Scattering on magnetic charge*

David G. Boulware, Lowell S. Brown, Robert N. Cahn, S. D. Ellis, and Choonkyu Lee
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 7 June 1976)

The nature of magnetic monopoles in an SO(3) gauge theory is explored by examining how they scatter charged particles. Peripheral scattering, where the momentum transfer is much less than the mass of the charged vector bosons, is just as it is in the earlier theory of Dirac. The scattering wave function is discussed with some rigor since its behavior is not uniform in the forward direction. The correspondence between the classical and quantum-mechanical scattering is displayed explicitly by relating scattering by the noncentral monopole force to the central force scattering by an attractive $1/r^2$ potential, for both the quantum-mechanical and classical systems. For deep scattering, the non-Abelian monopole exhibits features absent in the Dirac theory. The electromagnetic scattering departs from the result of the Dirac theory and charge-exchange processes occur, exciting the monopole into an electrically charged state. This process, which corresponds to the weak interaction of the unified SO(3) theory, is calculated in the distorted-wave Born approximation. The relation between the monopole in the non-Abelian gauge theory and that in Dirac's theory is investigated by carefully regulating the gauge transformation which connects them. This resolves some seeming paradoxes in the connection between the two theories.

I. INTRODUCTION

The existence of magnetic monopoles has remained one of the most intriguing and elusive questions of particle physics. In the context of ordinary electrodynamics the monopole, as envisaged by Dirac,[1] is essentially a bar magnet of semi-infinite length and infinitesimal thickness. The associated vector potential, magnetic field, and electromagnetic current may be expressed as[2]

$$\vec{A} = g\frac{\hat{r}\times\hat{z}}{r+z}\,, \tag{1}$$

$$\vec{B} = -g\left[\frac{\hat{r}}{r^2} + 4\pi\hat{z}\,\theta(-z)\delta(x)\delta(y)\right], \tag{2}$$

$$\begin{aligned}\vec{j} &= \vec{\nabla}\times\vec{B}\\ &= 4\pi g\,\theta(-z)\left[\hat{y}\delta'(x)\delta(y) - \hat{x}\delta(x)\delta'(y)\right].\end{aligned} \tag{3}$$

The Dirac charge-quantization condition, $eg = n$ (integer or half-integer), ensures that only the pole of the bar magnet is observable quantum mechanically. The singular "string" of magnetic flux along the negative z axis is not observable because it gives rise only to a phase change of $4\pi n$ in a charged-particle wave function when it is encircled. This phase would produce an Aharanov-Bohm effect[3a] were it not for the charge-quantization condition which fixes the phase change to be an integer multiple of 2π.

If one considers gauge transformations defined by multivalued gauge functions, new singularities are introduced in the vector potentials. In this way it is possible to change the location of the singular "string." To the extent that the string is not observable, this is a legitimate transformation. However, as the "string" singularity also appears in the magnetic field (2), it contributes (infinitely) to the energy-momentum densities and to the Maxwell stress tensor. Thus, in a more fundamental sense, the singular string should be observable, for example, by gravitational experiments. This difficulty was recognized[1] by Dirac and by Wentzel, who gave physical arguments for defining the electromagnetic field as the sum of the curl of the vector potential and an additional term which removes the string singularity. Dirac found that in order to incorporate this *ad hoc* definition of the field in a Hamiltonian theory, the charged particles could not cross the string singularity of the monopole. The same *ad hoc* definition was employed by Schwinger[3b] to construct a Lorentz-invariant field theory of magnetic charge in which the charged particles are not constrained to avoid the string.

The properties of the recently discovered monopole solution in certain spontaneously broken non-Abelian gauge theories[4,5] are in sharp contrast to those of the older theory. There is a scale in this static classical solution set by the (very short) Compton wavelength, $\lambda\!\!\!^{-}_{\rm ch}$, of the (very massive) charged vector fields. At distances large on this scale, this solution consists solely of a magnetic-monopole field and the Higgs scalar field associated with the broken symmetry. At short distances, the classical solution is nonsingular and the total field energy, the monopole mass, is finite. The solution is everywhere regular; it does not contain a singular "string." Charge quantization is a direct consequence of the non-Abelian group structure. The additional terms in the electromagnetic field which Dirac required to eliminate the string singularity occur naturally in the non-Abelian gauge theories. However, there being no "string," no condition on the path of the charged particles analogous to that of Dirac is required.

A characteristic feature of the new non-Abelian monopole solution is that it locks together physical space with the internal charge space. If the generators of the internal symmetry are T_a with $a = 1, 2, 3$ [following Refs. 4 and 5 we take the internal symmetry to be SO(3)], the charge operator is $\hat{r}\cdot\vec{T}$. The interlocking of the physical and isospin spaces can be removed by a gauge transformation which sends $\hat{r}\cdot\vec{T}$ into the more familiar charge operator T_3. This position-dependent gauge transformation is singular. With careful definition the transformation introduces no singularities in the field tensor. However, the vector potential does acquire a "string" singularity and becomes identical to the Dirac vector potential (1) in the asymptotic region.

Motivated by this development in non-Abelian gauge theories, we have investigated those aspects of the nature of magnetic charge which are revealed in its scattering of electrically charged particles. In particular, we study the scattering of electrically charged, spin-zero particles by a fixed, point magnetic charge. Our results apply to the scattering in the region outside the Compton wavelength, $\lambda\!\!\!^{-}_{\rm ch}$, in the new gauge theory, as well as to the scattering in the old theory. We shall also briefly discuss deep scattering, where charge-exchange reactions occur.

We first study the classical motion (Sec. II). It is well known[6] that this motion lies on the surface of a cone whose symmetry axis is parallel to the total angular momentum. We show that this conical motion in the monopole field can be projected onto the planar motion of a particle in a $1/r^2$ attractive potential. This projection relates the time development of the two systems as well as their orbit equations.

Section III comprises a review of the monopole solution in a non-Abelian gauge theory that unifies weak and electromagnetic interactions. These monopoles are extremely massive with $M_{\rm pole}$ being of order $m_{\rm ch}/\alpha$, where $m_{\rm ch}$ is the already-large mass of the charged weak vector boson. Thus the monopole can be treated as a fixed scattering center except for processes involving extraordinarily large momentum transfers. The structure of the fixed monopole is probed by coupling it to a multiplet of spin-zero particles. This problem is defined and solved in Sec. IV and its connection with the Dirac theory is discussed.[7]

The correspondence between the quantum-mechanical description and the classical motion is demonstrated in Sec. V. We show that a quantum-mechanical wave packet corresponding to a classical trajectory is concentrated on the cone of the classical motion. Moreover, the quantum-mechanical solution is shown to reproduce the classical mapping between the monopole and the $1/r^2$ potential problems in an appropriate approximation.

Section VI is devoted to a treatment of the quantum-mechanical scattering of an electrically charged particle by a fixed magnetic charge. This involves the discussion of wave functions whose behavior, as r goes to infinity for fixed polar angle θ, is not uniform in θ for small θ. This problem is solved by separating a piece of the wave function that contains the most singular behavior and computing it explicitly in closed form (Appendix A). Our results agree with the earlier work of Banderet,[8] where, however, the separation of the most singular part was not made, nor was a well-behaved expression for the scattering amplitude exhibited.

At distances small compared to $\lambda\!\!\!^{-}_{\rm ch}$, the monopole solution in the non-Abelian gauge theory exhibits new characteristics which are not present in the Dirac theory. Charged vector fields exist at small distances and produce charge-exchange scattering. Their interaction is treated approximately in Sec. VII using a distorted-wave Born approximation. In this approximation both the recoil and the electrical excitation of the monopole[9] are ignored. This is justified, except for extremely deep scattering, because the monopole is very massive and its electrical effects are much smaller than are its magnetic effects.

The nature of the singular gauge transformations appearing in the Dirac theory and in the relationship between the non-Abelian and Dirac theories is discussed carefully in Appendix B.

II. CLASSICAL SCATTERING

We consider the classical motion of a particle of charge e and mass m in the field of a fixed, point magnetic charge $-g$. The Lorentz force law is[2]

$$m\ddot{\vec{r}} = -e\dot{\vec{r}}\times\frac{g\hat{r}}{r^2}\,. \tag{4}$$

The sum of the particle's mechanical angular momentum and the angular momentum in the electromagnetic field, the total angular momentum, is given by

$$\vec{J} = m\vec{r}\times\dot{\vec{r}} + eg\hat{r}\,. \tag{5}$$

The kinetic energy of the particle is

$$E = \tfrac{1}{2}m(\dot{\vec{r}})^2 = \tfrac{1}{2}mv^2\,. \tag{6}$$

It is easy to show that these quantities are constants of the motion. It follows from Eq. (5) that the angle between the position $\vec{r}$ and the angular momentum $\vec{J}$ is a constant,

$$\vec{J}\cdot\hat{r} = eg\,. \tag{7}$$

Hence the trajectory lies on a cone whose half opening angle, $\frac{1}{2}\pi - \xi$, is given by

$$\sin\xi = \frac{eg}{J}\,. \tag{8}$$

The motion on the cone can be related to a simpler, central-force problem. Let us define a position vector $\vec{R}$ by

$$\vec{R} = \frac{\hat{J}\times(\vec{r}\times\hat{J})}{\cos\xi} = \frac{1}{\cos\xi}[\vec{r} - \hat{J}(\vec{r}\cdot\hat{J})]\,. \tag{9a}$$

Except for the scale factor $(\cos\xi)^{-1}$, $\vec{R}$ is the projection of $\vec{r}$ onto the plane perpendicular to $\vec{J}$. This scale factor is chosen so that $\vec{R}$ and $\vec{r}$ have the same length, for it follows from Eqs. (7), (8), and (9a) that

$$\vec{R}^2 = \vec{r}^2\,. \tag{10}$$

Thus, the new vector $\vec{R}$ is obtained by rotating $\vec{r}$ in the plane formed from $\vec{r}$ and $\vec{J}$ onto the plane perpendicular to $\vec{J}$. The mapping (9a) is easily inverted since according to Eqs. (7) and (8), $\hat{r}\cdot\hat{J} = \sin\xi$, and so

$$\vec{r} = \vec{R}\cos\xi + R\hat{J}\sin\xi\,. \tag{9b}$$

The geometry of this mapping is illustrated in Fig. 1.

Now as $\vec{r}$ moves on the cone, $\vec{R}$ moves on the plane perpendicular to $\vec{J}$, tracing out a path determined by Eqs. (4), (5), and (9a):

$$m\ddot{\vec{R}} = \frac{\hat{J}\times(\ddot{\vec{r}}\times\hat{J})}{\cos\xi} = \frac{eg}{mr^3}\frac{\hat{J}\times[(\vec{J}-eg\hat{r})\times\hat{J}]}{\cos\xi} = -\frac{e^2g^2}{mR^4}\vec{R}\,. \tag{11}$$

Thus $\vec{R}$ describes the position of a particle of mass m moving in the potential $V = -e^2g^2/2mR^2$. From Eqs. (5) and (9a) it is easy to show that the mechanical angular momentum of the projected motion is the same as the conserved total angular momentum of the motion in the monopole field,

$$m\vec{R}\times\dot{\vec{R}} = \vec{J}\,. \tag{12}$$

Moreover, the conserved energy of the projected motion is the same as the kinetic energy of the motion in the monopole field,

$$\tfrac{1}{2}m(\dot{\vec{R}})^2 - \frac{e^2g^2}{2mR^2} = \tfrac{1}{2}m\dot{R}^2 + \frac{J^2}{2mR^2} - \frac{e^2g^2}{2mR^2} = \tfrac{1}{2}mv^2 = E\,. \tag{13}$$

This follows immediately on comparing the constant energies for the two problems in the region where $|\vec{R}| = |\vec{r}|$ is large.

If the distance of closest approach of the particle to the monopole at the origin is b, then from Eq. (5)

$$J^2 = m^2v^2b^2 + e^2g^2\,. \tag{14}$$

On the other hand, since $|m\vec{r}\times\dot{\vec{r}}|^2$ and v^2 are constants of the motion, we can also identify b as the ordinary impact parameter of the motion in the monopole field.

We can now exploit the simple motion on the plane and the mapping (9b) to describe the more complex motion on the cone in the monopole problem. The planar orbit is obtained in the familiar manner by introducing a polar angle ψ and using

$$\dot{R} = \frac{dR}{d\psi}\dot{\psi} = \frac{dR}{d\psi}\frac{J}{mR^2}\,. \tag{15}$$

Inserting this into the energy equation (13) we find

$$\left(\frac{dR^{-1}}{d\psi}\right)^2 = \cos^2\xi(b^{-2} - R^{-2}) \tag{16}$$

and the integral

$$R = \frac{b}{\sin(\psi\cos\xi)}\,. \tag{17}$$

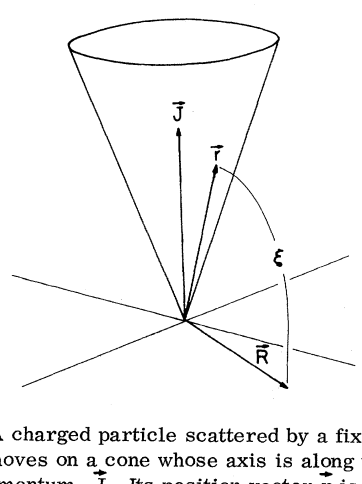

FIG. 1. A charged particle scattered by a fixed monopole moves on a cone whose axis is along the angular momentum, $\vec{J}$. Its position vector $\vec{r}$ is related by Eqs. (9a) and (9b) to a vector $\vec{R}$, of the same length as $\vec{r}$, but lying in the plane perpendicular to $\vec{J}$.

