Quantization condition for 't Hooft monopoles in compact simple Lie groups

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The quantization conditions for 't Hooft monopoles and their related conserved topological charges are obtained for all compact simple Lie groups. In addition, all possible Dirac monopoles are classified.

As shown by 't Hooft,[1] magnetic monopoles arise as fundamental solutions of non-Abelian gauge theories without imposing Dirac strings.[2] In O_3 the magnetic charge is not quantized in single Dirac units; in general, magnetic charge quantization depends on the underlying gauge group as seen from explicit evaluation in SU_3 (see Ref. 3) and SU_4 (see Ref. 4). In the present paper we shall obtain the quantization conditions for 't Hooft monopoles in all compact simple Lie Groups and for the related topologically conserved charges: The latter will be identified with the quantized magnetic charges or with a subset of these depending on the nature of the unbroken subgroup. For those groups which admit a nontrivial first homotopy group (e.g. O_3, $SU_3/Z_3, \ldots$), one may in addition introduce monopoles with genuine Dirac strings; their quantized magnetic charges will also be given.

I. GENERAL CHARACTERIZATION OF 't HOOFT MONOPOLES

Let us review the construction of a 't Hooft monopole for a compact simple Lie group G using the "Abelian gauge" approach of Arafune, Freund, and Goebel[5] (AFG). As a simple example we first consider the following Lagrangian invariant under G:

$$\mathcal{L} = -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \tfrac{1}{2}D_\mu\phi^a D^\mu\phi^a - V(\phi^a)\,, \tag{1}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ef^{abc}A^b_\mu A^c_\nu\,. \tag{2}$$

Here D_μ is the covariant derivative operator; the matter field ϕ^a belongs to the adjoint representation of G and $V(\phi^a)$ is a potential. A *point*-singular solution to (1) can be obtained from

$$D_\mu G^{a\mu\nu} = 0\,, \tag{3}$$

$$D_\mu \phi^a = 0\,, \tag{4}$$

$$\frac{\partial V(\phi^b)}{\partial \phi^a} = 0 \tag{5}$$

as follows. A nontrivial solution of (5), $\phi^a(x)$, defines a direction in the space of the adjoint representation. Performing a (possibly singular) gauge transformation it is always possible to rotate $\phi^a(x)$ for all x into a *given* maximal Abelian subgroup T of order l, where l is the rank of the group [e.g. in SU_3/Z_3 we may rotate $\phi^a(x)$ into the 3–8 plane]. From (4) we learn that $\phi^a(x)$ is x independent and that $A^b_\mu(x) = 0$ ($b \notin T$) if the little group of $\phi^a(x)$ is isomorphic with T. However, if the unbroken subgroup H is larger than T [as is generally the case in the above model (1) (see Ref. 6)], we may still impose $A^b_\mu(x) = 0$ ($b \notin T$), but this is not the only solution: In particular, solutions exist for which the nonvanishing $A^a_\mu(x)$ are in a subgroup T' conjugate to T under H and we shall consider these solutions. Equation (3) then reduces to $\Box A^a_\mu = 0$ ($a \in T'$), where the further allowable gauge transformation $\partial_\mu A^{a\mu} = 0$ ($a \in T'$) has been used. These linear equations admit monopole solutions as *line*-singular solutions:

$$\vec{A}^a(\vec{r}_p) = g^a \int_{-\infty}^{0} d\vec{s} \times \vec{\nabla}\left(\frac{-1}{|\vec{r} - \vec{r}_p|}\right) \quad (a \in T',\ T' \subset H)\,. \tag{6}$$

Here the integral is taken along an arbitrary Dirac string terminating at the origin 0, where the monopole is located; g^a ($a \in T'$) are the monopole charges. If the string can be removed in (6) by a gauge transformation in G, one obtains in this way a *point*-singular solution to (1) (singular 't Hooft monopole). As (3), (4), and (5) must necessarily be satisfied asymptotically for all finite-energy solutions, one may take the point-singular solution as a boundary condition of a possible regular solution in which the energy at 0 is smeared out (regular 't Hooft monopole); by construction, this regular monopole still has the string in the original Abelian gauge T'.

