomenology.

I. INTRODUCTION

In this article we will describe a model of all the hadrons as collective states of permanently bound quarks. Our approach will be to integrate several of the major ideas about the structure of hadrons.

The first idea is the remarkably successful quark model[1,2] in which hadrons are states of quarks and antiquarks bound in a potential well. This nonrelativistic atomic picture of hadrons, originally developed in the mid 1960's, has recently reemerged as the bag or sack[3] model of hadrons, which takes the principle of relativity and quark confinement into account.

The second idea about hadron structure we will incorporate has its origin in the Nambu–Jona-Lasinio model of the pion.[4] This model has as its inspiration the BCS theory of superconductivity.[5] The bound-state pion is a collective excitation, a Goldstone boson obeying partially conserved axial-vector current (PCAC) as a consequence of spontaneous breaking of γ_5 invariance of the vacuum. Like the quark model, PCAC has its dramatic experimental successes, although the underlying chiral symmetry is more difficult to test.[6] It would appear that any theory of the hadrons that does not incorporate PCAC is necessarily incomplete. The inclusion of both PCAC and the quark model into a model of the hadrons will be an insistence of our approach. In placing PCAC in a fundamental place we deviate from the current formulations of the quark model of hadrons (as a bag or sack) which fail to give a correct description of the pion.

A third insistence will be the incorporation of the assumption of quark confinement. No attempt will be made to prove quark confinement; quark confinement has never been proven in the context of a relativistic $(3+1)$-dimensional field theory. However, the spectrum indicates that hadrons are indeed built up out of quarks with the rule meson $\sim\bar{q}q$, baryon$\sim qqq$, as suggested by Gell-Mann and Zweig.[1] The further evidence for *confined* quarks is the experimental absence of quarks, linearly rising Regge trajectories, and pointlike scaling behavior of weak processes at high momentum transfer.

Remarkably, these three assumptions, the quark model, PCAC, and confinement, as applied to quantum chromodynamics (QCD) lead in a rather definite fashion to a picture of hadrons as collective excitations in the gap of a type-II superconductor. The vacuum is a color-singlet condensate with the hadrons as flavored excitations of the condensate. The hadrons have the quantum numbers of the quark model and the pion obeys PCAC. Further, the interactions of all the gap excitations (hadrons) with one another are completely specified as solutions to the Gor'kov equations for QCD.

The paradigm for our model is the BCS theory of superconductivity and the application of these ideas to particle physics by Nambu and Jona-Lasinio. This is in contrast with models for which the quarks are confined in a bag and for which PCAC must be put in by hand. The Nambu–Jona-Lasinio model has been developed as a renormalizable field theory so that the bound-state Goldstone phenomena is finite and cutoff independent.[7] We also draw on a recent extension of the Nambu–Jona-Lasinio model by Eguchi and Sugawara[8,9] who derived the Gor'kov equations[10] appropriate for this model. The Gor'kov equations essentially transcribe the microscopic theory (BCS or Nambu–Jona-Lasinio) into a phenomenological theory (London or Σ model).

An earlier version of some of these ideas is in a paper by D. G. Caldi and myself,[11] which integrates PCAC into the quark model as a solution to the ρ-π puzzle. There it was proposed that the ρ meson is also a collective excitation, a dormant Goldstone boson transforming as the same representation as the pion under the chiral group. Then the quark model which places the π and ρ in the same SU(6) multiplet and PCAC which requires the π to be a Goldstone state are not in conflict.

The fundamental "microscopic" theory we will use is QCD. The Lagrangian is

$$\mathcal{L}_{\rm QCD} = \mathcal{L}_{\rm YMS} + i\bar{q}\,\not{D} q\,,$$

with $\mathcal{L}_{\rm YMS}$ as the Lagrangian for a colored $SU_c(3)$ Yang-Mills-Shaw[12] gauge field theory and q as the quark field transforming as $\underline{3}$ under local $SU_c(3)$ and as the $(\underline{n},\underline{1})\times(\underline{1},\underline{n})$ representation of a global chiral $SU(n)\times SU(n)$ of flavor. The $SU_c(3)$ local symmetry is presumed exact, while the chiral $SU(n)\times SU(n)$ global symmetry may be explicitly broken by amending to $\mathcal{L}_{\rm QCD}$ a quark mass matrix. We will not consider such explicit breaking here.