As R goes from $+\infty$ to b and back to $+\infty$, ψ changes by an amount

$$\Delta\psi = \frac{\pi}{\cos\xi} = \pi\left(1+\frac{e^2g^2}{m^2v^2b^2}\right)^{1/2}. \tag{18}$$

With $\cos\xi \lesssim \frac{1}{2}$, the orbit circles the origin. As $\cos\xi$ decreases further, more and more loops about the origin are made. A planar orbit and its mapping on the cone giving the orbit in the monopole field are shown in Fig. 2.

If the initial and final velocities in the plane are $\vec{V}_i$ and $\vec{V}_f$, then the corresponding velocities on the cone are given by

$$\vec{v}_i = \vec{V}_i\cos\xi - v\hat{J}\sin\xi \tag{19a}$$

and

$$\vec{v}_f = \vec{V}_f\cos\xi + v\hat{J}\sin\xi\,, \tag{19b}$$

where $v = |\vec{v}_i| = |\vec{v}_f| = |\vec{V}_i| = |\vec{V}_f|$. Thus the scattering angle θ for the monopole problem is determined by

$$\cos\theta = \hat{v}_i\cdot\hat{v}_f = -\cos^2\xi\cos\left(\frac{\pi}{\cos\xi}\right) - \sin^2\xi \tag{20}$$

or

$$\cos^2\frac{\theta}{2} = \cos^2\xi\sin^2\left(\frac{\pi}{2\cos\xi}\right), \tag{21}$$

where we have used

$$\hat{V}_f\cdot\hat{V}_i = -\cos\Delta\psi\,. \tag{22}$$

The scattering angle θ is shown as a function of ξ in Fig. 3. Scattering angles near $\theta=\pi$ are achieved

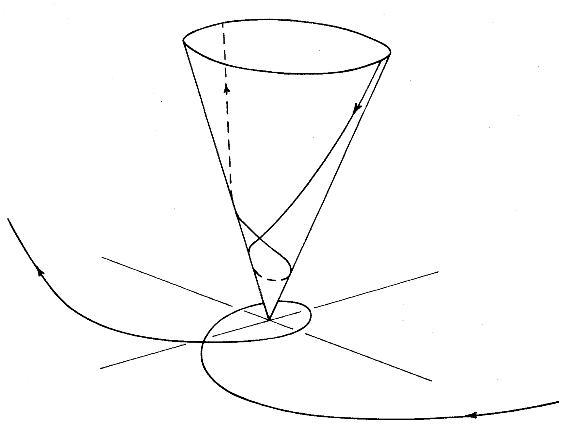

FIG. 2. While the position vector, $\vec{r}$, of the charged particle traces out its path on a cone, the related vector, $\vec{R}$, follows the path of a particle in an attractive inverse-square-law potential as shown.

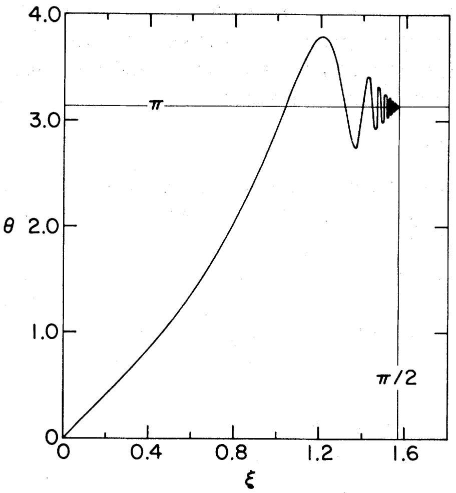

FIG. 3. The classical scattering angle, θ, for a charged particle in a pure monopole field as a function of ξ, defined by Eq. (8). As the cone becomes narrower, $\xi\to\frac{1}{2}\pi$ and the scattering angle oscillates about the backward direction.

for several values of ξ, corresponding to one encirclement of the origin, two encirclements, etc. As a result, the classical differential scattering cross section varies rapidly with many peaks near the backward direction. For small θ, however, we have from Eq. (19)

$$\theta \cong 2\xi \cong \frac{2eg}{mvb}\,. \tag{23}$$

At small angles an area of the incident beam $2\pi b\,db = 2\pi(2eg/\theta m v)^2(d\theta/\theta)$ will be scattered into a solid angle $d\Omega = 2\pi\theta\, d\theta$, giving a differential cross section

$$\theta \ll 1:\ \ \frac{d\sigma}{d\Omega} = \left(\frac{2eg}{mv}\right)^2\frac{1}{\theta^4}\,. \tag{24}$$

III. NON-ABELIAN GAUGE THEORY

Spontaneously broken non-Abelian gauge theories may support a classical solution which is asymptotically equivalent to a monopole magnetic field. In particular, 't Hooft[4] and Polyakov[5] have demonstrated this behavior for the system described by the Lagrange function

$$\mathfrak{L} = -\tfrac{1}{4}G_a^{\mu\nu}G_{a\mu\nu} - \tfrac{1}{2}(D_\mu h)_a^2 - \frac{\lambda}{4}(h_a^2 - f^2)^2\,. \tag{25}$$

Here $a = 1, 2, 3$ is an isotopic spin index, and

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\epsilon_{abc}A_b^\mu A_c^\nu\,, \tag{26}$$

$$(D_\mu h)_a = \partial_\mu h_a + e\epsilon_{abc}A_{b\mu}h_c\,. \tag{27}$$

In the solutions of interest the O(3) symmetry of the Lagrange function is spontaneously broken.[10] In the lowest mode (and in the tree approximation) some component of the scalar triplet, say h_3, has a constant, nonvanishing vacuum expectation value $\langle h_3\rangle = f$. The translated field $h_3' = h_3 - f$ describes a neutral scalar field of mass

$$\mu^2 = 2\lambda f^2 . \tag{28}$$

The remaining components $h_{1,2}$ are absorbed into the longitudinal components of the charged vector fields $A^\mu_{1,2}$ which have charge e and mass

$$m_{\mathrm{ch}}^2 = e^2 f^2 . \tag{29}$$

The neutral vector field A^μ_3 remains massless and may be identified with the photon.

The field equations which follow from (25) also admit a static classical solution with a structure that locks together the isotopic and physical spaces. In contrast to the earlier (Abelian) monopole structure of Dirac, this new solution appears to be a natural starting point for a true quantum-mechanical theory of monopoles. In particular, it is a classical (tree-approximation) solution to a well-defined dynamical system; the program required to find the quantum-mechanical corrections to this solution is well defined, if somewhat difficult technically.[11,12] Moreover, the stability of this solution under quantum fluctuations appears to be guaranteed by its topological character.[13,14] (This result, of course, assumes the nonexistence of any lower-energy, less-symmetrical static monopole solutions.) The study of such monopoles is closely related to the study of various topologically stable, special solutions ("lumps" or "solitary waves") in theories of one spatial dimension.

The electrically neutral monopole solution was extended by Julia and Zee[9] to the case of an electrically charged magnetic monopole. In this general case one has a static field structure, coupling physical and internal spaces, of the form

$$A^k_a = \epsilon_{akl}\, \frac{\hat{r}_l}{er}\,[1 - K(r)] , \tag{30}$$

$$A^0_a = \hat{r}_a N(r) , \tag{31}$$

and

$$h_a = \hat{r}_a f[1 - S(r)] . \tag{32}$$

The insertion of these forms into Eqs. (26) and (27) yields

$$G^{kl}_a = \frac{\epsilon_{klm}}{er}\left\{(\delta_{am} - \hat{r}_a\hat{r}_m)K'(r) + \frac{\hat{r}_a\hat{r}_m}{r}[K^2(r) - 1]\right\} , \tag{33}$$

$$G^{k0}_a = (\delta_{ak} - \hat{r}_a\hat{r}_k)\,\frac{K(r)N(r)}{r} + \hat{r}_a\hat{r}_k N'(r) , \tag{34}$$

$$(D^k h)_a = (\delta_{ak} - \hat{r}_a\hat{r}_k)\,\frac{K(r)f[1 - S(r)]}{r} - \hat{r}_a\hat{r}_k f S'(r) , \tag{35}$$

and

$$(D^0 h)_a = 0 . \tag{36}$$

Here the prime denotes a derivative with respect to r. Utilizing the orthogonality of the projection tensors $(\delta_{ak} - \hat{r}_a\hat{r}_k)$ and $\hat{r}_a\hat{r}_k$, it is a simple matter to compute the action and derive the field equations,

$$K'' = \frac{1}{r^2}K(K^2 - 1) + K[m_{\mathrm{ch}}^2(1 - S)^2 - N^2e^2] , \tag{37}$$

$$N'' + \frac{2}{r}N' = \frac{2}{r^2}NK^2 , \tag{38}$$

and

$$S'' + \frac{2}{r}S' = \frac{2K^2}{r^2}(S - 1) + \frac{\mu^2}{2}S(S - 1)(S - 2) . \tag{39}$$

The coupled, nonlinear differential equations (37)–(39) admit regular solutions that are everywhere finite. Near the origin the equations are compatible with the limits $K \to 1 + O(r^2)$, $N \to O(r)$, $S \to 1 + O(r)$ which ensure that the fields are analytic at $r = 0$. These equations also guarantee that, for finite-energy solutions with $\mu, m_{\mathrm{ch}} \neq 0$, the functions $K(r)$ and $S(r)$ vanish exponentially at large distances. To study these solutions in somewhat more detail consider the simplest case of an electrically neutral monopole where $N \equiv 0$ and the only parameters appearing in Eqs. (37) and (39) are the masses μ and m_{ch}. There are two extreme possibilities with respect to the two mass scales set by the mass of the Higgs scalar, μ, and by the mass of the charged vector meson, m_{ch}. The limit $\mu/m_{\mathrm{ch}} \to 0$ can be solved exactly[15] with the result that $K = m_{\mathrm{ch}}r/\sinh m_{\mathrm{ch}}r$ and $S = 1 + 1/m_{\mathrm{ch}}r - \coth m_{\mathrm{ch}}r$, which have the asymptotic forms $K \sim 2m_{\mathrm{ch}}re^{-m_{\mathrm{ch}}r}$ and $S \sim 1/(m_{\mathrm{ch}}r)$. In the opposite limit, $\mu \gg m_{\mathrm{ch}}$, it is straightforward to show that for $r > 1/m_{\mathrm{ch}}$ $K \sim Ae^{-m_{\mathrm{ch}}r}$ and $S \sim Be^{-2m_{\mathrm{ch}}r}/(m_{\mathrm{ch}}r)^2$. For intermediate values of the ratio μ/m_{ch}, the structure of the solutions is more complex but corresponds to a smooth variation between these two limits. In particular, it is clear that the scale of the distribution of the charged vector fields is given by m_{ch}.

Using the stress-energy tensor

$$T^{\mu\nu}=G_a^{\mu\lambda}G_{a\lambda}^{\nu}+(D^\mu h)_a(D^\nu h)_a+g^{\mu\nu}\mathfrak{L}\,, \tag{40}$$

the monopole mass

$$M_{\rm pole}=\int d^3r\, T^{00} \tag{41}$$

can be written in terms of the functions introduced above as

$$M_{\rm pole}=4\pi\int_0^\infty r^2dr\left[\frac{1}{e^2r^2}K'^2+\frac{1}{2e^2r^4}(K^2-1)^2+\tfrac{1}{2}N'^2+\frac{1}{r^2}K^2N^2+\frac{1}{2}\frac{m_{\rm ch}^{\;2}}{e^2}S'^2+\frac{1}{r^2}K^2\frac{m_{\rm ch}^{\;2}}{e^2}(1-S)^2+\frac{\mu^2m_{\rm ch}^{\;2}}{8e^2}S^2(2-S)^2\right]. \tag{42}$$

For the neutral case where N vanishes, the mass is of the general form ($\alpha=e^2/4\pi$)

$$M_{\rm pole}=\frac{m_{\rm ch}}{\alpha}f\left(\frac{\mu}{m_{\rm ch}}\right). \tag{43}$$

In the limit[15] $\mu=0$, $f(0)=1$. More complete studies for nonzero values of $\mu/m_{\rm ch}$ yield[4] the not-surprising result that $f(\mu/m_{\rm ch})$ is quite generally of order 1 as is appropriate for an essentially geometrical factor. When the monopole carries electrical charge, $N(r)$ behaves at large r ($r\gg 1/m_{\rm ch}$) as

$$N(r)\sim C-\frac{Q}{4\pi r}+O(e^{-2m_{\rm ch}r})\,, \tag{44}$$

where, as indicated in Eq. (50) below, Q is the electrical charge of the monopole. The change in the mass of the monopole due to the presence of the electrical charge is easily estimated to be of the form $\alpha(Q/e)^2m_{\rm ch}\tilde{f}(\mu/m_{\rm ch})$, where again $\tilde{f}$ is simply a geometrical factor of order 1. Hence the relative change in the mass of the monopole owing to adding the electrical charge is of order α^2 and can be safely ignored in our discussion of the charge-exchange process in Sec. VII.