The matter field in (1) is an important ingredient in the construction of a regular solution and provides a mechanism for spontaneous symmetry breaking. It is the latter that leads to topological stability.[7] However, the detailed form of the matter Lagrangian is not relevant: We shall only need to specify the unbroken subgroup H. We consider only the cases where H is large enough to contain a maximal Abelian subgroup T from which all Abelian gauges T' are obtained by conjugation in H; in other cases the monopole problem may be different owing to flux line formation.[8] When $H \supset T$ (including the case $H \equiv T$) the singular solution (6) can always be constructed when all matter fields do not vary in space; indeed, all A_μ^b ($b \notin H$) then acquire a mass and may consistently be set equal to zero, while the others satisfy the free field equation (3). From now on we therefore generalize $\mathcal{L}$ to include all possible matter Lagrangians. However, it is necessary to specify whether all matter fields in $\mathcal{L}$ occur only in representations of the adjoint group or whether other representations are present. In the first case we have $G \equiv G_A$, where $G_A = G_C/Z_C$; G_C is the universal covering group of the adjoint group G_A and Z_C is its discrete center; in the second case one has generally $G \equiv G_C$ but for some groups intermediate cases may exist.[9] (For the exceptional groups G_2, F_4, and E_8, $Z_C = 1$ and one has always $G \equiv G_A \equiv G_C$.)

II. QUANTIZATION CONDITION FOR 't HOOFT MONOPOLES

To construct a point-singular 't Hooft monopole we must now determine g^a in (6) so that the string can be gauged out. To obtain this condition we shall follow Wu and Yang.[10] According to their prescription one replaces (6) by an equivalent description in terms of two potentials $\vec{A}_{\rm I}^a$ and $\vec{A}_{\rm II}^a$ obtained from two strings I and II taken respectively from $-\infty$ to 0 and from $+\infty$ to 0 along the z axis. $\vec{A}_{\rm I}^a$ is defined in the polar angle range $0 \leq \theta < \pi/2 + \delta$ and $\vec{A}_{\rm II}^a$ in $\pi/2 - \delta < \theta \leq \pi$, $\delta > 0$. In the overlap region one easily evaluates

$$\vec{A}_{\rm I}^a - \vec{A}_{\rm II}^a = 2eg^a \vec{\nabla}\varphi \quad (a \in T')\,, \tag{7}$$

where φ is the azimuthal angle. To this difference corresponds a phase factor $\exp(2ie\varphi g^a t^a)$ for each matter field coupled to A_μ^a and belonging to the representation generated by t^a. The string will be unobservable if and only if this phase factor is single-valued so that (7) corresponds to a gauge transformation. Thus we must have

$$\exp(i4\pi e g^a t^{a(G)}) = 1 \quad (a \in T')\,, \tag{8}$$

where $t^{a(G)}$ generates a faithful representation of G. The condition (8) implies that a closed curve in space within the overlapping region ($\pi/2 - \delta < \theta < \pi/2 + \delta$) is mapped onto a closed curve in the group space of G starting and ending at the unit element. If this curve can be continuously shrunk to zero, δ will be reduced to 0 by a gauge transformation in G; in this case the Dirac string is removed. We see that if G is simply connected (8) already implies that the string can be gauged out; however, in general we must have the more stringent condition

$$\exp(i4\pi e g^a t^{a(G_C)}) = 1 \quad (a \in T')\,, \tag{9}$$

where $t^{a(G_C)}$ is a faithful representation of the universal covering group of G; indeed the closed curves in G which are homotopic to zero are those which correspond to closed curves in G_C.