The advantages of the model have been described in the literature.[13] There is no evident conflict with experiment [a possible exception is the $U_A(1)$ problem[14]] providing the spectrum of $\mathcal{L}_{\rm QCD}$ consists only of color singlets (confinement) and the chiral γ_5 symmetry is broken in the ground state (PCAC). A remarkable feature of this theory is that all the fundamental fields that appear in $\mathcal{L}_{\rm QCD}$ are to be confined and unobservable. It would be ironic if QCD were correct inasmuch as quantum mechanics was founded with the insistence that only operationally defined observables have meaning. From the present point of view, strong interactions are an epiphenomena of unobservable color dynamics.

QCD will be our "microscopic" theory like the BCS theory. The practical problem we address is the extraction of the phenomenological content of this theory. QCD can determine the equations of motion for phenomenological fields corresponding to the observed hadrons. This would be the "macroscopic" theory. These effective field equations are called the Gor'kov equations[10] in analogy with the same development in superconductivity. This development seems required if one is to make a contact between experiment and QCD. So we endeavor to obtain the Gor'kov equations for QCD.

Our derivation will be conventional, and it will be consistent with our primary assumptions. We will discuss only the mesons $\sim\bar{q}q$ and not the baryons $\sim qqq$ as a first step. The mesons are described completely in terms of the meson gap excitation function, a bilocal structure with the Dirac decomposition

$$\Sigma^*(x,y) = \sigma(x,y) I + \pi(x,y) i\gamma_5 + V_\mu(x,y)\gamma_\mu + A_\mu(x,y) i\gamma_5\gamma_\mu + T_{\mu\nu}(x,y)\sigma_{\mu\nu}\,.$$

Here x and y are quark and antiquark coordinates. Σ^* is a matrix in flavor space. Baryons would be described by a trilocal structure, the baryon gap excitation function, $\Sigma^*(x,y,z)$. The Fourier transform of $\Sigma^*(x,y)$ with respect to $x+y$ specifies the bound-state meson wave function $\Sigma^*(P,x-y)$. The equations of motion for $\Sigma^*(P,x-y)$, the Gor'kov equations, will be derived below.

The equations of motion for Σ^* will involve the quark propagator $S(p)$ and the color gauge field propagator $d(q^2)$. The quark propagator $S(p)$ is a γ_5-noninvariant solution to the gap equation, which is specified once $d(q^2)$ is given. So, given the single function $d(q^2)$, the motion is determined. The gauge propagation function $d(q^2)$ is in principle determined by complicated color dynamics. No attempt will be made to determine it; rather we treat it like a "potential" in Schrödinger theory and adopt a pragmatic attitude.

A knowledge of the fermion propagator $S(p)$ is important for this approach, and in the next section we will describe solutions to the gap equation for confining "potentials" $d(q^2)$. In the following section the meson Gor'kov equations are obtained.

II. THE GAP EQUATION FOR CONFINED FERMIONS

A. The gap equation

Before describing the meson gap excitation equation we will discuss the usual gap equation. The renormalized quark propagator is

$$S^{-1}(p) = \not{p} - \Sigma(p) = A(p^2)\not{p} - B(p^2)\,, \tag{1}$$

where $\Sigma(p)$ is the gap function. For QCD this obeys the Schwinger-Dyson or gap equation

$$\Sigma(p) = i\int \frac{d^4q}{(2\pi)^4}\Gamma^a_\alpha(p,p-q)\,S(p-q)\gamma_\beta \tfrac{1}{2}\lambda^b D^{ab}_{\alpha\beta}(q)\,. \tag{2}$$

Here $\Gamma^a_\alpha = \frac{1}{2}\lambda^a\Gamma_\alpha$ is the proper vertex function and satisfies a Ward identity:

$$c\Gamma_\beta(p,p) = \left.\frac{\partial S^{-1}(p+q)}{\partial q_\beta}\right|_{q=0} + \text{ghost field terms}\,, \tag{3}$$

where c is a constant. The quark propagator is gauge dependent, and it is convenient to work in the Landau gauge. In this case the gauge field propagator can be written as

$$D^{ab}_{\alpha\beta}(q) = \left(-g_{\alpha\beta} + \frac{q_\alpha q_\beta}{q^2}\right)\frac{\delta^{ab} d(q^2)}{q^2}\,,$$

so that

$$\Sigma(p) = \frac{4i}{3}\int \frac{d^4q}{(2\pi)^4}\frac{d(q^2)}{q^2}\left(-g_{\alpha\beta} + \frac{q_\alpha q_\beta}{q^2}\right) \times \Gamma_\beta(p, p-q) S(p-q)\gamma_\alpha\,, \tag{4}$$

where the gauge-coupling-constant dependence has been lumped into $d(q^2)$.

