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Renormalization of the σ model at finite temperature

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The temperature dependence of a many-body theory with the dynamics defined by the relativistic linear σ model is studied. The model has $SU(2)\times SU(2)$ chiral symmetry with fermions belonging to a $(1/2,0)+(0,1/2)$ representation interacting with the σ and π mesons belonging to the $(1/2,1/2)$ representation of the chiral symmetry group. The dimensional-regularization technique together with the renormalization procedure of 't Hooft is used to reproduce the well-known result that the counterterms of the symmetric theory remove the divergences of the theory with spontaneous symmetry breaking, without however the need for invoking auxiliary fermion fields. Renormalizability is maintained at finite temperatures by the cancellation of temperature-dependent infinities which appear at the two-loop level. This is shown explicitly for the ground-state expectation value of the scalar σ field at the two-loop level. When the symmetry is explicitly broken by the term $f_\pi m_\pi{}^2\sigma$ the symmetry of the original Lagrangian is never restored. In the absence of such a term a symmetry change with temperature is realized and the persistence of the Goldstone mode up to a critical temperature T_c, above which the original symmetry is restored, is verified. Thus below $T=T_c$ the low-energy theorems of current algebra associated with the existence of the Goldstone pions would be valid except that all parameters of the theory develop finite, temperature-dependent, corrections. A parallel discussion for density dependence of the symmetry is included. All calculations are done in the real-time formalism for the thermodynamic Green's functions.

I. INTRODUCTION

The temperature dependence of symmetries of physical systems has been the subject of interest in many areas of physics because of abrupt changes in symmetries that accompany phase transitions. Most of the models used to describe these phenomena are based in some way or the other on Landau's theory of phase transitions[1] in which the free energy of the system, expressed in terms of an order parameter, develops a minimum at a nonzero value of the order parameter. This gives rise to ground states which do not have the full symmetry of the original interaction. These ideas have been used in context of field theory to generate particle mass spectra closely resembling the actually observed particle spectra by requiring the vacuum state, which is the ground state of elementary particle physics, to exhibit spontaneous symmetry breaking.[2-5] The order parameter now is the vacuum expectation value of the field.

Recently it has been suggested that the broken symmetries of particle physics could be restored by raising the temperature of a many-body system of particles whose mutual interactions are governed by relativistic field theory.[6,7] Another possibility is that the symmetry of such a relativistic many-body system could be altered by a change of density. Both aspects of the problem of symmetry changes arise directly as a consequence of the statistical mechanics of the many-body system and are of particular interest because of their astrophysical implications. The derivation of an equation of state for a relativistic system of nucleons, for example, must necessarily include the effect of the density dependence of symmetries.[8]

Weinberg[7] has given a general theory of gauge symmetries at high temperature in which all quantities of interest may be calculated by the usual techniques of field theory using a perturbation expansion. The minima of the effective potential are determined in order to see whether a given symmetry is spontaneously broken or not and also to define the perturbation expansion in terms of the appropriate excitation spectrum. It is found that at very high temperatures the effective potential changes its shape. The leading terms which signal the breakdown of the perturbation expansion can be used to roughly estimate the critical temperature at which there is a change of symmetry. Dolan and Jackiw[9] have used functional techniques to show the same results.

In the present paper the temperature dependence of the σ model with chiral $SU(2)\times SU(2)$ symmetry is studied. The model[10] has two nucleons belonging to the $(\frac{1}{2},0)+(0,\frac{1}{2})$ representation interacting with a scalar σ meson and an isotriplet of π mesons belonging to the $(\frac{1}{2},\frac{1}{2})$ quartet representation of the symmetry group. The σ model incorporates all the basic ideas of spontaneous symmetry breaking and reproduces the results of current algebra.[11] The model has been used as a testing ground for many new ideas in strong-interaction physics over the past two decades. It has the additional feature of being a renormalizable field theory.[10] It is shown that this renormalizability is maintained at finite temperatures and that there is a critical temperature below which the symmetry is spontaneously broken. This spontaneous symmetry breaking signals the appearance of a Nambu-Goldstone[2,3] mode of excitation which persists at finite temperature up to the critical temperature. This is true for the theory with no explicit symmetry breaking. When a term $f_\pi m_\pi^2\sigma$, linear in the σ field, is added to the Lagrangian the pion picks up a mass. If the parameters determined at zero temperature by the physical masses are used to study the temperature behavior it is found that symmetry is not restored and the vacuum expectation value of the σ field remains nonzero. This is understandable if one uses an analogy based on ferromagnetism. If an external magnetic field is applied then the rotational symmetry in a ferromagnet is not restored even after the Curie temperature is exceeded since the external field introduces a special directionality. This paper includes a parallel discussion of the density dependence of the symmetry,[8] for the sake of completeness.