[As a parenthetical remark, we should like to point out an interesting alternative expression for the monopole mass. Consider the new stress-energy tensor[16]

$$\tilde{T}^{\mu\nu}=T^{\mu\nu}+\tfrac{1}{6}(g^{\mu\nu}\partial^2-\partial^\mu\partial^\nu)h_a^{\;2}\,, \tag{45}$$

which yields the same Poincaré generators $(P^\mu, J^{\mu\nu})$ as those obtained from Eq. (40) and which is also conserved. It will, of course, give the same value for the monopole mass, but it has the additional virtue that its trace directly measures the dimensional quantities in the Lagrange function,

$$\tilde{T}^\mu{}_\mu=\lambda f^2(h_a^{\;2}-f^2)\,. \tag{46}$$

Since $\tilde{T}^{\mu\nu}$ is divergenceless and time independent, we have (the integration by parts yields no surface terms if $\lambda>0$)

$$\int d^3r\,\tilde{T}^k{}_k=\int d^3r\,\tilde{T}^{kl}\partial_k r_l =-\int d^3r\, r_l\,\partial_k\tilde{T}^{kl} =0\,. \tag{47}$$

It follows then that

$$M_{\rm pole}=\int d^3r\,\tilde{T}^{00} =-\int d^3r\,\tilde{T}^\mu{}_\mu =\int d^3r\,\lambda f^2(f^2-h_a^{\;2}) =\frac{m_{\rm ch}^{\;2}}{\alpha}\int r^2dr\,\frac{\mu^2}{2}(2S-S^2)\,. \tag{48}$$

This expresses the monopole mass in terms of the only fundamental constant which carries a dimension, f, and thus in terms of the deviation of the Higgs field h_a, from its asymptotic limit.]

In order to elucidate more fully the character of the classical solution introduced above and, in particular, to indicate why it is to be identified with a magnetic monopole, we note that the direction of the field h_a in isotopic spin space specifies the electromagnetic charge direction. Since h_a is proportional to $\hat{r}_a$, the charge axis in isotopic spin space is dependent on the position in physical space, always pointing in the local radial direction. Although one is not generally accustomed to thinking in terms of a spatially dependent charge direction, one can still define a field which asymptotically corresponds to the electromagnetic field. This may be identified simply as the component of $G_a^{\mu\nu}$ along the direction of h_a, $G_a^{\mu\nu}h_a/(h_bh_b)^{1/2}$. To see how this is related to the more familiar case of a fixed charge direction, one can uncouple the link between the physical and internal spaces by performing an appropriate coordinate-dependent local gauge transformation.[13] For example, one may choose the transformation which rotates $\hat{r}_a$ at each point in space into δ_{a3}, thereby transforming the fields $G_a^{\mu\nu}$ into the new fields $\overline{G}_a^{\mu\nu}$. In this case both h_a and the charge direction will be along the 3-direction. However, care must be taken in performing such a gauge rotation since the azimuthal angle ϕ, which plays an important role in the rotation, is undefined along the 3-axis. This can lead to the appearance of new singularities (related to the "strings" of the Dirac monopole) which we shall discuss further in Sec. IV and in Appendix B. Here it is sufficient to note that with proper care these rotations can be performed without introducing any singularities in the fields $\overline{G}_a^{\mu\nu}$ (although the vector field $\overline{A}_a^\mu$ will necessarily acquire singularities). However, as far as the neutral component of the field is concerned, the rotation need not be performed explicitly since $G_a^{\mu\nu}h_a/(h_bh_b)^{1/2}$ is a scalar. Thus the electromagnetic components of $\overline{G}_a^{\mu\nu}$ are, using Eqs. (26), (30), and (31) and the previously discussed asymptotic forms of $K(r)$ and $N(r)$,

$$\begin{aligned}\overline{G}_3^{kl} &= \frac{h_a}{(h_bh_b)^{1/2}}G_a^{kl} \\ &= \epsilon^{klm}\hat{r}_m\frac{K^2(r)-1}{er^2} \\ &\to -\frac{1}{e}\epsilon^{klm}\frac{\hat{r}_m}{r^2}\end{aligned} \tag{49}$$

and

$$\begin{aligned}\overline{G}_3^{k0} &= \frac{h_a}{(h_bh_b)^{1/2}}G_a^{k0} \\ &= \hat{r}_k N'(r) \\ &\to \frac{Q}{4\pi}\frac{\hat{r}_k}{r^2}\,.\end{aligned} \tag{50}$$

For $r \gg 1/m_{\rm ch}$ the fields exponentially approach those of an electrically charged magnetic monopole with electric charge Q and magnetic charge $-1/e$. At distances comparable to $1/m_{\rm ch}$ these neutral fields deviate from their asymptotic forms and the charged fields simultaneously begin to play a significant role. Note that the field-strength tensor (49) cannot be expressed as the curl of a vector potential in any connected region of coordinate space which separates the origin from infinity. This is clear at short distances where the fields deviate significantly from the simple monopole forms. It is also true at large distances where such a representation would require the presence of singular strings which are not present in these solutions.

In this regard we note that 't Hooft[4] defined the gauge-invariant quantity

$$\begin{aligned}F^{\mu\nu} = {}& \frac{1}{(h_dh_d)^{1/2}}h_aG_a^{\mu\nu} \\ & -\frac{1}{e}\frac{1}{(h_dh_d)^{3/2}}\epsilon_{abc}h_a(D^\mu h)_b(D^\nu h)_c\end{aligned} \tag{51}$$

to represent the "physically observable electromagnetic fields." Although the asymptotic limit of this expression is identical to that of Eq. (49), the two expressions differ markedly for small distance. Using the explicit expressions for A_a^μ and h_a in Eqs. (30) and (32) one finds that

$$F^{kl} = -\epsilon^{klj}\frac{\hat{r}_j}{er^2} \tag{52}$$

in *all* spatial regions. This result, which is divergenceless except at the origin, indicates the utility of the expression (51) as an indicator of the topological structure of the theory.[13] However, in the context of a unified theory of weak and electromagnetic interactions, the scattering of particles which are coupled to the monopole in a fully gauge-invariant fashion exhibits much richer physics at small impact parameter than indicated by (52). In particular, the charged vector fields play an essential role in the scattering process for particles whose distance of closest approach to the monopole is of the order of or less than the charged vector meson's Compton wavelength. The charged fields not only produce charge-exchange processes, they also modify the short-distance behavior of the noncharge-exchange reactions. Because of the unified nature of the underlying theory, a "purely electromagnetic" piece of the short-distance interaction cannot be isolated. Any attempt to separate the effects of the charged vector fields from those of the neutral vector field will not respect gauge invariance. Hence one must differentiate between the topological structure of the monopole solution and the dynamics arising from interactions with the monopole.

IV. PROBING MONOPOLE STRUCTURE WITH SPINLESS PARTICLES

The nature of the classical monopole solution is clarified by investigating the scattering of charged particles by the monopole.[17] In the simplest situation, the monopole is coupled gauge invariantly to a charged multiplet of scalar fields $\{\phi_a\}$ with a Lagrange function

$$\begin{aligned}\mathcal{L}_\phi = {}& -\tfrac{1}{2}(D^\mu\phi)^\dagger D_\mu\phi - \tfrac{1}{2}m^2\phi^\dagger\phi \\ & -\phi^\dagger[Ah_aT_a + B(h_aT_a)^2 + Ch_a^2]\,\phi\,.\end{aligned} \tag{53}$$

Here T_a are the Hermitian generators of isospin rotations on ϕ, and

$$D_\mu\phi = (\partial_\mu - ieT_aA_{a\mu})\phi \tag{54}$$

is the gauge-covariant derivative of ϕ. We shall treat the field ϕ as a small perturbation; thus self-interaction terms such as $(\phi^+\phi)^2$ are omitted in $\mathcal{L}_\phi$. Since the monopole mass is very large ($M_{\text{pole}} \approx m_{\text{ch}}/\alpha$), we can treat the monopole as a fixed scattering center providing an external field A_a^l, even for large momentum transfers on the order of m_{ch}. At large distances, the Higgs field h_a behaves as a constant times $\hat{r}_a$, and its interactions in (53) split the masses of the members of the multiplet according to their electrical charge (which is defined by $\hat{r}_a T_a$). We shall take account of this by tacitly assigning different mass values to the different members of the multiplet. Near the monopole, the Higgs field departs from its asymptotic form, and it will contribute to the deep scattering. In this section, however, we shall focus attention on the long-range forces, which are purely electromagnetic. Thus, we shall neglect all direct interactions with the Higgs field. Furthermore, we shall assume that the monopole carries no electrical charge so that $A_a^0 = 0$. In any case, the effects of the electrical charge of a monopole would be of order $e^2/(v/c)$ ($A_a^0 \propto e$) which are very small in comparison to the magnetic effects of order 1 ($\vec{A}_a \propto 1/e$).

Within the approximations we have just discussed, the wave equation for the scalar field is given by

$$\left[\left(\frac{1}{i}\vec{\nabla} - e\vec{A}_a T_a\right)^2 - k^2\right]\phi = 0\,, \tag{55}$$

where $k^2 = \omega^2 = m^2$ is the appropriate channel momentum involving the appropriate mass value. We now specialize further to the case of peripheral scattering ($|\Delta\vec{k}| \ll m_{\text{ch}}$), where the use of the asymptotic form

$$eA_a^k = \epsilon_{kla}\hat{r}_l \frac{1}{r} \tag{56}$$

suffices. (The nature of the deep scattering will be discussed later in Sec. VII.) The wave equation (55) is easily solved by introducing a total angular momentum operator which is the sum of the orbital angular momentum and the isotopic spin,

$$\vec{J} = \vec{r} \times \frac{1}{i}\vec{\nabla} + \vec{T}\,. \tag{57}$$

Using this operator and the vector potential (56), the wave equation (55) can be written in the form

$$\left\{-\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}[J^2 - (\hat{r}\cdot\vec{T})^2] - k^2\right\}\phi = 0\,. \tag{58}$$

(There is some similarity between the following analysis of the non-Abelian theory in terms of the operator $\vec{T}$ and the earlier work on the Abelian theory by Goldhaber,[18] who introduced a formal spin variable $\vec{S}$.) We can define a simultaneous eigenfunction $\phi_{lt}^{mn}(\hat{r})$ of the commuting operators $\vec{J}^2$, J_3, $\vec{T}^2$, and $\hat{r}\cdot\vec{T}$:

$$\begin{Bmatrix} \vec{J}^2 \\ J_3 \\ \vec{T}^2 \\ \hat{r}\cdot\vec{T} \end{Bmatrix} \phi_{lt}^{mn}(\hat{r}) = \begin{Bmatrix} l(l+1) \\ m \\ t(t+1) \\ n \end{Bmatrix} \phi_{lt}^{mn}(\hat{r})\,. \tag{59}$$

This function depends only upon the angular variables specified by $\hat{r}$. A solution of the wave equation, $\phi(\vec{r})$, can now be written as the product of the angular function $\phi_{lt}^{mn}(\hat{r})$ and a radial function. Inserting this form into Eq. (58) we may identify the radial function as a spherical Bessel function, and conclude that the partial-wave solutions are given by

$$\phi(\vec{r}) = \phi_{lt}^{mn}(\hat{r})\, j_{l'}(kr)\,, \tag{60}$$

where the index l' is the positive root of

$$l'(l'+1) = l(l+1) - n^2\,. \tag{61}$$

The construction of the angular eigenfunctions is facilitated by the use of a spatially dependent unitary matrix which rotates $\hat{r}\cdot\vec{T}$ into T_3:

$$U(-\phi, -\theta, \phi) = e^{-i\phi T_3} e^{-i\theta T_2} e^{i\phi T_3}\,. \tag{62}$$

Thus

$$\hat{r}\cdot\vec{T}\,U(-\phi, -\theta, \phi) = U(-\phi, -\theta, \phi)T_3\,. \tag{63}$$

Moreover, a little calculation using the rotation-group property of $\vec{T}$ verifies that

$$\vec{J}\,U(-\phi, -\theta, \phi) = U(-\phi, -\theta, \phi)\vec{\mathcal{J}}\,, \tag{64}$$

where

$$\vec{\mathcal{J}} = \vec{r} \times \left(\frac{1}{i}\vec{\nabla} - e\vec{\mathfrak{a}}\,T_3\right) + \hat{r}T_3\,, \tag{65}$$

with

$$e\vec{\mathfrak{a}} = \frac{\hat{r}\times\hat{z}}{r+z}\,. \tag{66}$$

We see that Eqs. (59) will be satisfied with

$$\phi_{lt}^{mn}(\hat{r}) = U(-\phi, -\theta, \phi)\chi_t^n \mathfrak{D}(\hat{r}) \tag{67}$$

if χ_t^n is a charge eigenvector,

$$T_3\chi_t^n = n\chi_t^n\,, \tag{68}$$

and if the function $\mathfrak{D}(\hat{r})$ obeys

$$\begin{Bmatrix} \mathcal{J}^2 \\ \mathcal{J}_3 \end{Bmatrix} \mathfrak{D}(\hat{r}) = \begin{Bmatrix} l(l+1) \\ m \end{Bmatrix} \mathfrak{D}(\hat{r})\,, \tag{69}$$

where T_3 in Eq. (65) is now replaced by its eigenvalue, n. Equations (69), when written in terms of the angular variables θ, ϕ, are the differential equations which define the representation functions of the rotation group. Thus we have[19]

$$\mathfrak{D}(\hat{r}) = \mathfrak{D}^{(l)}_{nm}(-\phi, \theta, \phi) = \langle l, n| e^{-i\phi J_3} e^{i\theta J_2} e^{i\phi J_3} |l, m\rangle . \tag{70}$$