A few remarks are in order. First, the asymptotic solution to (4) is known;

$$S^{-1}(p) \underset{p^2\to-\infty}{\sim} \not p\,, \tag{5}$$

on account of the asymptotic freedom of QCD.[15] This asymptotic region is really not relevant to our problem for low-lying hadron phenomenology. Second, it is possible that the gap equation and the relevant Green's functions may not exist, on account of infrared divergences of QCD. This, in fact, is what occurs for two-dimensional QCD.[16] In the presence of infrared singularities we assume that the gap equation can be regulated. An example of this is given below.

We will assume that quarks are confined. This means that $S(p)$ does not have a pole. The perturbative solution to (4) does not have this property, so one must go beyond perturbation theory What confinement suggests is that $d(q^2)$ is singular as $q^2\to 0$. This would imply that the low-q^2 region of the integral (4) is important. We will assume that it is the low-q_μ region of (4) that is important. This renders the problem tractable since in the integrand we may then approximate

$$c\Gamma_\beta(p, p-q) \simeq \Gamma_\beta(p,p) = [A'(p^2)\not p - B'(p^2)]\,2p_\beta + A(p^2)\gamma_\beta\,, \tag{6}$$

where the prime denotes differentiation with respect to p^2. This is consistent with the Ward identity (3) providing the ghost field terms decouple at $q_\mu = 0$. Then the gap equation is

$$\Sigma(p) = \frac{4i}{3c}\int \frac{d^4q}{(2\pi)^4}\frac{d(q^2)}{q^2}\left(-g_{\alpha\beta} + \frac{q_\alpha q_\beta}{q^2}\right) \times\{A(p^2)\gamma_\alpha + [A'(p^2)\not p - B'(p^2)]\,2p_\alpha\} \times S(p-q)\gamma_\beta\,, \tag{7}$$

a nonlinear integro-differential equation for $S(p)$.

The gauge field propagator $d(q^2)$ also obeys integral equations.[17] These integral equations involve the proper vertices of gauge field self-couplings and ghost-field–gauge-field coupling. In principle, $d(q^2)$ is determined from this complex color dynamics of QCD; in practical terms no progress has been made to determine $d(q^2)$. Consequently we adopt the pragmatic attitude that $d(q^2)$ is to be specified in conformity with fits to hadron phenomenology and quark confinement. In or approach it plays the role of a potential with the gap equation (7) as the Schrödinger equation for QCD. The solution of (7) $S(p^2, d(q^2))$, it turns out, is all we require for the Gor'kov equations.

B. An example of confined fermion solutions to the gap equations

It is illuminating to construct examples of confined solutions to the gap equation. For quark confinement $d(q^2)$ must be singular at $q^2 = 0$, as emphasized by Johnson.[18] [$d(q^2) = 1$ corresponds to a Coulomb force law.] If we have a linearly rising potential in configuration space this corresponds to

$$d(q^2) = \lim_{\epsilon\to 0}\left(\frac{\mu^2}{q^2}\right)^{1-\epsilon} = \frac{\mu^2}{q^2}\,, \tag{8}$$

where ϵ, an infrared regulator, is to be set to zero at the end of the calculation and μ is a mass characterizing the confinement. This appears to be a physically reasonable example. Remarkably, the complete gap equation can be solved analytically in this case. No assumption on the vertex $\Gamma_\beta(p, p-q)$ need be made since in the infrared limit $\epsilon\to 0$ only $\Gamma_\beta(p,p)$ will contribute, and this is specified by the Ward identity.

Substituting (8) in the gap equation (4) or (7) and letting $\epsilon\to 0$, one learns the integral diverges like ϵ^{-1} and

$$\Sigma(p) = -\frac{m^2}{\epsilon}c\Gamma_\mu(p,p)S(p)\gamma_\mu + \text{ft}\,, \tag{9}$$

where $m^2 = \mu^2/64\pi^2 c$ and ft = finite terms as $\epsilon\to 0$. These infrared-finite pieces can be dropped in what follows for $|p^2/m^2| < \epsilon^{-1}$. Using equations (1) and (6) and introducing $x = p^2/2m^2$ and $C(x) = B^2(x)/2m^2$, Eq. (9) is

$$\epsilon[1 - A(x)][xA^2(x) - C(x)] = A^2(x) - xA'(x)A(x) + \tfrac{1}{2}C'(x)\,,$$
$$\epsilon C(x)[xA^2(x) - C(x)] = -2A(x)C(x) - xA'(x)C(x) + \tfrac{1}{2}xA(x)C'(x)\,, \tag{10}$$