Particular attention is paid to the details of renormalization, and the real-time formalism[12,13] is used for the temperature Green's functions, which has the feature of isolating the many-body effects right from the beginning. Products of these Green's functions cause no difficulty when they appear in the evaluation of Feynman diagrams which contain more than one loop. These features are shown while demonstrating that there are no temperature-dependent infinities at the two-loop level. Recently Kislinger and Morley[14] have shown that for a scalar φ^4 field theory the temperature-dependent infinities are absent at the two-loop level for self-energy corrections by using the imaginary-time formalism which employs frequency sums in Green's functions.

In discussing renormalizability use is made of the dimensional regularization of Feynman integrals at the intermediate stages of the calculation.[15,16] The integrals are analytically continued in n, the number of dimensions, and their divergences as $n\to 4$ are realized as poles in the complex n plane. The advantages of dimensional regularization are well appreciated by now. The fact that it does not introduce additional mass parameters or cutoff parameters in integrals has been of crucial importance in gauge theories, where gauge invariance is maintained at all stages of the calculation by this procedure.[17] This is also useful in the context of chiral symmetries in the presence of fermions, and it allows us to renormalize the σ model without introducing auxiliary fermion regulator fields.

It is convenient to use the 't Hooft[18] prescription for renormalization when using dimensional regularization. Counterterms are introduced into the Lagrangian order by order to cancel the pole terms in n appearing in the regulated Feynman integrals. The pole terms at $n=4$ have residues which are always polynomials in external momenta. This means that if subtractions have been made to a given order in perturbation theory, the new pole terms in the next higher order are polynomials in the external momenta. The residues can also be shown to be polynomials of a degree given by power-counting rules. For this renormalization scheme to work the residues of the pole terms must remain polynomials in momenta and no other functional dependence must be present. It has been shown by 't Hooft and Veltman[15] that poles with residues which are logarithmic functions of momenta which arise at the l-loop level are canceled by counterterms of the $(l-1)$-loop diagrams. In the case of many-body theory, again a similar cancellation should occur among the pole terms which have temperature-dependent, or density-dependent, residues in order that the renormalizability of the theory be maintained. It is shown that this nontrivial cancellation indeed occurs by actually evaluating the 2-loop diagram for the vacuum expectation value of the σ field, i.e., for the σ-tadpole terms. Such a result is of interest in the case of spontaneously broken symmetries with massless excitations for the following reason: Dimensional regularization has its own set of self-consistent rules. For massless particles the integrals of $\delta(k^2)$ and $(k^2)^\alpha$ $(\alpha \geq -1)$ over the n momenta k_n are set equal to zero. This has been shown to be necessary[19] by properly defining the limits $m^2\to 0$ and $n\to 4$ so that ambiguities associated with the infrared divergences are removed. These rules are applied to the σ-model calculations.

In Sec. II the σ model for the many-body problem of N nucleons interacting with σ and π mesons in the tree-diagram approximation is described. It is shown that the change in the phase space due to the presence of many particles affects the usual results in this approximation. In Sec. III the details of the renormalization procedure at the one-loop level are presented and it is shown that the pions remain massless at finite temperatures in the absence of explicit symmetry breaking.

In Sec. IV it is shown that the divergences of the σ-tadpole diagrams have no log (m^2) residues and that the temperature-dependent inifinites also cancel. In Sec. V the relation of the present paper to other problems is discussed. For the sake of completeness a brief review of the temperature Green's functions is included in Appendix A, and in Appendix B some useful relations for the dimensional-regularization scheme are given. The temperature- and density-dependent corrections to the renormalized meson coupling constant are given in Appendix C.