In summary, a specific partial wave is given by

$$\phi(\vec{r}) = U(-\phi, -\theta, \phi)\chi^n_t \mathfrak{D}^{(l)}_{nm}(-\phi, \theta, \phi) j_{l'}(kr) . \tag{71}$$

The unitary matrix $U(-\phi, -\theta, \phi)$ is not well defined along the negative z axis. This is reflected in the "string" singularity along the negative z axis in the function $\vec{\mathfrak{a}}(\vec{r})$ [Eq. (66)]. It is also reflected in the function $\mathfrak{D}(\hat{r})$ which is not analytic along the negative z axis. However, the wave equation (58) is regular along the negative z axis and so are its solutions. There are compensating singularities in $U(-\phi, -\theta, \phi)$ and $\mathfrak{D}(\hat{r})$. This lack of singularity is explicitly exhibited by writing Eq. (67) in the form

$$\begin{aligned}\phi^{mn}_{lt}(\hat{r}) &= \sum_{n'} \chi^{n'}_t \mathfrak{D}^{(t)}_{n'n}(-\phi, -\theta, \phi)\mathfrak{D}^{(l)}_{nm}(-\phi, \theta, \phi) \\ &= \sum_{n'} \chi^{n'}_t (-1)^{n-n'} \mathfrak{D}^{(t)}_{-n,-n'}(-\phi, \theta, \phi) \\ &\quad \times \mathfrak{D}^{(l)}_{nm}(-\phi, \theta, \phi)\end{aligned} \tag{72}$$

and reducing the direct product by the Clebsch-Gordan series to obtain

$$\begin{aligned}\phi^{mn}_{lt}(\hat{r}) = \sum_{n'} \chi^{n'}_t (-1)^{n-n'} &\times \sum_j \langle t, -n, l, n | j0\rangle \\ &\times \mathfrak{D}^{(j)}_{0, m-n'}(-\phi, \theta, \phi) \\ &\times \langle j, m-n' | t, -n', l, m\rangle .\end{aligned} \tag{73}$$

The single representation function that appears in Eq. (73), $\mathfrak{D}^{(j)}_{0,m-n'}(-\phi, \theta, \phi)$, is proportional to the ordinary spherical harmonic $Y^{m-n'}_l(\theta, \phi)$, which is a finite polynomial in $x/r, y/r, z/r$, and is therefore analytic everywhere. (The amplitudes here, which are eigenfunctions of $\hat{r}\cdot\vec{T}$, are akin to the usual helicity amplitudes which are eigenfunctions of $\hat{k}\cdot\vec{S}$.)

The unitary matrix $U(-\phi, -\theta, \phi)$ may be viewed as a spatially dependent gauge rotation which transforms the original spatially dependent charge direction in isotopic spin space $\hat{r}_a$ into the constant direction δ_{a3}. It is instructive to consider the effect of this gauge transformation

$$\phi(\vec{r}) = U(-\phi, -\theta, \phi)\psi(\vec{r}) \tag{74}$$

in more detail. Using Eqs. (63) and (64), the gauge transformation of the wave equation in the version of Eq. (58) is immediate,

$$\left[-\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}(\mathcal{J}^2 - T_3^2) - k^2\right]\psi = 0 . \tag{75}$$

Inserting the explicit form of the operator $\vec{\mathcal{J}}$ [Eqs. (65) and (66)], reverting to Cartesian coordinates, and assuming that ψ diagonalizes T_3 with eigenvalue n gives

$$\left[\left(\frac{1}{i}\vec{\nabla} - \frac{n\hat{r}\times\hat{z}}{r+z}\right)^2 - k^2\right]\psi(\vec{r}) = 0 . \tag{76}$$

This is the wave equation for a charged particle in the magnetic field of a monopole of the kind envisaged by Dirac.[1] Moreover, we now see that the angular momentum operator $\vec{\mathcal{J}}$ [Eq. (65)] is the sum of the particle's mechanical angular momentum and the angular momentum in the electromagnetic field. We should emphasize, however, that this correspondence holds only in the asymptotic region $r \gg m_{\rm ch}^{-1}$ where Eq. (56) is valid.

In deriving the wave equation (76), we have used a singular gauge transformation, Eq. (62). It is the singular structure of this transformation which has introduced the "string" singularity into the vector potential of Eq. (76). The character of such singular gauge transformations is explored in some detail in Appendix B.

V. CLASSICAL LIMIT

Here we briefly discuss the manner in which the quantum-mechanical motion is connected with the classical motion outlined in Sec. II. For this purpose, we shall consider the simplest case of the nonrelativistic limit of the wave equation which, in the gauge where $h_a \propto \delta_{a3}$, is the ordinary Schrödinger equation with a monopole potential. Thus, we shall consider time-dependent wave functions of the form

$$\Psi(\vec{r}, t) = \sum_{l,m}\int_0^\infty dk\, F^m_l(k) e^{-i(k^2/2m)t} \times \mathfrak{D}^{(l)}_{n,m}(-\phi, \theta, \phi) j_{l'}(kr) . \tag{77}$$

Suppose that the quantum wave packet (77) corresponds to the motion of a classical particle whose total angular momentum lies along the positive z axis. This wave packet will have $\mathcal{J}_3$ eigenvalues in a small range about $m = l$. It will involve essentially the angular function

$$\mathfrak{D}^{(l)}_{n,l}(-\phi, \theta, \phi) \propto e^{i(l-n)\phi}\frac{\sin^{l+n}\theta}{(1-\cos\theta)^n} . \tag{78}$$

The modulus of this function is at a maximum on the cone where $\theta = \theta_m$, with

$$\cos\theta_m = \frac{n}{l}. \tag{79}$$

Remembering that l denotes the total angular momentum, we see that $\theta_m = (\frac{1}{2}\pi - \xi)$ is precisely the half-angle of the cone of the classical motion, Eq. (8). Thus the quantum wave packet is indeed concentrated on the surface of the cone of the classical motion. For large l we have

$$l \gg 1: \ \mathfrak{D}^{(l)}_{n,l}(-\phi, \theta, \phi) \propto e^{i(l-n)\phi} e^{-l(\theta-\theta_m)^2/2}. \tag{80}$$

[With $n=0$ the familiar description of the planar motion of a particle in a central force is recovered with an angular function $Y^l_l(\theta, \phi)$.]

We can make further contact with the classical limit. Let us define

$$\mathcal{Y}^m_l(\theta, \phi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} \mathfrak{D}^{(l)}_{n,m}(-\phi, \theta, \phi) \tag{81}$$

so that the angular function $\mathcal{Y}^m_l(\theta, \phi)$ has the same normalization as does the spherical harmonic $Y^m_l(\theta, \phi)$ of central force problems,[19]

$$\int d\Omega\, \mathcal{Y}^{m'}_{l'}(\theta, \phi)^* \mathcal{Y}^m_l(\theta, \phi) = \int d\Omega\, Y^{m'}_{l'}(\theta, \phi)^* Y^m_l(\theta, \phi) = \delta^{m'm}\delta_{l'l}. \tag{82}$$

The matrix element

$$\int d\Omega\, \mathcal{Y}^{m_1}_{l_1}{}^* \vec{r}\, \mathcal{Y}^{m_2}_{l_2}$$

vanishes unless $l_1 = l_2$ or $l_1 = l_2 \pm 1$. We write $\vec{r}$ in terms of the spherical harmonics $Y^{m_3}_1$ and compute the angular integrals in terms of 3-j symbols to establish the relationship

$$\int d\Omega\, \mathcal{Y}^{m_1}_{l_1}{}^* \vec{r}\, \mathcal{Y}^{m_2}_{l_2} = \int d\Omega\, Y^{m_1}_{l_1}{}^* (\vec{r}\, c_{l_1 l_2} + \vec{L} r d_{l_1 l_2}) Y^{m_2}_{l_2}, \tag{83}$$

where

$$c_{l_1 l_2} = 0, \quad l_1 \neq l_2 \pm 1 \tag{84a}$$

$$c_{l+1,l} = c_{l,l+1} = \left[1 - \frac{n^2}{(l+1)^2}\right]^{1/2}, \tag{84b}$$

$$d_{l_1 l_2} = 0, \quad l_1 \neq l_2 \tag{84c}$$

$$d_{ll} = \frac{n}{l(l+1)}. \tag{84d}$$

This relationship is the quantum-mechanical analog of the classical projection (9b). The motion in the monopole field described by a wave packet constructed from the functions $j_l \mathcal{Y}^m_l$ can be projected onto the central-n^2/r^2 potential problem by replacing the $\mathcal{Y}^m_l$ functions with the ordinary spherical harmonics Y^m_l. Then, assuming that the l values are sufficiently large so that the noncommutativity of the angular momentum operator $\vec{L}$ and $\vec{r}$ can be neglected and that l can be replaced by $l+1$, we have the following relationship between the expectation values in the two wave packets:

$$\langle \vec{r}(t) \rangle_{\text{monopole}} = \left\langle \vec{r}(t) \left(1 - \frac{n^2}{L^2}\right)^{1/2} + nr(t)\frac{\vec{L}}{L^2} \right\rangle_{-n^2/r^2 \text{ potential}} \tag{85}$$

This is indeed the quantum-mechanical analog of the projection (9b).

VI. QUANTUM-MECHANICAL SCATTERING

From the partial-wave solutions to the wave equation [Eqs. (58) and (75)] we can construct scattering solutions, solutions which at large distances from the origin consist of a plane wave plus an outgoing spherical wave. If the particle enters from $z=-\infty$, the z component of its total angular momentum is given by $J'_3 = T'_3 = -(\vec{T}\cdot\hat{r})' = -n$. The appropriate angular eigenfunctions are $\phi^{-nn}_{lt}(\hat{r})$. We shall take $n = eg$ to be positive to make the notation simple since the reflection $n \to -n$ corresponds to a spatial inversion and just produces an overall phase change [cf. Eq. (A2)]. Guided by this angular momentum information and by the familiar expansion of e^{ikz} in spherical Bessel functions, we consider the partial-wave sum

$$\phi^{(+)}(\vec{r}) = e^{-i\pi n} \sum_{l=n}^{\infty} (2l+1) e^{i\pi l} e^{-i\pi l'/2} j_{l'}(kr) \times [U(-\phi, -\theta, \phi)\chi^n_t] \times \mathfrak{D}^{(l)}_{n,-n}(-\phi, \theta, \phi). \tag{86}$$

The factor in square brackets represents the charge eigenstate which is not altered in the peripheral, elastic scattering described by this wave function. The remaining factor is the physically significant wave function

$$\psi^{(+)}(\vec{r}) = e^{-i\pi n} \sum_{l=n}^{\infty} (2l+1) e^{i\pi l} e^{-i\pi l'/2} \times j_{l'}(kr) e^{-2in\phi} d^{(l)}_{n,-n}(\theta). \tag{87}$$

[The matrix $U(-\phi, \theta, \phi)$ transforms J_3 into $(1/i)(\partial/\partial\phi) + n$. Since J_3 takes on the value $-n$, this accounts for the factor $e^{-2in\phi}$ in Eq. (87).]

The wave function in Eq. (87) is not mathematically well-behaved near the forward direction $\theta = 0$: The limits $r\to\infty$ and $\theta\to 0$ are not uniform and cannot be interchanged. The wave function vanishes in the exact forward direction $\theta = 0$, for all distances r, but it is sharply peaked near the forward direction at large distances. It is convenient to decompose the wave function into two parts

$$\psi^{(+)}(\vec{r}) = e^{-2in\phi}[\psi_{\rm I}(\vec{r}) + \psi_{\rm II}(\vec{r})], \tag{88}$$

where

$$\psi_{\rm I}(\vec{r}) = e^{-in\pi} \sum_{l=n}^{\infty} (2l+1) e^{i\pi l'/2} j_l(kr) d_{n,-n}^{(l)}(\theta) \tag{89a}$$

contains the most singular behavior while the sum

$$\begin{aligned}\psi_{\rm II}(\vec{r}) = e^{-in\pi} \sum_{l=n}^{\infty} (2l+1) e^{i\pi l} \\ \times [e^{-i\pi l'/2} j_{l'}(kr) - e^{i\pi l/2} j_l(kr)] \\ \times d_{n,-n}^{(l)}(\theta)\end{aligned} \tag{89b}$$

converges faster and is better behaved. We show in Appendix A that $\psi_{\rm I}(\vec{r})$ can be written in closed form [Eq. (A19)],

$$\begin{aligned}\psi_{\rm I}(\vec{r}) = e^{-i\pi n/2} e^{ik(r+z)/2} \tfrac{1}{2} k(r-z) \\ \times [j_{n-1}(\tfrac{1}{2}k(r-z)) - i j_n(\tfrac{1}{2}k(r-z))].\end{aligned} \tag{90}$$

In the limit when $k(r-z) = kr(1-\cos\theta)$ becomes large, this function describes an incident plane wave plus an outgoing spherical wave [Eq. (A21)],

$kr(1-\cos\theta) \to \infty$:

$$\psi_{\rm I}(\vec{r}) \approx e^{ikz} - \frac{ine^{-in\pi}}{k(1-\cos\theta)} \frac{1}{r} e^{ikr}. \tag{91}$$

This proves that the wave function $\psi^{(+)}(\hat{r})$ is indeed the correct scattering wave function since the $\psi_{\rm II}(\hat{r})$ piece contains only outgoing waves. Note that the absolute square of the outgoing wave amplitude in Eq. (91), $[n/k(1-\cos\theta)]^2$, is just the classical cross section [Eq. (24)] in the small-scattering-angle limit.