where the prime denotes differentiation with respect to x. These are the differential equations for the quark propagator. To solve them we will look for infrared-finite functions defined by

$$\overline{C}(x)=\epsilon^2 C(x),\quad \overline{A}(x)=\epsilon A(x)\,, \tag{11}$$

which, using (10) in the limit $\epsilon\to 0$, obey the nonlinear equations

$$-\overline{A}(x)[x\overline{A}^2(x)-\overline{C}(x)]=\overline{A}^2(x)-x\overline{A}'(x)\overline{A}(x)+\tfrac{1}{2}\overline{C}'(x)\,,$$

$$\overline{C}(x)[x\overline{A}^2(x)-\overline{C}(x)] \tag{12}$$

$$=-2\overline{A}(x)\overline{C}(x)-x\overline{A}'(x)\overline{C}(x)+\tfrac{1}{2}x\overline{A}(x)\overline{C}'(x)\,.$$

A solution to these equations is

$$\overline{A}(x)=\frac{-2}{x-2\lambda^2}\,,\quad \overline{C}(x)=\frac{8\lambda^2}{(x-2\lambda^2)^2}\,, \tag{13}$$

where λ is an arbitrary constant. Using (13), (11), (1), and (6) we find the exact solution to the gap equation as $\epsilon\to 0$:

$$S(p)=\frac{\epsilon(\not{p}-2m\lambda)}{-4m^2}\,,\quad \left|\frac{p^2}{m^2}\right|<\frac{1}{\epsilon} \tag{14}$$

and

$$c\epsilon\Gamma_\beta(p,p)=\left(\frac{4m^2}{p^2-4m^2\lambda^2}\right)\left[\frac{2p_\beta(\not{p}+2m\lambda)}{p^2-4m^2\lambda^2}-\gamma_\beta\right]. \tag{15}$$

A few remarks are in order. First, our solution (14) is compatible with the usual analyticity properties of a fermion propagator in a completely trivial way since $\mathrm{Im}S=0$. Further, $S(p)$ vanishes as $\epsilon\to 0$; quarks do not exist. However, the combination

$$\Gamma_\beta(p,p)S(p)=-\frac{1}{\not{p}-M}\gamma_\beta\,,\quad M=2m\lambda \tag{16}$$

is infrared finite and it is this combination that enters bound-state integral equations. Interestingly, (16) looks just like a fermion propagator with a pole at $p^2=M^2$ times a free vertex function. But there is no free quark; nor do such states conbribute to the unitarity sums of color-singlet scattering processes.

The solution (14) valid for $|p^2/M^2|<\epsilon^{-1}$ is not incompatible with the result of asymptotic freedom (5) valid in a different region $|p^2/M^2|>\epsilon^{-1}$. In the infrared limit $\epsilon\to 0$ it is (14) that is the relevant solution.

Finally, we observe that $\lambda^2=0$ corresponds to a γ_5-invariant vacuum and PCAC will not be satisfied in this case. The appearance of an arbitrary constant like λ in the solution is interesting. The gap equation is a consequence of equations of motion following from a variational principle $\delta L=0$. However, one also requires stability $\delta^2 L<0$, a requirement difficult to check, and this, we speculate, may impose the PCAC phase condition $\lambda^2>0$.

Much of what we find here is qualitatively the same as two-dimensional QCD, especially the studies of 't Hooft[16] and of Callan, Coote, and Gross.[16] We expect the same development for scattering processes, etc., found in two dimensions will also apply for the physically relevant four-dimensional QCD.

III. THE GOR'KOV EQUATIONS OF QCD

Next we consider the fluctuations about the ground state. These excitations are to be identified with the hadrons. As a first step consider only the mesons $\sim\overline{q}q$ and not the baryons $\sim qqq$. There could also be excitations of the type $\overline{q}q\overline{q}q$, etc., but they are unstable, decaying into mesons. We denote by $|\Omega\rangle$ the ground state of the meson subspace and this is the vacuum in the presence of sources for all meson excitations. Our formal procedure will be essentially equivalent to a Hartree-Fock Bethe-Salpeter treatment of the bound-state problem.