II. THE σ MODEL

Consider a system of N nucleons whose interactions with σ and π mesons are given by

$$\mathcal{L}=\bar{\Psi}i\gamma^{\mu}\partial_{\mu}\Psi-g\bar{\Psi}(\sigma-i\gamma_5\vec{\tau}\cdot\vec{\pi})\Psi+\tfrac{1}{2}(\partial_{\mu}\sigma\,\partial^{\mu}\sigma+\partial_{\mu}\vec{\pi}\cdot\partial^{\mu}\vec{\pi})-\frac{\mu_0^2}{2}(\sigma^2+\vec{\pi}^2)-\frac{\lambda}{4!}(\sigma^2+\vec{\pi}^2)^2+C\sigma. \quad (1)$$

Apart from the term $C\sigma$ the Lagrangian (1) is invariant under the isospin transformations

$$\delta_{\alpha}\sigma=0,\quad \delta_{\alpha}\vec{\pi}=\vec{\alpha}\times\vec{\pi},\qquad \delta_{\alpha}\Psi=-i\vec{\alpha}\cdot\frac{\vec{\tau}}{2}\Psi,\quad \delta_{\alpha}\bar{\Psi}=i\bar{\Psi}\frac{\vec{\tau}}{2}\cdot\vec{\alpha} \quad (2)$$

and under the axial transformations

$$\delta_{\beta}\sigma=\vec{\beta}\cdot\vec{\pi},\quad \delta_{\beta}\vec{\pi}=-\vec{\beta}\sigma,\qquad \delta_{\beta}\Psi=-i\vec{\beta}\cdot\gamma_5\frac{\vec{\tau}}{2}\Psi,\quad \delta_{\beta}\bar{\Psi}=-i\bar{\Psi}\frac{\vec{\tau}}{2}\cdot\gamma_5\vec{\beta}, \quad (3)$$

which are generated by the vector and axial-vector currents

$$\vec{V}^{\mu}\equiv\frac{\delta\mathcal{L}}{\delta\partial_{\mu}\alpha}=\bar{\Psi}\gamma^{\mu}\frac{\vec{\tau}}{2}\Psi+\vec{\pi}\times\partial^{\mu}\vec{\pi},\qquad \vec{A}^{\mu}\equiv\frac{\delta\mathcal{L}}{\delta\partial_{\mu}\beta}=\bar{\Psi}\gamma^{\mu}\gamma_5\frac{\vec{\tau}}{2}\Psi+\vec{\pi}\,\partial^{\mu}\sigma-\sigma\partial^{\mu}\vec{\pi}. \quad (4)$$

The term $C\sigma$ in the Lagrangian breaks axial symmetry and gives rise to the relation

$$\partial^{\mu}\vec{A}_{\mu}(x)=\frac{\delta\mathcal{L}}{\delta\vec{\beta}}=C\vec{\pi}(x)=f_{\pi}m_{\pi}^2\vec{\pi}(x), \quad (5)$$

where f_{π} is the pion decay constant.

We shall assume that even in the absence of the symmetry-breaking term $C\sigma$ and also at zero density and temperature we have spontaneous symmetry breaking and that

$$\langle 0|\sigma|0\rangle=\sigma_0\neq 0, \quad (6)$$

where $|0\rangle$ is the vacuum state. The relation (6) implies that the σ field develops classical parts besides the usual second-quantized parts, and gives rise to finite nucleon masses at zero temperature. This may be compared with the Bogoliubov[20] picture of a condensate in which the nonzero ground-state expectation value of a boson field arises because of the very large number of particles in one particular mode, the condensate mode. All particles interacting with the boson species which forms a condensate have contributions to their masses because of interactions with the condensate background.

It should be noted that it is possible to generate spontaneous symmetry breaking by changing the density of the nucleons. This would give rise to pair correlations among nucleons leading to Cooper pairing of nucleons and holes as in the case of electronic superconductivity.[21,22] The attractive forces would be those arising from σ and π exchanges. The present paper concerns only the temperature dependence and the density dependence of the usual σ model with $\sigma_0\neq 0$ as defined by (6), rather than by $\langle G|\sigma_0|G\rangle\neq 0$ for a many-body ground state $|G\rangle$.