Let $m^a, m^b, \ldots$ be a set of l eigenvalues belonging to a common eigenvector of the $t^{a(G_C)}$; these are the Cartesian components of a weight vector $\vec{m}$ in an l-dimensional Euclidean space E_l (E_l may be viewed as the space of the universal covering group of T'). The totality of weights of *all* representations of G_C (including of course those of G) is contained in a weight lattice in E_l, which is related to the root lattice by Eq. (10). We recall that the root lattice is generated from l linearly independent simple roots and that the root lattice is contained in the weight lattice as a sublattice; the simple roots and the vectors obtained from them by a Weyl reflection (reflection in the hyperplanes perpendicular to those roots) are the elementary roots which are the weights of the adjoint representation; thus the elementary roots are formed out of the f^{abc} connecting T' to the other group generators.[9] As an example the weight lattice of SU_2 and SU_3 is depicted in Fig. 1. For all $\vec{m}$ and all simple roots $\vec{\alpha}_i$ we have

$$\vec{m}\cdot\vec{\alpha}_i = \frac{n}{2}\,\vec{\alpha}_i\cdot\vec{\alpha}_i \quad (n \text{ integer})\,. \tag{10}$$

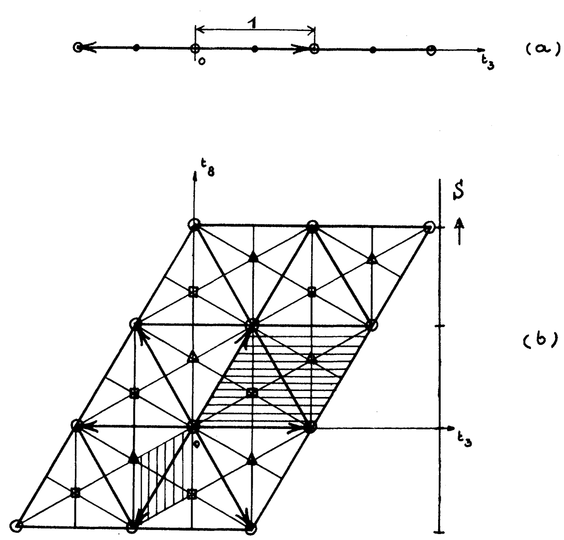

FIG. 1. (a) Weight lattice of SU_2. → represents elementary roots, ○ represents 't Hooft monopoles in SU_2 and O_3, ● represents Dirac monopoles in O_3. (b) Weight lattice of SU_3. → represents elementary roots, ○ 't Hooft monopoles in SU_3 and SU_3/Z_3, □ first-class Dirac monopoles in SU_3/Z_3, △ second-class Dirac monopoles in SU_3/Z_3, ≡ elementary root-lattice cell, ||| elementary weight-lattice cell, and S "soliton charge" axis when $H \equiv U_2$.

Let $\vec{g}$ be a vector in E_l with Cartesian components g^a. From (9) we must have for all $\vec{m}$

$$e\vec{g}\cdot\vec{m}=\frac{n}{2}\quad (n\text{ integer})\,. \tag{11}$$

Comparing (11) with (10) we see that

$$e\vec{g}=\sum_{i=1}^{l} n_i\vec{\alpha}_i^*\quad (n_i\text{ integer})\,, \tag{12}$$

$$\vec{\alpha}_i^*=\vec{\alpha}_i/\vec{\alpha}_i\cdot\vec{\alpha}_i\,, \tag{13}$$

where the sum is over the l simple roots.

From (13) we have $\vec{\alpha}_i^*\cdot\vec{\alpha}_j^*=(n'/2)\vec{\alpha}_j^*\cdot\vec{\alpha}_j^*$, where n' is some integer and thus the $\vec{\alpha}_i^*$'s form a system of simple roots.[9] In fact for all simple groups except B_l and C_l, the α^* lattice is isomorphic to the initial root lattice α; for B_l (C_l), the α^* lattice is isomorphic to the α lattice of C_l (B_l). Note that it follows from (10) that if the smallest simple root is normalized to unity, the α lattice falls always on a sublattice of the α^* lattice and in particular, for $A_l\,(SU_{l+1}), D_l, E_6, E_7, E_8$, the α and α^* lattices coincide. The fact that the lattice α^*, which is reciprocal to the weight lattice of a group, is sometimes different from the initial root lattice α was pointed out to us by Nuyts and Olive, who explicitly solved Eq. (8) for all simple groups G in a different physical context.[11]

The quantization condition (12) is valid for point-singular 't Hooft monopoles and thus for regular ones if the latter can be constructed. Note that (12) depends only on the Lie algebra and not on global properties; e is always defined from the coupling (2).