The quark propagator in this ground state is

$$S(x',x)_{\beta\alpha}=-i\langle\Omega|\,T(q_\beta(x')\overline{q}_\alpha(x))\,|\Omega\rangle \tag{17}$$

and

$$S(p',p)=\int d^4x'd^4x\,e^{i(p'\cdot x'-p\cdot x)}S(x',x)\,. \tag{18}$$

The bare propagator is

$$S^0(p',p)=(2\pi)^4\delta^4(p'-p)/\not{p} \tag{19}$$

and the self-energy $\Sigma(p',p)$ is defined as the solution to

$$S(p',p)=S^0(p',p)+\int\frac{d^4p_1}{(2\pi)^4}S(p',p_1)\Sigma(p_1,p)\frac{1}{\not{p}}\,. \tag{20}$$

We remove the singular part of $\Sigma(p',p)$ by the usual displacement familiar in the Σ model:

$$\Sigma(p',p)=(2\pi)^4\Sigma(p)\delta^4(p'-p)+\Sigma^*(p',p)\,, \tag{21}$$

where $\Sigma(p)$ is the usual gap function and $\Sigma^*(p',p)$ is the gap excitation function. $\Sigma^*(p',p)$ will describe the mesons. The quark propagator is as in (1),

$$S^{-1}(p)=\not{p}-\Sigma(p)\,,$$

so we may, resuming, express (20) as

$$S(p',p)=(2\pi)^4\delta^4(p'-p)S(p)+\int\frac{d^4p_1}{(2\pi)^4}S(p',p_1)\Sigma^*(p_1,p)S(p)\,. \tag{22}$$

The self-consistent propagator is determined from the Schwinger-Dyson equation

$$\Sigma(p',p) = \frac{i4}{3}\int \frac{d^4q}{(2\pi)^4} D_{\alpha\beta}(q)\,\Gamma_\alpha(p',p'-q) \times S(p'-q,p-q)\gamma_\beta\,, \quad (23)$$

where in the Landau gauge $D_{\alpha\beta}(q) = (-g_{\alpha\beta} + q_\alpha q_\beta / q^2)\,d(q^2)/q^2$. Substituting Eq. (22) in (23), identifying the singular part of (23) proportional to $\delta^4(p'-p)$, one obtains the two equations

$$\Sigma(p) = \frac{i4}{3}\int \frac{d^4q}{(2\pi)^4} D_{\alpha\beta}(q)\,\Gamma_\alpha(p,p-q)\,S(p-q)\gamma_\beta\,, \quad (24a)$$

$$\Sigma^*(p',p) = \frac{i4}{3}\int \frac{d^4q}{(2\pi)^4} D_{\alpha\beta}(q)\,\Gamma_\alpha(p',p'-q) \times \int \frac{d^4p_1}{(2\pi)^4} S(p'-q,p_1)\,\Sigma^*(p_1,p)\,S(p)\gamma_\beta\,. \quad (24b)$$

Equation (24a) is the gap equation which we have already discussed. Our interest now is Eq. (24b) for Σ^*. Substituting (22) in the right side of (24b) one obtains, upon iterating (22) in a Neumann series,

$$\Sigma^*(p',p) - \frac{4i}{3}\int \frac{d^4q}{(2\pi)^4} D_{\alpha\beta}(q)\,\Gamma_\alpha(p',p'-q)\,S(p'-q)\,\Sigma^*(p'-q,p)\,S(p)\gamma_\beta = J^*(p',p)\,, \quad (25a)$$

where the source function is

$$J^*(p',p) = \frac{4i}{3}\int \frac{d^4q}{(2\pi)^4} D_{\alpha\beta}(q)\,\Gamma_\alpha(p',p'-q)\int \frac{d^4p_1}{(2\pi)^4} S(p'-q)\,\Sigma^*(p'-q,p_1)\,S(p_1)\,\Sigma^*(p_1,p)\,S(p)\gamma_\beta + O(\Sigma^{*3}) + \cdots\,. \quad (25b)$$

Equation (25) is the meson gap excitation or Gor'kov equation for QCD. Again, confinement suggests that we may use $\Gamma_\alpha(p,p-q) \simeq \Gamma_\alpha(p,p)$ given by (6) so that (25) is specified by $S(p)$ and $d(q^2)$.

Comparing (25) with the Bethe-Salpeter equation one sees that

$$\Sigma^*(x,P) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot x}\,\Sigma^*(p',p)\,, \quad (26)$$

with $q = p' + p$, $P = p' - p$, is the bound-state wave function of a meson of mass P^2. $\Sigma^*(x,P)$ is further decomposed from (26), with

$$\Sigma^*(p',p) = \sigma(p',p)I + \pi(p',p)\,i\gamma_5 + V_\mu(p',p)\gamma_\mu + A_\mu(p',p)\,i\gamma_5\gamma_\mu + T_{\mu\nu}(p',p)\,\sigma_{\mu\nu} \quad (27)$$

to exhibit its Dirac spin structure.