In order to proceed with the quantization of the σ fields we separate out the "condensate" part or the classical part σ_0 and substitute $\sigma\to\sigma_0+\sigma$ in (1) to obtain

$$\mathcal{L}=\bar{\Psi}(i\gamma^{\mu}\partial_{\mu}-g\sigma_0)\Psi-g\bar{\Psi}(\sigma-i\gamma_5\vec{\tau}\cdot\vec{\pi})\Psi+\tfrac{1}{2}(\partial_{\mu}\sigma)^2+\tfrac{1}{2}(\partial_{\mu}\vec{\pi})^2-\tfrac{1}{2}(\mu_0^2+\tfrac{1}{2}\lambda\sigma_0^2)\sigma^2-\tfrac{1}{2}(\mu_0^2+\tfrac{1}{6}\lambda\sigma_0^2)\vec{\pi}^2$$
$$-\frac{\lambda}{4!}[\sigma^4+4\sigma_0\sigma(\sigma^2+\vec{\pi}^2)+2\sigma^2\vec{\pi}^2+(\vec{\pi}^2)^2]-\sigma[\sigma_0(\mu_0^2+\tfrac{1}{6}\lambda\sigma_0^2)-C]-\tfrac{1}{2}\mu_0^2\sigma_0^2-\frac{\lambda}{4!}\sigma_0^4+C\sigma_0. \quad (7)$$

In Eq. (7) all parameters are unrenormalized.

The relevant perturbation expansion in the presence of spontaneous symmetry breaking may be shown to be in terms of the number of loops in Feynman diagrams,[10] and it corresponds to an expansion to a given power in λ and in g, but to all orders in $\lambda\sigma_0^2$ and $g\sigma_0$, with the true minimum of the effective potential being used to define the ground state. For such a ground state the expectation value of the σ field appearing in (7) should be zero. This would be ensured by the absence of the terms linear in the σ field to all orders of perturbation theory. These terms give rise to the tadpole graphs. In the tree-diagram approximation we have

$$T_0=-i[\sigma_0(\mu_0^2+\tfrac{1}{6}\lambda\sigma_0^2)-C], \quad (8)$$

and σ_0 is defined by $T_0=0$ to be

$$\sigma_0(\mu_0^2+\tfrac{1}{6}\lambda\sigma_0^2)=C. \quad (9)$$

The masses of the particles are given by

$$m_{\pi}^2=\mu_0^2+\frac{\lambda\sigma_0^2}{6}=\frac{C}{\sigma_0}=\frac{f_{\pi}m_{\pi}^2}{\sigma_0}, \quad (10)$$

which shows that $\sigma_0 = f_\pi$, and

$$m_\sigma^2 = \mu_0^2 + \tfrac{1}{2}\lambda\sigma_0^2 , \tag{11}$$

$$M_N = g\sigma_0 . \tag{12}$$

The parameters of the theory may be expressed in terms of the masses:

$$\lambda = \frac{3}{f_\pi^2}(m_\sigma^2 - m_\pi^2) , \tag{13a}$$

$$\mu_0^2 = -\tfrac{1}{2}(m_\sigma^2 - 3m_\pi^2) , \tag{13b}$$

and

$$g = M_N / f_\pi . \tag{13c}$$

For a positive coupling constant λ, the mass m_σ will be larger than m_π. Also, μ_0^2 is negative if the symmetry-breaking term $C\sigma$ is absent. (A parallel situation obtains in Landau's theory of phase transitions.[23]) The Goldberger-Treiman relation[24] in this model is given by (13c).

Using the notation explained in Appendix A the propagators for these excitations are given by

$$iS_F(p) = \frac{i(\not p + M)}{[\![p^2 - M^2 + i\epsilon]\!]} , \tag{14}$$

$$iD_\sigma(k) = i/[\![k^2 - m_\sigma^2 + i\epsilon]\!] , \tag{15}$$

$$iD_\pi(k) = i/[\![k^2 - m_\pi^2 + i\epsilon]\!] , \tag{16}$$

where the double brackets on the denominators denote the presence of temperature-dependent terms in addition to the usual Feynman poles in these propagators. These propagators are used to evaluate higher-order effects in Sec. III, and temperature-dependent corrections to the parameters of the theory are obtained.

The scattering amplitudes are also modified in the Born approximation since the presence of many particles now affects the phase space available in scattering. As a first example consider $\pi\pi \to \pi\pi$ scattering. With the above changes in the propagators the scattering amplitude for $\pi_a\pi_b \to \pi_c\pi_d$ is

$$T_{cd,ab} = \left[\delta_{ac}\delta_{bd}\left(\frac{\lambda}{3} + \frac{\lambda^2 f_\pi^2}{3(m_\sigma^2 - s + i\epsilon)} + \frac{2\pi i\,\delta(s - m_\sigma^2)}{e^{\beta E_\sigma} - 1}\right) + \delta_{ad}\delta_{bc}(s \to u) + \delta_{ab}\delta_{cd}(s \to t) \right] . \tag{17}$$

The additional terms proportional to δ functions in the (energy)^2 variable in each channel contribute to the imaginary part of the amplitude, and this corresponds to the thermal background of σ mesons contributing to the scattering amplitude a finite amount due to absorption and emission. However, this does not affect the $\pi\pi$ scattering lengths as long as $m_\sigma^2 \neq 4m_\pi^2$. The expressions for cross sections will now have temperature-dependent factors.