III. TOPOLOGICAL STABILITY AND "SOLITON CHARGE" QUANTIZATION

A. *H* isomorphic to *T*

In this case the Abelian gauge T' is uniquely determined to coincide with H. Thus each $\vec{g}$ satisfying (12) determines entirely a singular 't Hooft monopole and hence can be assigned uniquely to a corresponding regular one. As all components of $\vec{g}$ are discrete they cannot be changed by continuous motions of matter. Thus $\vec{g}$ is strictly conserved for finite-energy solutions and may be identified to a "soliton charge" quantized according to (12).

B. *H* contains *T* as a subgroup

Any T' conjugate to a fixed T under H is an acceptable Abelian gauge in which (12) is valid. Thus the previous argument applies only to those components of $\vec{g}$ which are common to all T'. This common subgroup is the Lie subgroup t of T which is an invariant subgroup of H. The "soliton charge" is thus the orthogonal projection of $\vec{g}$ in the subspace of E_l which is the image of t. [For instance, if SU_3 has U_2 as an unbroken subgroup the "soliton charge" is quantized along the 8-direction, or any equivalent charge direction; see Fig. 1(b).] In particular, if the symmetry is not broken ($H\equiv G$), t is the unit element and topological stability is lost.[7]

IV. CLASSIFICATION OF DIRAC MONOPOLES

We now examine the significance of (6) if the string cannot be gauged away. Such a solution can be obtained only by adding to $\mathcal{L}$ a Dirac string term. The acceptable Dirac monopoles are given by the solution of (8), which is not a solution of (9).

Let us consider the extreme cases $G\equiv G_C$ and $G\equiv G_A$, the latter containing the largest set of available Dirac monopoles. If $G\equiv G_C$ no Dirac monopole can be introduced. When $G\equiv G_A$, a faithful representation of $t_a^{(G_A)}$ is generated by the simple roots $\vec{\alpha}_i$ and (8) yields $e\vec{g}_D\cdot\vec{\alpha}_i=n/2$, where $\vec{g}_D$ is the magnetic charge of the Dirac monopole. This result may also be written as $e\vec{g}_D\cdot\vec{\alpha}_i^*=(n/2)\times(\vec{\alpha}_i^*\cdot\vec{\alpha}_i^*)$, and thus it follows from (10) that

$$e\vec{g}_D=\sum_{i=1}^{l} n_i\vec{m}_i^*\,,$$
$$e\vec{g}_D\neq\sum_{i=1}^{l} n_i\vec{\alpha}_i^*\quad (n_i\text{ integer})\,, \tag{14}$$

where $\vec{m}_i^*$ are the l primitive translations of the weight lattice m^* corresponding to the root lattice α^*. Thus the Dirac monopoles in G_A are given by the points of m^* which do not belong to the sublattice α^*.

We may decompose the weight lattice m^* in p root sublattices obtained from the original α^* sublattice by $(p-1)$ translations; these translations are the $(p-1)$ weight vectors m^* with lie within an elementary cell of the α^* lattice. These $(p-1)$ points represent the nontrivial elements of the center of G_C and thus the $(p-1)$ nontrivial elements of the first homotopy group of G_A. While the original root sublattice α^* corresponds to 't Hooft monopoles, each of the remaining $(p-1)$ sublattices defines a class of Dirac monopoles [see for example Fig. 1(b)]. These p different classes of monopoles are topologically inequivalent and cannot be transformed one into another by continuous motions of matter as way already pointed out by Wu and Yang[10]; this fact is true even in the absence of symmetry breaking where no topological stability condition is available for 't Hooft monopoles.

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