The Gor'kov equations (25) in principle establish the spectrum and the interactions of all the mesons once the "potential" $d(q^2)$ is specified. The procedure is as follows. Given $d(q^2)$ one first solves the usual gap equation for the quark propagator $S(p)$. Here we must use $\Gamma_\alpha(p,p-q) \simeq \Gamma_\alpha(p,p)$ in conformity with the Ward identity as discussed. The solution for $S(p)$ is specified by PCAC and quark confinement. With $S(p)$ established for a given $d(q^2)$ one has the information needed to solve the meson gap excitation equation for $\Sigma^*(x,P)$. As a first step one may ignore meson interactions and set $J^*(p',p) = 0$. Then (25) is the "free" meson Gor'kov field equation and an eigenvalue problem for the meson spectrum.

In this interpretation the mesons are collective modes, self-consistent vacuum fluctuations of the color-singlet bilocal operator $\bar{q}(x)\,q(y)$. We have given here a simple framework which in practice determines phenomenological meson dynamics from the specification of a single function, the "potential" $d(q^2)$. In principle $d(q^2)$ could be determined from QCD, but this seems an undertaking more difficult than actually solving the Gor'kov equations. If one is optimistic, the Gor'kov equation (25) provides the link between QCD and meson phenomenology. This method can provide an integrated picture of the hadrons that combines the quark model, PCAC, and quark confinement.

A soluble example

It turns out that the Gor'kov equations are easily established for the previous example of the infrared-singular potential, Eq. (8), $d(q^2) = (\mu^2/q^2)^{1-\epsilon}$. Substitutions of this potential into the gap excitation equation (25), letting $\epsilon \to 0$, and symmetrizing, according to $\Gamma_\mu\gamma_\mu \to \frac{1}{2}(\Gamma_\mu\gamma_\mu + \gamma_\mu\Gamma_\mu)$, one has

$$\Sigma^*(p',p) + \frac{m^2}{2\epsilon}[\Gamma_\mu(p',p')\,S(p')\,\Sigma^*(p',p)\,S(p)\gamma_\mu + \gamma_\mu S(p')\,\Sigma^*(p',p)\,S(p)\,\Gamma_\mu(p,p)] = J^*(p',p)\,. \quad (28)$$

As a first step we ignore the interactions and set $J^* = 0$, so that using our results for the fermion propagator and vertex (14) and (15), respectively, Eq. (28) becomes

$$8\Sigma^*(p',p)+\frac{1}{\not p'-M}\gamma_\mu\Sigma^*(p',p)(\not p-M)\gamma_\mu + \gamma_\mu(\not p'-M)\Sigma^*(p',p)\gamma_\mu\frac{1}{\not p-M}=0\,, \tag{29}$$

where $M=2m\lambda$ and all dependence on ϵ has disappeared.

We introduce the variables $P_\mu=(p'-p)_\mu$, which is the meson total 4-momentum, and $q_\mu=(p'+p)_\mu$, which is the momentum conjugate to the relative coordinate of the quark-antiquark pair. An algebraic simplification is accomplished if we assume that the meson bound-state amplitudes in the meson rest frame, $\vec{P}=0$, are independent of the relative time of the two constituents. Such relative time dependence, a feature of a relativistic treatment, has traditionally been difficult to interpret. This means $p'^2-p^2=P\cdot q\sim -iM\partial/\partial t=0$; so we look for solutions with this property. One could retain the $P\cdot q$ dependence but it is not instructive to do so. With $P\cdot q=0$ (29) becomes

$$4(p^2-M^2)\Sigma^*(p'p)-(\not p'+M)\gamma_\mu\Sigma^*(p',p)\gamma_\mu(\not p+M) + (\not p'+M)\not p\Sigma^*(p',p)+\Sigma^*(p',p)\not p'(\not p+M)=0\,. \tag{30}$$

Using the general decomposition

$$\Sigma^*(p',p)=\sigma(p',p)I+\pi(p',p)i\gamma_5+V_\mu(p',p)\gamma_\mu + i\gamma_5\gamma_\mu A_\mu(p',p)+\sigma_{\mu\nu}T_{\mu\nu}(p',p)\,, \tag{31}$$

$$\sigma_{\mu\nu}=\tfrac{1}{2}i[\gamma_\mu,\gamma_\nu]$$

$$A_\mu(p',p)=A_+(p',p)(p'+p)_\mu+A_-(p',p)(p'-p)_\mu\,,$$

$$V_\mu(p',p)=V_+(p',p)(p'+p)_\mu+V_-(p',p)(p'-p)_\mu\,,$$

$$T_{\mu\nu}(p',p)=i(p'_\mu p_\nu-p'_\nu p_\mu)T(p',p) + i\epsilon_{\mu\nu\lambda\delta}p'_\lambda p_\delta B(p',p)\,,$$