As a second example consider $\sigma \to \pi\pi$. The decay width for this process is

$$\Gamma = \frac{\lambda^2\sigma_0^2}{3}\,\frac{(m_\sigma^2 - 4m_\pi^2)^{1/2}}{16\pi m_\sigma^2}\,[2n_B(E_k) + 1]^2 = \frac{3(m_\sigma^2 - m_\pi^2)^2}{16\pi f_\pi^2}\left(\frac{m_\sigma^2 - 4m_\pi^2}{m_\sigma^4}\right)^{1/2}\left(\frac{2}{e^{\beta m_\sigma/2} - 1} + 1\right)^2 . \tag{18}$$

The additional factor dependent on temperature represents the thermal broadening of the σ width.

As a third example consider $\pi N \to \pi N$ where the pions and the nucleons are the excitations in the presence of many particles. The appearance of nucleon and nucleon-hole quasiparticles requires us to consider pions scattering off a specified quasiparticle, and the initial conditions of the problem select the appropriate term in the fermion propagator:

$$iS_F(p) = \frac{i(\not p + M)}{2E_p}\left[\frac{\theta(E_p - \mu_F)}{p_0 - E_p + i\epsilon} + \frac{\theta(\mu_F - E_p)}{p_0 - E_p - i\epsilon} - \frac{1}{p_0 + E_p - i\epsilon}\right] = i(\not p + M)\left[\frac{\theta(E_p - \mu_F)}{p_0^2 - (E_p - i\epsilon)^2} + \frac{\theta(\mu_F - E_p)}{(p_0 - i\epsilon)^2 - E_p^2}\right], \tag{19}$$

which appears in the Born-approximation amplitude. The θ functions in (19) take account of the Pauli exclusion principle and the resultant changes in the phase space. In Eq. (19) and in the rest of this paper μ_F denotes the chemical potential of the nucleons.

III. RENORMALIZATION AT THE ONE-LOOP LEVEL

A. The tadpole graphs for $C \neq 0$

We begin by evaluating the one-loop corrections to the tadpole term shown in Fig. 1. If the symmetry-breaking term $C\sigma$ is present in the Lagrangian (1) then m_π^2 is not zero. Using the dimensional regularization described in Appendix B we obtain

$$T_1 = -i\sigma_0\left[\frac{\lambda}{2(2\pi)^4}\int d^4k\left(\frac{i}{[\![k^2-m_\sigma^2+i\epsilon]\!]}+\frac{i}{[\![k^2-m_\pi^2+i\epsilon]\!]}\right)-\frac{2g^2}{(2\pi)^4}\int d^4p\frac{4i}{\{\!\{p^2-M^2+i\epsilon\}\!\}}\right]$$

$$= -i\sigma_0\left[\frac{\lambda}{16\pi^2(n-4)}(m_\sigma^2+m_\pi^2)-\frac{16g^2M^2}{16\pi^2(n-4)}+\cdots\right]$$

$$-i\sigma\left[\frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\left(\frac{n_B((\vec{k}^2+m_\sigma^2)^{1/2})}{(\vec{k}^2+m_\sigma^2)^{1/2}}+\frac{n_B((\vec{k}^2+m_\pi^2)^{1/2})}{(\vec{k}^2+m_\pi^2)^{1/2}}\right)+8g^2\int\frac{d^3p}{(2\pi)^3}\frac{n_F((\vec{p}^2+M^2-\mu_F)^{1/2})}{2(\vec{p}^2+M^2)^{1/2}}\right] \tag{20}$$

where the leading terms with poles at $n=4$ have been shown.