When n is an integer, the closed form (90) for $\psi_{\rm I}(\vec{r})$ involves a finite sum of elementary functions. In particular, when $n=1$ we have

$$[\psi_{\rm I}(\vec{r})]_{n=1} = e^{ikz}\left[1 - \frac{i}{k(r-z)}\right] + \frac{ie^{ikr}}{k(r-z)}. \tag{92}$$

This result exhibits clearly the nonuniform behavior of the wave function as kr becomes large and θ becomes small. We see that the asymptotic scattering description is valid near the forward direction only for distances that are large compared to $(k\theta^2)^{-1}$.

The asymptotic value of the $\psi_{\rm II}$ piece of the wave function follows immediately from the asymptotic limit of the spherical Bessel function,

$kr(1-\cos\theta) \to \infty$:

$$\begin{aligned}\psi_{\rm II}(\vec{r}) \approx e^{-i\pi n} \frac{1}{2ik} \\ \times \sum_{l=n}^{\infty} (2l+1)[e^{-i\pi(l'-l)} - 1] d_{n,-n}^{(l)}(\theta) \frac{e^{ikr}}{r}.\end{aligned} \tag{93}$$

Combining the asymptotic values of $\psi_{\rm I}(\vec{r})$ and $\psi_{\rm II}(\vec{r})$ gives

$$\psi^{(+)}(\vec{r}) \approx e^{-2in\phi}\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right], \tag{94}$$

with the scattering amplitude $f(\theta)$ identified as

$$\begin{aligned}f(\theta) = \frac{e^{-in\pi}}{2ik}\Bigg[\frac{n}{\sin^2(\frac{1}{2}\theta)} \\ + \sum_{l=n}^{\infty} (2l+1)(e^{-i\pi(l'-l)} - 1) d_{n,-n}^{(l)}(\theta)\Bigg].\end{aligned} \tag{95}$$

The sum appearing here, which is the contribution from $\psi_{\rm II}$, is not uniformly convergent as θ approaches 0. It produces a singularity in the forward direction which is weaker than that of the $1/\sin^2(\frac{1}{2}\theta)$ term which was isolated in the function $\psi_{\rm I}(\vec{r})$. The convergence of the l sum can be further improved by using the completeness of the $d_{n,-n}^{(l)}(\theta)$ to write

$$\begin{aligned}\sin^{\nu}\frac{\theta}{2} = \left[\frac{1-\cos\theta}{2}\right]^{\nu/2} \\ = \sum_{l=n}^{\infty} a_l d_{n,-n}^{(l)}(\theta).\end{aligned} \tag{96}$$

We use the orthogonality relation[19]

$$\int_{-1}^{1} d(\cos\theta) d_{n,-n}^{(l_1)}(\theta) d_{n,-n}^{(l_2)}(\theta) = \frac{2}{2l_1+1}\delta_{l_1,l_2}, \tag{97}$$

and the generalized Rodriguez formula[19]

$$\begin{aligned}d_{n,-n}^{(l)}(\theta) = \frac{(-1)^{l-n}}{2^l(l-n)!}(1-\cos\theta)^{-n} \\ \times \left(\frac{d}{d\cos\theta}\right)^{l-n}[(1-\cos\theta)^{l+n}(1+\cos\theta)^{l-n}],\end{aligned} \tag{98}$$

and then integrate by parts $l-n$ times to obtain

$$\begin{aligned}\frac{2}{2l+1}a_l &= \frac{1}{2^{l+\nu/2}(l-n)!}\int_{-1}^{1} d(\cos\theta)[(1-\cos\theta)^{l+n}(1+\cos\theta)^{l-n}]\left(\frac{d}{d\cos\theta}\right)^{l-n}(1-\cos\theta)^{-(n-\nu/2)} \\ &= \frac{1}{2^{l+\nu/2}(l-n)!}\frac{\Gamma(l-\frac{1}{2}\nu)}{\Gamma(n-\frac{1}{2}\nu)}\int_{-1}^{1} d(\cos\theta)(1-\cos\theta)^{n+\nu/2}(1+\cos\theta)^{l-n}.\end{aligned} \tag{99}$$

The trivial variable change $1+\cos\theta=2t$ puts the integral in the form of the standard integral representation for the beta function, and we find that

$$a_l=(2l+1)\frac{\Gamma(n+\frac{1}{2}\nu+1)}{\Gamma(n-\frac{1}{2}\nu)}\frac{\Gamma(l-\frac{1}{2}\nu)}{\Gamma(l+\frac{1}{2}\nu+2)}. \tag{100}$$

Setting $\nu=-1,0,1$ and using Eqs. (96) and (100) gives the sums

$$\left\{\begin{matrix}\dfrac{1}{\sin(\frac{1}{2}\theta)}\\ 1\\ \sin(\frac{1}{2}\theta)\end{matrix}\right\}=\sum_{l=n}^{\infty}(2l+1)\left\{\begin{matrix}\dfrac{1}{l+\frac{1}{2}}\\ \dfrac{n}{l(l+1)}\\ \dfrac{n^2-\frac{1}{4}}{(l+\frac{3}{2})(l^2-\frac{1}{4})}\end{matrix}\right\}d^{(l)}_{n,-n}(\theta). \tag{101}$$

We now use Eqs. (101) to remove the first three terms in the $n^2/(2l+1)$ expansion of the exponential of the phase shift $-\frac{1}{2}\pi(l'-l)=-\pi\{[(l+\frac{1}{2})^2-n^2]^{1/2}-(l+\frac{1}{2})\}$ in the partial-wave sum (95) and secure

$$e^{i n\pi}f(\theta)=\frac{-in}{2k\sin^2(\frac{1}{2}\theta)}\left\{1+\left(\frac{i\pi n}{2}\sin\frac{\theta}{2}\right)+\frac{1}{2}\left(\frac{i\pi n}{2}\sin\frac{\theta}{2}\right)^2+\frac{1}{6}\left(\frac{i\pi n}{2}\sin\frac{\theta}{2}\right)^3-\frac{1-\frac{1}{24}\pi^2}{n^2-\frac{1}{4}}\left(\frac{i\pi n}{2}\sin\frac{\theta}{2}\right)^3\right\}$$
$$+\frac{1}{2ik}\sum_{l=n}^{\infty}\left[(2l+1)(e^{-i\pi(l'-l)}-1)-i\pi n^2+\frac{\pi^2n^4}{8}\frac{2l+1}{l(l+1)}-\frac{i\pi n^4}{4}\frac{(1-\frac{1}{6}n^2\pi^2)}{(l-\frac{1}{2})(l+\frac{3}{2})}\right]d^{(l)}_{n,-n}(\theta). \tag{102}$$

(If $n=\frac{1}{2}$ we can subtract only the first two terms and not the first three terms as we have done.) The factor in square brackets now behaves as n^8/l^3 for large l. This ensures that the sum converges rapidly if n is small. For $n=1$ the sum is in fact negligible. The situation when n is large requires a different treatment. When n is large we have essentially a classical limit and the partial-wave sum can be evaluated by a stationary phase method.[20] (While completing this work, we received a report by Schwinger *et al.*[21] which contains extensive evaluations of the scattering cross section, to which we refer the reader.)

In the following section, we shall require the scattering solution for a wave incident along an arbitrary direction. This is most easily obtained by rotating our previous scattering solution which has the wave incident along the negative z axis. Such a rotation must not alter the wave equation which couples the physical and isospin spaces. Hence, the appropriate infinitesimal rotation operators are the total angular momentum operators of Eq. (57), and a finite rotation characterized by the Euler angles (α,β,γ) is given by

$$R(\alpha,\beta,\gamma)=e^{i\alpha J_3}e^{i\beta J_2}e^{i\gamma J_3}. \tag{103}$$

Under this rotation, the angular eigenfunction $\phi^{mn}_{lt}(\hat{r})$ becomes

$$\begin{aligned}R(\alpha,\beta,\gamma)\,\phi^{mn}_{lt}&=\sum_{m'}\phi^{m'n}_{lt}(\hat{r})\mathfrak{D}^{(l)}_{m'm}(\alpha,\beta,\gamma)\\&=\sum_{m'}[U(-\phi,-\theta,\phi)\chi^n_t]\mathfrak{D}^{(l)}_{nm'}(-\phi,\theta,\phi)\\&\quad\times\mathfrak{D}^{(l)}_{m'm}(\alpha,\beta,\gamma)\\&=\sum_{m'}[U(-\phi,-\theta,\phi)\chi^n_t]\mathfrak{D}^{(l)}_{nm}(\Psi,\Theta,\Phi),\end{aligned} \tag{104}$$

where (Ψ,Θ,Φ) denote the Euler angles of the combined rotations $(-\phi,\theta,\phi)$ and (α,β,γ). In order to rotate the wave vector $k\hat{z}$ into an arbitrary direction

$$k\hat{z}\to\vec{k}=k(\hat{z}\cos\theta_k+\hat{x}\sin\theta_k\cos\phi_k+\hat{y}\sin\theta_k\sin\phi_k), \tag{105}$$

we set $(\alpha,\beta,\gamma)=(-\phi_k,-\theta_k,\phi_k)$, choosing $\alpha=-\gamma$ so that the rotation continuously approaches the identity when the polar angle θ_k vanishes. The Eulerian angles for the combined rotation may be read off from the explicit spin-$\frac{1}{2}$ representation of the rotation group with the result that

$$\begin{aligned}\cos\Theta&=\cos\theta\cos\theta_k+\cos(\phi-\phi_k)\sin\theta\sin\theta_k\\&=\hat{r}\cdot\hat{k},\end{aligned} \tag{106a}$$

$$e^{i(\Psi-\Phi)}=\frac{e^{-i\phi_k}\cos(\frac{1}{2}\theta)\sin(\frac{1}{2}\theta_k)-e^{-i\phi}\sin(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta_k)}{e^{i\phi_k}\cos(\frac{1}{2}\theta)\sin(\frac{1}{2}\theta_k)-e^{i\phi}\sin(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta_k)}, \tag{106b}$$

and

$$e^{i(\Psi+\Phi)}=\frac{\cos(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta_k)+e^{i(\phi_k-\phi)}\sin(\frac{1}{2}\theta)\sin(\frac{1}{2}\theta_k)}{\cos(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta_k)+e^{-i(\phi_k-\phi)}\sin(\frac{1}{2}\theta)\sin(\frac{1}{2}\theta_k)}. \tag{106c}$$

The scattering solution, Eq. (86), is now rotated into

$$\phi_{\vec{k}}^{(+)}(\vec{r})=e^{-i\pi n}\sum_{l=n}^{\infty}(2l+1)e^{i\pi l}e^{-i\pi l'/2}j_{l'}(kr)\times[U(-\phi,-\theta,\phi)\chi_t^n]\times\sum_{m'}\mathfrak{D}_{n,m'}^{(l)}(-\phi,\theta,\phi)\mathfrak{D}_{m',-n}^{(l)}(-\phi_k,-\theta_k,\phi_k). \tag{107}$$

It is instructive to examine the manner in which this rotation acts in the Dirac monopole theory. We recall that the gauge transformation which sends $\hat{r}_a$ into δ_{a3},

$$\phi(\vec{r})=U(-\phi,-\theta,\phi)\psi(\vec{r}), \tag{74}$$

transforms $\phi(\vec{r})$ into the function $\psi(\vec{r})$ which is, except for the isospinor factor χ_t^n, the wave function for the Dirac monopole. We have effectively made this transformation in passing from the wave function $\phi^{(+)}(\vec{r})$ of Eq. (86) to the wave function $\psi^{(+)}(\vec{r})$ of Eq. (87). Thus, the rotated Dirac monopole scattering wave function is given by

$$\psi_k^{(+)}(\vec{r})=e^{-i\pi n}\sum_{l=n}^{\infty}(2l+1)e^{i\pi l}e^{-i\pi l'/2}j_{l'}(kr)\times\mathfrak{D}_{n,-n}^{(l)}(\Psi,\Theta,\Phi)$$
$$=e^{in(\Psi-\Phi)}\left[e^{-i\pi n}\sum_{l=n}^{\infty}(2l+1)e^{i\pi l}e^{-i\pi l'/2}\times j_{l'}(kr)d_{n,-n}^{(l)}(\Theta)\right]. \tag{108}$$