where σ, π, $V_\pm$, $A_\pm$, T, and B are invariant functions of $p'^2, p^2, p'\cdot p$, one obtains from (30) the wave equations for the meson amplitudes. After an exercise in γ algebra one finds

$$(\tfrac{3}{2}P^2+\tfrac{1}{2}q^2-8M^2)\sigma+3q^2MV_+ +P^2q^2T=0\,,$$
$$-3M\sigma+(P^2+2q^2-4M^2)V_+ +P^2MT=0\,, \tag{32a}$$
$$2\sigma-4MV_+ +(P^2+3q^2-8M^2)T=0\,,$$

$$(\tfrac{3}{2}P^2+\tfrac{1}{2}q^2)\pi+3P^2MA_- +P^2q^2B=0\,,$$
$$-3M\pi+(4M^2-q^2)A_- -q^2MB=0\,, \tag{32b}$$
$$2\pi+4MA_- +(3P^2+q^2-8M^2)B=0\,,$$

$$(q^2-4M^2)V_-=0\,,$$
$$(-P^2+2q^2+4M^2)A_+=0\,. \tag{32c}$$

Here $(P^2)^{1/2}$ is the meson mass operator and q^2 is the Laplacian in the meson rest frame.

For the ground-state mesons evidently $q^2=0$ and (32) are elementary free-field wave equations for which one can read off the mass spectrum. For (32b) the physically relevant solution is $\pi=\frac{4}{3}MA_-$, $B=0$, and so $M_\pi{}^2=0$. This is just the Goldstone pseudoscalar state and it is an automatic feature of our general approach. We may similarly inspect (32a) for the ground-state spectrum. Disregarding complex eigenvalues for P^2 and the trivial solution $\sigma=V_+=T=0$, one has either $V_+=0$, $\sigma, T\neq 0$ and the scalar and skew-tensor masses given by

$$M_\sigma{}^2=\tfrac{16}{3}M^2,\quad M_T{}^2=\tfrac{24}{5}M^2\,, \tag{33}$$

or $T=0$, $\sigma, V_+\neq 0$ and

$$M_\sigma{}^2=\tfrac{16}{3}M^2,\quad M_{V_+}{}^2=10M^2\,. \tag{34}$$

The skew tensor is, of course, a vector excitation, so that in either case the vector excitations are more massive than the axial vector at

$$M_{A_+}{}^2=4M^2 \tag{35}$$

as follows from (32c). An alternative solution to (32c) is $A_+=0$. Of course only mass ratios have meaning, since M^2 cannot be absolutely specified in a theory like QCD, which has no mass scale. Using the solution (33) the ground-state mass-squared ratios are $0^-:0^+:1^-=0:10:9$.

The solution given above for the ground-state meson mass spectrum is given in the absence of explicit flavor symmetry breaking. Such symmetry breaking can be incorporated by adding $\mathcal{L}_{QCD}$ a quark mass matrix. Then the gap eqaution will have an inhomogeneous term. The influence on the solution (14) for $S(p)$ will be to have λ diagonal but not proportional to the identity matrix in flavor space, at least to leading order in symmetry breaking. Then the Gor'kov equations imply that the ground-state pseudoscalars will acquire mass with the mass squared proportional to the quark mass. Other flavor supermultiplets also have their degeneracy removed.

Even if one incorporates explicit chiral symmetry breaking as a quark mass term, this is well known not to solve the $U_A(1)$ problem.[14] In the SU(2) symmetry limit one will have four, not three Goldstone states. It is now known[14] that the extra $U_A(1)$ symmetry is broken by the pseudoparticle solution, and one has an *effective* $U_A(1)$ symmetry-breaking term of the form $\det M + \det M^\dagger$, $M=\bar{q}(1+\gamma_5)q$, a six-quark flavor- and color-singlet interaction. Such an interaction necessarily alters the Gor'kov equations in the flavor-singlet sector and one will avoid the $U_A(1)$ problem. Exactly how to incorporate this modification of the Gor'kov gap wave equations (32) is not known.