The meson-scattering graphs of Fig. 2 and the self-energy graphs of Figs. 3(a), 3(b), and 3(d) for the symmetric theory (with $\sigma_0=0$) require the counterterms

$$\delta\mu_0^2=-(2\lambda\mu_0^2+8g^2\mu_0^2)/16\pi^2(n-4),\quad Z_\Phi-1=8g^2/16\pi^2(n-4),\qquad \delta\lambda=-(4\lambda^2-96g^4)/16\pi^2(n-4), \tag{21}$$

where Z_Φ is the wave-function renormalization constant and $\delta\mu_0^2$ and $\delta\lambda$ are the mass and coupling-constant renormalization counterterms. These counterterms of the symmetric theory give rise to an additional term to this order, for the σ tadpole:

$$-i\sigma_0\left[\delta\mu_0^2+\frac{\delta\lambda}{6}\sigma_0^2+\frac{\lambda\sigma_0^2}{6}(Z_\Phi^{-1}-1)-\frac{C}{\sigma_0}(Z_\Phi^{-1}-1)\right]. \tag{22}$$

Using the relations (10)–(12) for the masses it is seen that these counterterms of the symmetric theory cancel[25] the pole terms of T_1 in (20).

The condition determing σ_0, viz., $T_0+T_1=0$, is now given by

$$\sigma_0\left(\mu_0^2+\frac{\lambda\sigma_0^2}{6}-\frac{C}{\sigma_0}\right)+\frac{\lambda\sigma_0}{2}\int\frac{d^3k}{(2\pi)^3}\left[\frac{n_B((\vec{k}^2+m_\sigma^2)^{1/2})}{(\vec{k}^2+m_\sigma^2)^{1/2}}+\frac{n_B((\vec{k}^2+m_\pi^2)^{1/2})}{(\vec{k}^2+m_\pi^2)^{1/2}}\right]+8g^2\int\frac{d^3p}{(2\pi)^3}\frac{n_F((\vec{p}^2+M^2-\mu_F)^{1/2})}{2(\vec{p}^2+M^2)^{1/2}}=0. \tag{23}$$

The integrals appearing in (23) can be evaluated in various limits. At very low temperatures, where the nucleon chemical potential plays an important role, consider the integral

$$I_1\equiv\frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+m^2)^{1/2}}[e^{\beta(k^2+m^2)^{1/2}}-1]^{-1}$$

$$=\frac{\lambda m}{4\pi^2}\int_0^\infty\frac{dx\,x^2}{(x^2+1)^{1/2}}\sum_{s=1}^\infty e^{-\beta ms(x^2+1)^{1/2}},\quad s=1,2,3,\ldots$$

$$=\frac{\lambda}{4\pi^2\beta}\sum_{s=1}^\infty\frac{1}{s}K_1(\beta ms)$$

$$\underset{\beta\to\infty}{\sim}\frac{\lambda}{4\pi^2}\left(\frac{\pi}{2m\beta^3}\right)^{1/2}\sum_{s=1}^\infty\frac{e^{-\beta ms}}{s^{3/2}}. \tag{24}$$

The sum over s in the last expression for I_1 is less than $\zeta(\frac{3}{2})\cong 2.612$, and the integral is convergent. For $m=0$ we return to the original expression in (24) to obtain

$$I_1(m=0)=\frac{\lambda}{24\beta^2}. \tag{25}$$

Similarly, in the case of fermions, consider

$$I_2\equiv 8g^2\int\frac{d^3p}{(2\pi)^3}\frac{n_F((p^2+M^2)^{1/2}-\mu_F)}{(p^2+M^2)^{1/2}}$$

$$=\frac{8g^2}{4\pi^2}\left\{\mu_F(\mu_F^2-M^2)^{1/2}-M^2\ln\left[\frac{\mu_F+(\mu_F^2-M^2)^{1/2}}{M}\right]+\frac{2\pi^2\mu_F}{3\beta^2(\mu_F^2-M^2)^{1/2}}-\frac{7\pi^4\mu_F}{60\beta^4(\mu_F^2-M^2)^{3/2}}+\cdots\right\}. \tag{26}$$