Aside from a phase factor, the quantity which appears here in the square brackets is just the wave function $\psi^{(+)}(\vec{r})$ of Eq. (87) with the angle θ defined by $\cos\theta=\hat{z}\cdot\hat{r}$ replaced with the angle Θ defined by $\cos\Theta=\hat{k}\cdot\hat{r}$. This part corresponds to the result of a simple geometrical rotation $\hat{z}\to\hat{k}$. The phase factor $e^{-2in\phi}$ representing the total angular momentum $\mathcal{J}_3'=-n$ is now replaced by $e^{in(\Psi-\Phi)}$. By rotating the wave function in the gauge with $h_a\propto\hat{r}_a$ and then extracting the wave function in the gauge where $h_a\propto\delta_{a3}$, we have obtained a solution which is equivalent in the Dirac picture to rotating the wave function and the vector potential by a redefinition of variables, and then returning the "string" singularity in the vector potential to its original orientation by a gauge transformation. This is exhibited explicitly by decomposing the phase factor in Eq. (108) into two pieces

$$e^{in(\Psi-\Phi)}=e^{-2in\alpha}e^{-in\Omega}. \tag{109}$$

The angle α is the azimuthal angle defined relative to the rotated incident wave direction $\vec{k}$. Thus, the phase factor $e^{-2in\alpha}$ represents the angular momentum $(\hat{k}\cdot\vec{\mathcal{J}})'=-n$ appropriate to the altered incident wave direction. The second phase factor $e^{-in\Omega}$ is a gauge transformation which moves the "string" singularity along the line $r=\vec{r}\cdot\hat{k}$ in the factor $e^{-2in\alpha}$ back to its original position along the negative z axis. (The gauge function Ω is the solid angle subtended by the planar surface bordered by the two singularity lines as viewed from the position $\vec{r}$.[22]) Since this overall phase has no physical significance, we have demonstrated explicitly that the orientation of the Dirac string is not observable in scattering experiments.

VII. DEEP SCATTERING

So far we have concentrated on the peripheral scattering of a charged particle, where the potential is just that of a pure magnetic monopole. However, in non-Abelian gauge theory, the exact classical solution for the vector potential deviates from that of a pure magnetic monopole at short distances characterized by the Compton wavelength of the heavy charged vector bosons. In fact, the vector potential approaches zero at the origin, thus providing a finite total energy for the monopole system in the tree approximation. Hence, if the patticles of the charged field ϕ scatter deeply on the non-Abelian monopole, we must use the complete vector potential

$$A_a^k=\epsilon_{akl}\frac{\hat{r}_l}{er}[1-K(r)]. \tag{30}$$

This vector potential describes charged as well as neutral fields, and the particle's charge is no longer conserved in the scattering process. Since charge is conserved in the total system, deep scattering will induce charge-exchange reactions between the monopole system and the scattered particle. The perturbation of the monopole system by the charge exchange gives rise to two small effects: The mass of a charged monopole state is larger than that of the uncharged state by an amount of relative order α^2. This alters the energy balance of the scattered particle, but by a negligible amount. The electrical charge of the final monopole state also gives rise to a Coulomb final-state interaction which alters the scattering amplitude, but again by a negligible amount of order $\alpha/(v/c)$ relative to the magnetic final-state interaction. We note that deep scattering, where the momentum transfer Δk is on the order of $m_{\rm ch}$ but where $\Delta k \ll M_{\rm pole} \sim m_{\rm ch}/\alpha$, is consistent with the approximation of a fixed scattering center (the neglect of recoil and retardation effects). Thus we can describe the deep charge-exchange scattering by the wave equation (55) which, with the full vector potential (30), can be expressed in the form

$$\left\{-\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{1}{r^2}[\vec{\mathrm{J}}^2-(\hat{r}\cdot\vec{\mathrm{T}})^2]+V_2-k^2\right\}\phi(\vec{\mathrm{r}})=0\,, \tag{110}$$

with

$$V_2=\frac{1}{r^2}\{-2K[\vec{\mathrm{T}}\cdot\vec{\mathrm{J}}-(\hat{r}\cdot\vec{\mathrm{T}})^2]+K^2[T^2-(\hat{r}\cdot\vec{\mathrm{T}})^2]\}. \tag{111}$$

We are interested in the term $-(2K/r^2)[\vec{\mathrm{T}}\cdot\vec{\mathrm{J}}-(\hat{r}\cdot\vec{\mathrm{T}})^2]$ in Eq. (111) that induces charge-exchange reactions. (The other term in V_2 will modify the deep-scattering amplitude for the charge preserving processes.) These processes are also modified by effects from self-coupling of the charged scalar field and by its interactions with the Higgs field h_a. Since the direct-coupling strength of this latter process is not fixed *a priori*, we shall concentrate on the charge-exchange process whose structure is fixed by the original non-Abelian theory.

We can use the distorted-wave Born approximation to find the charge-exchange amplitude by the deviation term, $-(2K/r^2)[\vec{\mathrm{T}}\cdot\vec{\mathrm{J}}-(\hat{r}\cdot\vec{\mathrm{T}})^2]$. In this approximation, the charge-exchange amplitude is given by

$$f(\vec{\mathrm{k}}_f,n_f;\vec{\mathrm{k}}_i,n_i)_{\text{charge exchange}}=-\frac{1}{4\pi}\left(\phi^{(-)}_{\vec{\mathrm{k}}_f,n_f},\left\{-\frac{2K}{r^2}[\vec{\mathrm{T}}\cdot\vec{\mathrm{J}}-(\hat{r}\cdot\vec{\mathrm{T}})^2]\right\}\phi^{(+)}_{\vec{k}_i,n_i}\right), \tag{112}$$

where $\vec{\mathrm{k}}_{i(f)}$ and $n_{i(f)}$, are the initial (final) momentum and charge. Since the mass of the scattered particle may depend upon its electrical charge (through its coupling to the asymptotic Higgs field $h_a\to f\hat{r}_a$), the initial and final momenta may have different magnitude. The wave function $\phi^{(+)}$ asymptotically approaches a plane wave plus an outgoing spherical wave, while the wave function $\phi^{(-)}$ asymptotically approaches a plane wave plus an incoming spherical wave. Taking $\vec{\mathrm{k}}_i=k_i\hat{z}$, we have

$$\phi^{(+)}_{k;\hat{z},n_i}=e^{-i\pi n_i}\sum_{l=|n_i|}^{\infty}(2l+1)e^{i\pi(l-l_i'/2)}j_{l_i'}(k_ir)\times\phi^{-n_i n_i}_{lt}(\hat{r})\,, \tag{113}$$

where the index l_i' is the positive root of

$$l_i'(l_i'+1)=l(l+1)-n_i^{\,2}. \tag{114}$$

A wave function asymptotically describing a plane wave traveling up the z axis plus an incoming spherical wave is easily constructed using the methods of the previous section. We transform this wave function by a rotation that sends $\hat{z}$ into $\hat{\mathrm{k}}_f$ to obtain the final-state wave function $\phi^{(-)}_{\vec{\mathrm{k}}_f,n_f}$. With

$$\vec{\mathrm{k}}_f=k_f(\cos\theta_f\hat{z}+\sin\theta_f\cos\phi\hat{x}+\sin\theta_f\sin\phi_f\hat{y}) \tag{115}$$

we have

$$\begin{aligned}\phi^{(-)}_{\vec{\mathrm{k}}_f,n_f}&=e^{-i\phi_fJ_3}e^{-i\theta_fJ_2}e^{i\phi_fJ_3}\times\sum_{l=|n_f|}^{\infty}(2l+1)e^{i\pi l_f'/2}j_{l_f'}(k_fr)\phi^{n_fn_f}_{lt}(\hat{r})\\&=\sum_{l,m}(2l+1)e^{i\pi l_f'/2}j_{l_f'}(k_fr)\times\mathfrak{D}^{(l)}_{mn_f}(-\phi_f,-\theta_f,\phi_f)\phi^{mn_f}_{lt}(\hat{r})\,,\end{aligned} \tag{116}$$

where the index l_f' satisfies an equation analogous to Eq. (114) with n_i replaced by n_f. In Eq. (116), the phase factor $e^{i\pi l_f'/2}$ has been chosen to ensure the proper asymptotic behavior, the plane wave plus an incoming spherical wave. Since the total angular momentum $\vec{\mathrm{J}}$ is conserved, the charge-exchange scattering amplitude is now given by

$$\begin{aligned}f(\vec{\mathrm{k}}_f,n_f;k_i\hat{z},n_i)_{\text{charge exchange}}=\frac{1}{4\pi}e^{-i\pi n_i}\sum(2l+1)^2e^{i\pi(l-l_i'/2-l_f'/2)}\left[\int dr\,K(r)j_{l_i'}(k_ir)j_{l_f'}(k_fr)\right]\\\times\mathfrak{D}^{(l)}_{n_f,-n_i}(-\phi_f,\theta_f,\phi_f)\int d\Omega[\phi^{-n_in_f}_{lt}(\hat{r})]^\dagger 2[\vec{\mathrm{T}}\cdot\vec{\mathrm{J}}-(\hat{r}\cdot\vec{\mathrm{T}})^2][\phi^{-n_in_i}_{lt}(\hat{r})]\,.\end{aligned} \tag{117}$$

The angular integral in Eq. (117) can be evaluated with the aid of the two operators

$$X_\pm=\vec{\mathrm{T}}\cdot\vec{\mathrm{J}}-(\hat{r}\cdot\vec{\mathrm{T}})^2\pm i\vec{\mathrm{T}}\times\hat{\mathrm{r}}\cdot\vec{\mathrm{J}}\,. \tag{118}$$

They are raising (X_+) and lowering (X_-) operators for the charge $\hat{r}\cdot\vec{\mathrm{T}}$ since

$$[\hat{r}\cdot\vec{\mathrm{T}},X_\pm]=\pm X_\pm\,. \tag{119}$$

(Note that the $X_\pm$ are essentially the same as helicity-raising and -lowering operators.) The $\vec{T}\times\hat{r}\cdot\vec{J}$ term in Eq. (118) is Hermitian because J_k commutes with $(\vec{T}\times\vec{r})_k$. Hence

$$X_\pm^\dagger = X_\mp . \tag{120}$$

A rather lengthy calculation yields the normalization

$$X_\pm^\dagger X_\pm = [J^2 - \hat{r}\cdot\vec{T}(\hat{r}\cdot\vec{T}\pm 1)][T^2 - \hat{r}\cdot\vec{T}(\hat{r}\cdot\vec{T}\pm 1)] . \tag{121}$$

Therefore, with the spinor factors χ_t^n of unit norm, and with the $\mathfrak{D}_{n,m}^{(l)}$ normalization given in Eqs. (81) and (82), we conclude that, with an appropriate choice of phase,

$$\int d\Omega[\phi_{lt}^{-n_i n_f}(\hat{r})]^\dagger 2[T\cdot J - (\hat{r}\cdot\vec{T})^2]\phi_{lt}^{-n_i n_i}(\hat{r})] = \int d\Omega[\phi_{lt}^{-n_i n_f}(\hat{r})]^\dagger (X_+ + X_-)[\phi_{lt}^{-n_i n_i}(\hat{r})]$$
$$= \frac{4\pi}{2l+1}\{[l(l+1)-n_i(n_i-1)]^{1/2}[t(t+1)-n_i(n_i-1)]^{1/2}\delta_{n_f,n_i-1} + [l(l+1)-n_i(n_i+1)]^{1/2}[t(t+1)-n_i(n_i+1)]^{1/2}\delta_{n_f,n_i+1}\}. \tag{122}$$

The charge-exchange scattering amplitude thus has the form

$$f(\vec{k}_f, n_f; k_i\hat{z}, n_i)_{\text{charge exchange}} = -e^{-i\pi n_f}\sum_l (2l+1)e^{i\pi(l-l_i'/2-l_f'/2)}\left[\int dr\, K(r) j_{l_i'}(k_i r) j_{l_f'}(k_f r)\right]$$
$$\times\{\delta_{n_f,n_i+1}\mathfrak{D}_{n_f,-n_i}^{(l)}(-\phi_f,-\theta_f,\phi_f)[l(l+1)-n_i(n_i+1)]^{1/2}$$
$$\times[t(t+1)-n_i(n_i+1)]^{1/2}$$
$$+\delta_{n_f,n_i-1}\mathfrak{D}_{n_f,-n_i}^{(l)}(-\phi_f,-\theta_f,\phi_f)[l(l+1)-n_i(n_i-1)]^{1/2}$$
$$\times[t(t+1)-n_i(n_i-1)]^{1/2}\}. \tag{123}$$

The factor in curly brackets indicates the single-charge-exchange behavior of this approximate amplitude. As explained in Sec. III, the function $K(r)$ behaves essentially as $Ae^{-m_{\text{ch}}r}$ with the constant A being, in general, of order 1. The use of this approximation is clearly sensible for $k\ll m_{\text{ch}}$, and, moreover, in this limit only the lowest (allowed) l term need be retained in the sum. The overlap integral in Eq. (123) now behaves as $(k_f/m_{\text{ch}})^{l_f'}(k_i/m_{\text{ch}})^{l_i'}$, and we obtain the following conclusions: Let N be the larger of $|n_i|$ or $|n_f|$. Then, ignoring the mass splittings so that $k_i=k_f=k$, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{charge exchange}} \propto \left(\frac{k}{m_{\text{ch}}}\right)^{2[(N+1/4)^{1/2}+(3N-3/4)^{1/2}-1]} \times\sin^2\theta_f(1-\cos\theta_f)^{2N-2}. \tag{124}$$

From Eq. (124) it is clear that for small but nonzero incident charge, the reaction occurs predominantly in the direction of decreasing the magnitude of the particle's charge.