IV. MESON SCATTERING AMPLITUDES

The gap excitation function $\Sigma^*(p',p)$ is of order $1/\epsilon$ as is seen by inspecting (25). So we define

$$\bar{\Sigma}^*(p',p) = \epsilon\Sigma^*(p',p)\,,$$
$$\epsilon\bar{S}^*(p',p) = S(p',p) - (2\pi)^4\delta^4(p'-p)S(p), \tag{36}$$

which are finite as $\epsilon \to 0$. Then (22) and (24b), in the limit $\epsilon \to 0$, are

$$\bar{S}^*(p',p) = \bar{S}(p')\bar{\Sigma}^*(p',p)\bar{S}(p) + \int \frac{d^4p_1}{(2\pi)^4}\,\bar{S}^*(p',p_1)\bar{\Sigma}^*(p_1,p)\bar{S}(p)\,, \tag{37}$$

$$\bar{\Sigma}^*(p',p) = -m^2\bar{\Gamma}_\alpha(p',p')\bar{S}^*(p',p)\gamma_\alpha\,, \tag{38}$$

where the bar denotes that the factors of ϵ have been removed so the result is infrared finite. Consequently, combining (37) and (38) we obtain as a result a nonlinear integral equation for $\bar{S}^*$, or equivalently, using (38) for $\bar{\Sigma}^*$, we obtain

$$\bar{S}^*(p',p) + m^2\bar{S}(p')\bar{\Gamma}_\alpha(p',p')\bar{S}^*(p',p)\gamma_\alpha\bar{S}(p) + m^2\int\frac{d^4p_1}{(2\pi)^4}\,\bar{S}^*(p',p_1)\bar{\Gamma}_\alpha(p_1,p_1)\bar{S}^*(p_1,p)\gamma_\alpha\bar{S}(p) = 0\,. \tag{39}$$

This is the generalization of the free Gor'kov wave equations (32) to include interactions. No attempt will be made to solve these equations for $\bar{\Sigma}^*$. The solutions will serve to establish the appropriate boundary conditions on the free field equations (32).

An elementary (but approximate) procedure for the construction of meson amplitudes now suggests itself. The n-meson amplitudes are constructed by joining meson-quark-antiquark vertices $\Sigma^*(p',p)$ with quark propagators $S(p)$ as in the standard duality diagrams. The factors of $1/\epsilon$ from the $\Sigma^*(p',p)$ vertices just cancel the factors of ϵ from the quark propagators joining the vertices so that the n-meson amplitude is finite as $\epsilon \to 0$.

We have not proven that there is no further divergence associated with the quark loop integration. This depends on the behavior of $\bar{\Sigma}^*(p',p)$. However, this procedure is suggestive of how the dynamics of confinement yields nontrivial scattering amplitudes in the color-singlet sector. Importantly, if one cuts the diagram corresponding to such an n-meson amplitude across the quark propagators, the absorptive amplitude must vanish. This follows from the Cutkosky cutting rule and the fact that quark propagator (14) has no absorptive part. Hence, thresholds corresponding to quarks never appear in the absorptive part of amplitudes in this construction. Whether such amplitudes are unitary or approximately unitary is much more difficult to establish.

V. CONCLUSION

We have described a method for extracting the dynamics and spectrum of hadrons from a fundamental theory like QCD. This can be accomplished in a manner consistent with PCAC, quark confinement, and the principle of relativity. This investigation is preliminary and incomplete. Outstanding problems remain to be solved.

First is the problem of determining the gauge field propagator $d(q^2)$ for small q^2. This is a problem in QCD which we have not addressed and which is associated with the strong-coupling behavior of the Callan-Symanzik function. We have given as an example the infared-singular potential $d(q^2) = (\mu^2/q^2)^{1-\epsilon}$, for which the gap equation is exactly soluble in the infared limit $\epsilon \to 0$. If $d(q^2)$ is more singular, then it may not be possible to regulate the infared singularity. So conceivably this could be the right answer for small q^2 if the theory exists at all. A remarkable feature of the infared limit is that the integral equations are exactly soluble. Other simplifications, already learned from two-dimensional QCD, also result. So the infared limit, rather than being a problem, may be a virtue. Another important problem is to explicitly demonstrate the gauge invariance of this approach.

Whether the simplicity in the bound-state problem found for the example above will be retained for more complicated potentials is not now known. The spectrum of the hadrons is relatively simple, a linear Regge trajectory. Presumably the Regge trajectory is the rotational spectrum of the gap excitations with confined quarks and this should follow in a simple way from the Gor'kov equations.

Finally, we have found contact, at least in part, with two-dimensional QCD. This suggests that the scaling behavior, form factors, etc., found there have an analog for four-dimensional QCD. These and other problems will form the subject of a future investigation.

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