In the limit of extremely high temperatures the chemical potential of the nucleons may be neglected. The relevant energy scale in this theory is given by the σ-meson mass, and this is very much higher than the degeneracy temperature defined by μ_F for any reasonable nucleon density. In this limit of very high temperatures I_1 and I_2 have been evaluated by Dolan and Jackiw,[9] and

$$I_1 \underset{\beta\to 0}{\sim} \frac{\lambda}{2}\left[\frac{1}{12\beta^2} - \frac{m}{4\pi\beta} + \frac{m^2}{4\pi^2}\left(\tfrac{1}{2}\ln 4\pi + \tfrac{1}{4} - \tfrac{1}{2}\gamma - \ln\beta m\right) + \cdots\right] \tag{27}$$

and

$$I_2 \underset{\beta\to 0}{\sim} \frac{8g^2}{2\pi^2}\left\{\frac{\pi^2}{12\beta^2} + \frac{M^2}{8}\left[2\ln M^2\beta^2 - 1 + 2(\gamma - \ln\pi)\right] + \cdots\right\}, \tag{28}$$

where γ is Euler's constant. It should be noted that the masses appearing in the expression for I_1 and I_2 are all positive and real.

At zero temperature Eq. (23), which determines the value of σ_0, reduces to

$$\sigma_0\left(\mu_0^2 + \frac{\lambda\sigma_0^2}{6}\right) - C + \frac{g^2\sigma_0}{\pi^2}\left\{\mu_F(\mu_F^2 - g^2\sigma_0^2)^{1/2} - g^2\sigma_0^2 \ln\left[\frac{\mu_F + (\mu_F^2 - g^2\sigma_0^2)^{1/2}}{g\sigma_0}\right]\right\}. \tag{29}$$

For large values of μ_F, (29) takes the form

$$\sigma_0\left(\mu_0^2 + \frac{g^2\mu_F^2}{\pi^2}\right) + \frac{\lambda\sigma_0^3}{6} - C = 0, \tag{30}$$

and at high temperature σ_0 is determined by the relation

$$\sigma_0\left(\mu_0^2 + \frac{\lambda + 2g^2}{12\beta^2}\right) + \frac{\lambda\sigma_0^3}{6} - C = 0, \tag{31}$$

where only the leading terms have been retained in (30) and (31).

It is clear from Eqs. (30) and (31) that the introduction of many-body effects changes the nature of the solutions for σ_0. At zero temperature and density, σ_0 is given by a cubic equation with three real roots for

$$\frac{9C^2}{\lambda^2} + \frac{8\mu_0^6}{\lambda^3} < 0.$$

Recall that in order to have a σ meson heavier than $2m_\pi$ we required $\mu_0^2 < 0$, and since C is proportional to m_π^2, which is small, the above inequality is satisfied. As long as $C \neq 0$ there is always one positive root, so that $\sigma_0 \neq 0$ as the temperature or the nucleon density is increased. This means that chiral symmetry is never restored with increasing temperature or density. This is to be expected since the explicit-symmetry-breaking term $C\sigma$ continues to pick out a specific direction in the internal-symmetry space.

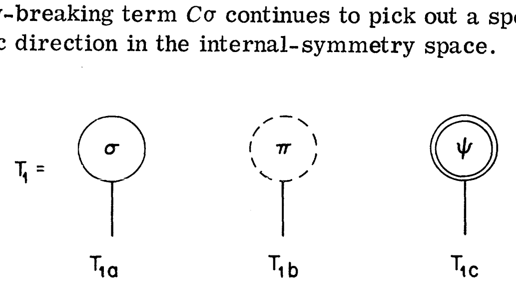

FIG. 1. The one-loop tadpole diagrams in the σ model.

As we are interested in symmetry changes with increasing temperature or density, we shall neglect the presence of such a term in the rest of this paper.

B. The tadpole graphs for $C = 0$

When the term $C\sigma$ is absent from the Lagrangian (1) the pion mass is zero and the axial-vector current is conserved. In the dimensional-regularization scheme the zero-temperature terms of the tadpole graphs at the one-loop level are zero for massless particles in the sense that they require no counterterms.[16] However, the finite-temperature terms survive as usual. Thus the one-loop tadpole terms are

$$\begin{aligned} T_1 = &-i\sigma_0\left[\frac{\lambda m_\sigma^2}{16\pi^2(n-4)} - \frac{16g^2M_N^2}{16\pi^2(n-4)} + \cdots\right] \\ &-i\sigma_0\left[\frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\left(\frac{n_B((\vec{k}^2 + m_\sigma^2)^{1/2})}{(\vec{k}^2 + m_\sigma^2)^{1/2}} + \frac{n_B(|\vec{k}|)}{|\vec{k}|}\right)\right. \\ &\left.+ 8g^2\int\frac{d^3p}{(2\pi)^3}\,\frac{n_F((\vec{p}^2 + M^2 - \mu_F)^{1/2})}{2(\vec{p}^2 + M^2)^{1/2}}\right] \\ &+ \text{counterterms}. \end{aligned} \tag{32}$$