APPENDIX A

We shall obtain a closed form for the sum

$$\Psi_{\text{I}}(\vec{r}) = e^{-i\pi n}\sum_{l=|n|}^{\infty}(2l+1)e^{i\pi l/2}j_l(kr)d_{n,-n}^{(l)}(\theta), \tag{A1}$$

where l and n are both either integers or half-integers. Here $j_l(kr)$ is a spherical Bessel function and $d_{n,-n}^{(l)}(\theta)$ a representation function of the rotation group.[23] We shall assume that the index n is positive since the result for negative n is obvious from the symmetry

$$d_{-n,n}^{(l)}(\theta) = (-1)^{2n}d_{n,-n}^{(l)}(\theta). \tag{A2}$$

We shall prove that

$$\psi_{\text{I}}(\vec{r}) = e^{-i\pi n/2}\frac{\Gamma(n+1)}{\Gamma(2n+1)}e^{ikr}[kr(1-\cos\theta)]^n \times\Phi(n+1;2n+1;-ikr(1-\cos\theta)), \tag{A3}$$

where $\Phi(a;c;z)$ is the regular, confluent hypergeometric function.[24]

First we note that the Bessel and rotation functions obey the ordinary differential equations

$$\left[\frac{1}{r}\frac{d^2}{dr^2}r-\frac{l(l+1)}{r^2}+k^2\right]j_l(kr)=0 \tag{A4}$$

and[23]

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+l(l+1)-\frac{2n^2}{1-\cos\theta}\right]d^{(l)}_{n,-n}(\theta)=0. \tag{A5}$$

Hence, any function of the form

$$F(r,\theta)=\sum_{l=n}^{\infty}c_l j_l(kr)d^{(l)}_{n,-n}(\theta) \tag{A6}$$

is a regular solution of the partial-differential equation

$$\left[\nabla^2+k^2-\frac{2n^2}{r^2(1-\cos\theta)}\right]F(r,\theta)=0. \tag{A7}$$

On the other hand, since the functions $d^{(l)}_{n,-n}(\theta)$ are complete, any regular solution of Eq. (A7) can be expressed in the form (A6) with constant coefficients c_l. The confluent hypergeometric function in Eq. (A3) obeys the differential equation[24]

$$\left[\zeta\frac{d^2}{d\zeta^2}+(2n+1-\zeta)\frac{d}{d\zeta}-n-1\right]\times\Phi(n+1;2n+1;\zeta)=0. \tag{A8}$$

Therefore it follows that $\psi_{\text{I}}(\vec{r})$, as defined in Eq. (A3), satisfies the partial-differential equation (A7). Since this definition of $\psi_{\text{I}}(\vec{r})$ is regular at the origin, it can be expressed in the form of the sum (A6). All that remains to be shown now is that the coefficients c_l have the values $(2l+1)e^{-i\pi n}e^{i\pi l/2}$ displayed in Eq. (A1).

The regular, confluent hypergeometric function in Eq. (A3) may be written as an infinite sum[24]

$$\frac{\Gamma(n+1)}{\Gamma(2n+1)}\Phi(n+1;2n+1;\zeta)=\sum_{p=0}^{\infty}\frac{\Gamma(n+p+1)}{\Gamma(2n+p+1)}\frac{\zeta^p}{p!}. \tag{A9}$$

This enables us to identify easily the coefficient of the term in Eq. (A3) involving the particular power of (kr) and $(1-\cos\theta)$, $[e^{-i\pi/2}kr(1-\cos\theta)]^L$:

$$\frac{\Gamma(L+1)}{\Gamma(L+n+1)}\frac{1}{\Gamma(L-n+1)}. \tag{A10}$$

To compare this with the sum (A6), we note that[25]

$$d^{(l)}_{n,-n}(\theta)=\left(\frac{1-\cos\theta}{2}\right)^n\frac{\Gamma(n+l+1)}{\Gamma(l-n+1)\Gamma(2n+1)}\times F(n-l,l+n+1;2n+1;\tfrac{1}{2}(1-\cos\theta)), \tag{A11}$$

where $F(a,b;c;z)$ is the ordinary hypergeometric function which, with a negative first index $-(l-n)$, is a finite polynomial in its argument. Thus

$$d^{(l)}_{n,-n}(\theta)=\frac{(-1)^{l-n}}{\Gamma(l-n+1)}\frac{\Gamma(2l+1)}{\Gamma(l+n+1)}\left(\frac{1-\cos\theta}{2}\right)^l+[\text{lower powers of }(1-\cos\theta)]. \tag{A12}$$

Moreover,

$$j_l(kr)=2\frac{\Gamma(l+2)}{\Gamma(2l+3)}(2kr)^l+[\text{higher powers of }(kr)]. \tag{A13}$$

Accordingly, the term involving the particular power $[e^{-i\pi/2}kr(1-\cos\theta)]^L$ in the general sum (A6) arises only from the $l=L$ term and has the coefficient

$$c_L e^{i\pi L/2}(-1)^{L-n}\frac{1}{2L+1}\frac{\Gamma(L+1)}{\Gamma(L+n+1)\Gamma(L-n+1)}. \tag{A14}$$

Comparing (A10) with (A14), we see that the function defined by Eq. (A3) has the expansion (A6) with the coefficient

$$c_L=e^{-\pi in}(2L+1)e^{i\pi L/2}. \tag{A15}$$

This proves our assertion that Eqs. (A1) and (A3) define the same function.

It is convenient to express $\psi_{\text{I}}(\vec{r})$ in terms of the familiar spherical Bessel functions. The confluent hypergeometric function appearing in Eq. (A3) may be written as[24]

$$\Phi(n+1;2n+1;\zeta)=2\frac{d}{d\zeta}\Phi(n;2n;\zeta), \tag{A16}$$

where[24]

$$\Phi(n;2n;-2ix)=\frac{\Gamma(2n)}{\Gamma(n)}e^{-ix}(2x)^{1-n}j_{n-1}(x). \tag{A17}$$

Using

$$x\frac{d}{dx}x^{1-n}j_{n-1}(x)=-x^{1-n}j_n(x), \tag{A18}$$

we obtain

$$\psi_{\text{I}}(\vec{r})=e^{-i\pi n/2}e^{(i/2)kr(1+\cos\theta)}\tfrac{1}{2}kr(1-\cos\theta)\times[j_{n-1}(\tfrac{1}{2}kr(1-\cos\theta))-ij_n(\tfrac{1}{2}kr(1-\cos\theta))]. \tag{A19}$$

Note that this expression is especially simple for integer n where the spherical Bessel functions contain a finite sum of trigonometric functions. We note that, in the general case, the limit $x\to\infty$,

$$j_l(x)\simeq\frac{1}{x}\sin\left(x-\frac{\pi l}{2}\right)+\frac{l(l+1)}{2x^2}\cos\left(x-\frac{\pi l}{2}\right), \tag{A20}$$

yields the asymptotic evaluation $r\to\infty$:

$$\psi_{\rm I}(\vec{r}) \simeq e^{ikr\cos\theta} - \frac{ine^{-i\pi n}}{kr(1-\cos\theta)}\, e^{ikr}. \tag{A21}$$

APPENDIX B

Here we shall discuss carefully the properties of various singular transformations which arise in the context of theories with magnetic monopoles. There are two cases of interest: the familiar Abelian gauge transformations of ordinary electrodynamics and the non-Abelian gauge transformations of the more recently studied unified theories of weak and electromagnetic interactions. In both cases singular-valued and multivalued functions arise which must be regulated in some way with the final results being obtained as limits.

In the Abelian case the vector potential for the magnetic monopole is given by

$$\begin{aligned}\vec{A} &= g\frac{\vec{r}\times\hat{z}}{r(r+z)} \\ &= g\frac{y\hat{x}-x\hat{y}}{r(r+z)}\,.\end{aligned} \tag{B1}$$

Since this potential is singular on the negative z axis and at the origin, it is necessary to study a regulated form, for example,

$$\vec{A}_R = g\frac{y\hat{x}-x\hat{y}}{R(R+z)}\,, \tag{B2}$$

where

$$\begin{aligned}R &= (r^2+\epsilon^2)^{1/2} \\ &= [(x^2+y^2+\epsilon^2)+z^2]^{1/2}\,.\end{aligned} \tag{B3}$$

The regulated magnetic field $\vec{B}_R$ is easily found to have the form

$$\begin{aligned}\vec{B}_R &= \vec{\nabla}\times\vec{A}_R \\ &= -g\left[\frac{\vec{r}}{R^3} + \frac{\epsilon^2\hat{z}}{R^3(R+z)} + \frac{\epsilon^2\hat{z}}{R^2(R+z)^2}\right].\end{aligned} \tag{B4}$$

In the limit $\epsilon^2\to 0$, the regulated magnetic field approaches

$$\vec{B}_R \underset{\epsilon^2\to 0}{\sim} -g\left[\frac{\vec{r}}{r^3} + \frac{2\epsilon^2\hat{z}\,\theta(-z)}{r^2(x^2+y^2+\epsilon^2)} + \frac{4\epsilon^2\hat{z}\,\theta(-z)}{(x^2+y^2+\epsilon^2)^2}\right]. \tag{B5}$$

Clearly, in this limit, the second and third terms vanish except on the negative z axis. Moreover, if we integrate Eq. (B5) over a vanishingly small surface element which intersects the negative z axis, we find that while the third term yields a finite result, the second term gives a vanishing contribution. Hence the second term can be dropped while the third term produces a "string" singularity, and we obtain the limit

$$\vec{B} = -g\left[\frac{\hat{r}}{r^2} + \hat{z}\,4\pi\delta(x)\delta(y)\theta(-z)\right]. \tag{B6}$$

Note that the magnetic flux in the "string" singularity precisely cancels that in the $1/r^2$ field,

$$\oint d\vec{S}\cdot\vec{B} = -4\pi g + 4\pi g = 0\,. \tag{B7}$$

Thus, the magnetic field is that of a semi-infinitely long bar magnet of infinitesimal thickness rather than that of an isolated magnetic pole.

The vector potential not only defines the magnetic field through its curl but, together with the gradient, it forms the gauge-covariant derivative

$$\vec{D}\psi(\vec{r}) = [\vec{\nabla} - ie\,\vec{A}(\vec{r})]\psi(\vec{r})\,. \tag{B8}$$

By means of the wave equation, $\vec{D}$ determines the electromagnetic interaction of the particle described by the wave function $\psi(\vec{r})$ with the potential $\vec{A}$. Under a gauge transformation defined by the gauge function $\lambda(\vec{r})$ the vector potential and wave function transform as

$$\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\lambda(\vec{r})\,, \tag{B9a}$$

$$\psi'(\vec{r}) = e^{ie\lambda(\vec{r})}\psi(\vec{r})\,. \tag{B9b}$$

As a result of Eq. (B9a), the gauge-covariant derivative $\vec{D}$ of ψ will transform like ψ [Eq. (B9b)]. The gauge function $\lambda(\vec{r})$ is usually taken to be single-valued. However, we may also consider gauge transformations $e^{i\lambda(\vec{r})}$ which are single-valued almost everywhere but with a multivalued gauge function $\lambda(\vec{r})$. In this case, although $\vec{\nabla}\lambda(\vec{r})$ is single-valued almost everywhere, it will be singular along a line that joins together the various sheets of the multivalued function λ. Such a gauge function will introduce singularities in the gauge-transformed vector potential $\vec{A}'$. This singularity structure in $\vec{A}'$ is exactly matched by the multivalued behavior of $\vec{\nabla}\psi'$ induced by the transformation $e^{ie\lambda(\vec{r})}$. The magnetic field $\vec{B}'$ defined by $\vec{A}'$ is also altered by such transformations. This can be seen easily by considering the change in the magnetic flux through a surface S induced by the function λ:

$$\begin{aligned}\Delta\Phi_S &= \int d\vec{S}\cdot(\vec{B}'-\vec{B}) \\ &= \int d\vec{S}\cdot\vec{\nabla}\times(\vec{\nabla}\lambda) \\ &= \oint d\vec{l}\cdot\vec{\nabla}\lambda\,.\end{aligned} \tag{B10}$$

Clearly, the necessary and sufficient condition for $\Delta\Phi_S$ to vanish is that λ be single-valued around the edge of the surface S. For a multivalued λ, $\Delta\Phi_S$ will, in general, not vanish.

As an explicit example consider the Dirac vector potential (B2) and magnetic field (B6). As already noted, these fields initially contain a singular string which is, however, not observable in a purely quantum-mechanical context for the case where eg is an integer or a half-integer. The multivalued function

$$\lambda(\vec{r}) = 2g\tan^{-1}\left(\frac{y}{x}\right) = 2g\phi \tag{B11}$$

has the gradient

$$\vec{\nabla}\lambda = 2g\frac{x\hat{y} - y\hat{x}}{x^2 + y^2} \tag{B12}$$

and changes the vector potential into

$$\vec{A}' = g\frac{x\hat{y} - y\hat{x}}{r(r-z)}. \tag{B13}$$

Thus the "string" has moved from the negative z axis to the positive z axis, and the flux line in the magnetic field now runs along the positive z axis,

$$\vec{B} = -g\left[\frac{\hat{r}}{r^2} - 4\pi\delta(x)\delta(y)\theta(z)\hat{z}\right]. \tag{B14}$$

The question remains as to whether or not these manipulations involving multivalued gauge functions are truly legitimate. If the starting point involved a purely regular magnetic field, the introduction of unphysical "strings" would certainly not be acceptable. The situation for the Dirac monopole is more ambiguous since the theory contains a "string" initially. Although the "string" is not observable from the standpoint of quantum-mechanical scattering processes (with $eg = n$), the magnetic field is itself a physical quantity. It contributes to the energy and momentum densities, to the Maxwell stress, and, thereby, to the gravitational field. Hence a change in the magnetic field corresponding to moving the "string" is, in principle, observable, and multivalued gauge functions are not truly legitimate. We may, of course, choose a definition such that the gauge function (B11) is single-valued,

$$\lambda_S(\vec{r}) = 2g\tan^{-1}\left(\frac{y}{x}\right), \quad 0 \le \tan^{-1}\left(\frac{y}{x}\right) < 2\pi. \tag{B15}$$

The gradient will now exhibit a surface discontinuity in the $y=0$, $x>0$ plane along with the previous "string,"

$$\vec{\nabla}\lambda_S(\vec{r}) = \frac{2g(x\hat{y} - y\hat{x})}{x^2 + y^2} - 4\pi g\hat{y}\,\delta(y)\theta(x). \tag{B16}$$

Thus the transformed vector potential, $\vec{A}'$, will not only have a relocated "string" but also a surface δ function. This latter contribution guarantees that the curl of the vector potential is unchanged by the transformation ($\vec{\nabla}\times\vec{\nabla}\lambda_S = 0$) and the "string" in the magnetic field is unchanged. However, once we have admitted that the string in $\vec{B}$ contributes to the energy density, a contribution that is infinite not only at the origin but along the semi-infinite string as well, we conclude that the Dirac monopole is not a fully acceptable construct. This feature makes the monopole of the non-Abelian theory appealing, for it is a solution of the field equations without singular "strings." We turn now to the question of gauge transformations in such theories.

In non-Abelian theories, gauge transformations have the form

$$\phi'(\vec{r}) = U(\vec{r})\phi(\vec{r}) \tag{B17}$$

and

$$\frac{1}{i}\vec{\nabla} - e\vec{A}'_a T_a = U\left[\frac{1}{i}\vec{\nabla} - e\vec{A}_a T_a\right]U^{-1}. \tag{B18}$$

Here the matrices U act in the internal isospin space and are generally taken to be single-valued everywhere. However, as mentioned briefly in Sec. IV, the transformation which connects the fully regular monopole solution in the gauge with $\hat{h}_a = \hat{r}_a$ to the more familiar gauge where $\hat{h}_a = \delta_{a3}$ is singular along the negative z axis. Note that in this discussion we are explicitly referring only to the region outside of the complicated central monopole core ($r > 1/m_{ch}$) since the interior region poses no further conceptual difficulties. The specific rotation in question is

$$U(\vec{r}) = e^{-iT_3\phi}e^{iT_2\theta}e^{i\phi T_3}, \tag{B19}$$

where $\hat{r} = \hat{z}\cos\theta + \sin\theta(\hat{x}\cos\phi + \hat{y}\sin\phi)$. In this case we could again require ϕ to be single-valued and ensure that the magnetic field is unchanged by the transformation. However, in the non-Abelian theory, one has the simpler option of regulating θ. We define a regulated polar angle by

$$\Theta(\theta) = \theta\frac{1+\cos\theta}{1+\cos\theta+\epsilon^2}. \tag{B20}$$

The monopole vector potential

$$e\vec{A}_a T_a = \frac{\vec{r}\times\vec{T}}{r^2} \tag{B21}$$

becomes after the transformation (B19), with Θ replacing θ,

$$\vec{A}_3^R(\vec{r}) = \frac{\hat{\phi}}{er\sin\theta}\left[\cos\Theta - 1 + \sin\theta\sin(\Theta - \theta)\right] \tag{B22}$$

and

$$\vec{A}_1^R + i\vec{A}_2^R = \frac{e^{i\phi}}{er}\left[\hat{\phi}\left(\cos(\Theta-\theta) - \frac{\sin\Theta}{\sin\theta}\right) + i\hat{\theta}(\Theta'-1)\right], \tag{B23}$$

where $\Theta' = (d/d\theta)\Theta$. Since this regulated gauge transformation is well defined, the field

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\epsilon_{abc}A_b^\mu A_c^\nu \tag{B24}$$

is simply rotated by the transformation. In more detail, the "magnetic field"

$$B_a^k = \tfrac{1}{2}\epsilon^{klm}G_a^{lm} = -\frac{\hat{r}^k\hat{r}_a}{er^2} \tag{B25}$$

in the initial gauge becomes

$$B_a^{k'} = -\frac{\hat{r}^k}{er^2}[\delta_{a3}\cos(\Theta-\theta) - \sin(\Theta-\theta)(\delta_{a1}\cos\phi + \delta_{a2}\sin\phi)] \tag{B26}$$

after the transformation. In the limit $\epsilon^2 \to 0$, this goes smoothly into the expected monopole field

$$B_a^{k'} = -\frac{\hat{r}^k\delta_{a3}}{er^2} \tag{B27}$$

with no "string" (recall again this is only for $r \gg 1/m_{\rm ch}$). It is instructive to compare this result with the result of calculating the magnetic field directly from Eqs. (B24) and (B25) using the regulated potentials (B22) and (B23) and then taking the limit $\epsilon^2 \to 0$. In this limit $\Theta \to \theta$, and we have

$$\vec{A}_3^R \to \frac{\hat{\phi}(\cos\theta - 1)}{er\sin\theta} = \frac{y\hat{x} - x\hat{y}}{er(r+z)} \tag{B28}$$

and

$$\vec{A}_1^R + i\vec{A}_2^R \to 0\,. \tag{B29}$$

However, the limit (B29) is a subtle one for the vector potential is a distribution that must be defined relative to the measure $\sin\theta\, d\theta$. Clearly the limit is zero for $\theta \neq \pi$. Furthermore, while Θ' is singular, the integral of $\Theta' - 1$ with the weight $\sin\theta\, d\theta$ vanishes. Thus $\Theta' - 1$ approaches zero (as $\epsilon \to 0$) in the sense of a distribution for the angular measure $\sin\theta\, d\theta\, d\phi$.

One might naively expect that the magnetic field in the new gauge will arise solely from the $\vec{\nabla}\times\vec{A}_3$ term in Eq. (B24) and thus possess a "string" singularity. This, however, is incorrect. We must calculate $\vec{B}_3$ directly from the regulated fields and then take the limit. In particular, the contribution to $\vec{B}_3$ from the second term in Eq. (B24) has the form

$$\begin{aligned}\Delta\vec{B}_3 &= \lim_{\epsilon^2\to 0} e\,\vec{A}_1^R \times \vec{A}_2^R \\ &= \lim_{\epsilon^2\to 0} \frac{ie}{2}(\vec{A}_1^R + i\vec{A}_2^R)\times(\vec{A}_1^R - i\vec{A}_2^R) \\ &= \lim_{\epsilon^2\to 0} \frac{-\hat{r}}{er^2}(\Theta'-1)\left[\cos(\Theta-\theta) - \frac{\sin\Theta}{\sin\theta}\right],\end{aligned} \tag{B30}$$

which vanishes for $\theta < \pi$. Now because of the $\sin\theta$ factor in the denominator, we have

$$\begin{aligned}\lim_{\delta\to 0}\lim_{\epsilon^2\to 0} r^2\int_{\pi-\delta}^{\pi}\sin\theta\, d\theta\int_0^{2\pi} d\phi\,\Delta\vec{B}_3 &= -\frac{2\pi\hat{z}}{e}\lim_{\delta\to 0}\lim_{\epsilon^2\to 0}\int_{\pi-\delta}^{\pi} d\theta\,\Theta'\sin\Theta \\ &= \frac{2\pi\hat{z}}{e}\lim_{\delta\to 0}\lim_{\epsilon^2\to 0}\cos[\Theta(\theta)]\Big|_{\pi-\delta}^{\pi} \\ &= \frac{4\pi\hat{z}}{e}\,.\end{aligned} \tag{B31}$$

Thus, although $\vec{A}_1$ and $\vec{A}_2$ separately vanish as distributions, their product yields a δ-function distribution and

$$\Delta\vec{B}_3 = \frac{4\pi\hat{z}}{e}\delta(x)\delta(y)\theta(-z)\,, \tag{B32}$$

which exactly cancels the "string" in the $\vec{\nabla}\times\vec{A}_3$ contribution to $\vec{B}_3$. In this gauge then, the vector potential (B28) is just as in the Dirac theory. However, when care is taken to treat the vector potential properly as a limiting distribution resulting from a single-valued transformation, the physical fields are given by Eq. (B27) and they exhibit no unphysical "string" singularities.

---

*Work supported in part by the U. S. Energy Research and Development Administration.

[1]P. A. M. Dirac, Proc. R. Soc. London A133, 60 (1931); Phys. Rev. 74, 817 (1948); see also the comments of G. Wentzel, Prog. Theor. Phys. Suppl. 37–38, 163 (1966).

[2]We use the Heaviside-Lorentz (rationalized Gaussian) cgs system of electromagnetic units. Thus, the magnetic charge carried on the monopole is given by $4\pi g$. Also, we use the metric $g_{\mu\nu}$ with signature (– + + +).

[3](a) Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959); (b) J. Schwinger, *ibid.* 144, 1087 (1966).

[4]G. 't Hooft, Nucl. Phys. B79, 276 (1974).

[5]A. M. Polyakov, Zh. Eksp. Teor. Fiz. Pis'ma Red. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].

[6]H. Poincaré, C. R. Acad. Sci. Ser. B 123, 530 (1896);

M. Fierz, Helv. Phys. Acta. 17, 27 (1944); I. R. Lapidus and J. L. Pietempole, Am. J. Phys. 28, 17 (1960); G. Nadeau, *ibid*. 28, 566 (1960).

[7]The wave function for the Dirac theory was found by J. Tamm, Z. Phys. 71, 141 (1931).

[8]P. P. Banderet, Helv. Phys. Acta. 19, 503 (1946).

[9]Classical monopole solutions to the non-Abelian gauge theory that also bear electrical charge have been found by B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975).

[10]P. Higgs, Phys. Lett. 12, 132 (1966); T. W. B. Kibble, Phys. Rev. 155, 1554 (1967).

[11]J. L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975); J. L. Gervais, A. Jevicki, and B. Sakita, *ibid*. 12, 1038 (1975); see also J. Goldstone and R. Jackiw, *ibid*. 11, 1486 (1975); N. Christ and T. D. Lee, *ibid*. 12, 1606 (1975).

[12]Detailed quantization of the non-Abelian monopole system has been performed by E. Tomboulis and G. Woo, Nucl. Phys. B107, 221 (1976).

[13]J. Arafune, P. Freund, and C. Goebel, J. Math. Phys. 16, 433 (1975); see also A. Patrascioiu, Phys. Rev. D 12, 523 (1975).

[14]S. Coleman, Lectures delivered at the 1975 International School of Subnuclear Physics "Ettore Majorana" (unpublished).

[15]M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[16]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).

[17]Recently, the Dirac equation for a charged fermion in the non-Abelian monopole field has been discussed by T. Dereli, J. H. Swank, and L. J. Swank, Phys. Rev. D 12, 1096 (1975), and by R. Jackiw and C. Rebbi, Phys. Rev. D 13, 3398 (1976).

[18]A. S. Goldhaber, Phys. Rev. 140, B1407 (1965).

[19]For the representation functions of the rotation group, we follow the notation by A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, 1957).

[20]K. Ford and J. A. Wheeler, Ann. Phys. (N.Y.) 7, 259 (1959).

[21]J. Schwinger, K. A. Milton, Wu-yang Tsai, Lester L. DeRaad, Jr., and D. C. Clark, UCLA report, 1976 (unpublished).

[22]J. D. Jackson, *Classical Electrodynamics*, second edition (Wiley, New York, 1975), p. 259.

[23]See, for example, Ref. 19, Chap. 4.

[24]See, for example, *Higher Transcendental Functions* (Bateman Manuscript Project) edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, Chap. VI.

[25]See, Ref. 19, Chap. 4 and Ref. 24, Vol. II, Chap. X.