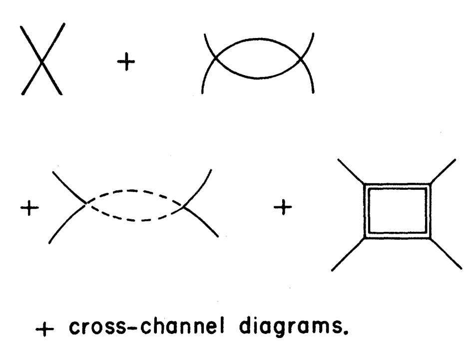

FIG. 2. Meson-meson scattering diagrams.

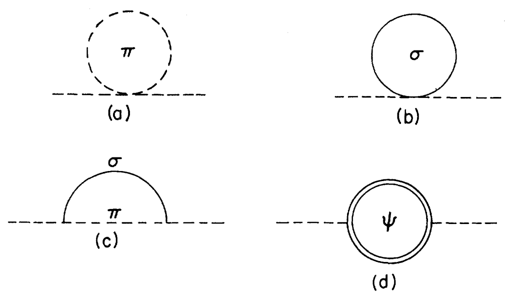

FIG. 3. Self-energy graphs for pions at the one-loop level.

Since

$$m_\pi^2=\mu_0^2+\frac{\lambda}{6}\sigma_0^2=0,$$

$$m_\sigma^2=\mu_0^2+\frac{\lambda\sigma_0^2}{2}=\frac{\lambda\sigma_0^2}{3},$$

we see that the counterterms of the symmetric theory again eliminate the pole terms in (32). Also, the conditions defining σ_0 are of the form

$$(\tilde{\mu}_0^2+\tfrac{1}{6}\lambda\sigma_0^2)\sigma_0=0, \tag{33}$$

where

$$\tilde{\mu}_0^2=\mu_0^2+\frac{g^2\mu_F^2}{\pi^2}, \tag{34}$$

in the high-density limit at zero temperature, and

$$\tilde{\mu}_0^2=\mu_0^2+\frac{\lambda+2g^2}{12\beta^2} \tag{35}$$

in the high-temperature limit.

Initially, at zero temperature and density

$$\tilde{\mu}^2=\mu_0^2=-\tfrac{1}{2}m_\sigma^2. \tag{36}$$

As the density increases $\tilde{\mu}_0^2$ continues to increase till, at a critical density ρ_c and a corresponding critical chemical potential $\mu_{F,c}$ given by

$$\mu_{F,c}^2=-\mu_0^2\left(\frac{\pi^2}{g^2}\right)=\frac{\pi}{16}\left(\frac{2m_\sigma^2}{g^2/4\pi}\right), \tag{37}$$

the quantity σ_0 vanishes. The number density corresponds to

$$\frac{\rho_c}{M_N^3}=\frac{2}{3\pi^2}\left(\frac{\pi}{16}\right)^{3/2}\left(\frac{8\pi m_\sigma^2}{g^2M_N^2}\right)^{3/2}$$

or

$$\rho_c=\frac{\pi}{12}\left(\frac{m_\sigma(2)^{1/2}}{g}\right)^3. \tag{38}$$

Above this density the original symmetry is restored and the nucleons are massless and the mesons are degenerate. This is the abnormal-nuclear-matter phase proposed first by Lee and Wick.[8]

In the high-temperature limit $\tilde{\mu}_0^2$ increases with temperature until, at a critical temperature given by

$$\frac{1}{\beta_c^2}\equiv T_c^2=-\mu_0^2\left(\frac{12}{\lambda+2g^2}\right)=\frac{6m_\sigma^2}{\lambda+2g^2}, \tag{39}$$

the vacuum expectation value of the σ field vanishes. In the absence of fermions (39) reduces to the expression obtained by Weinberg[7] for the symmetry group O(N) with $N=4$. Above this critical temperature the system reverts to the normal, symmetric phase in which the mesons are degenerate. Near the critical temperature the mesons and the nucleons are massless, while much above the critical temperature the particles have temperature-dependent masses. From (39) it is also seen that increasing the number of species of particles interacting with the σ mesons lowers the critical temperature.

The infrared divergences associated with the massless particles may be expected to lead to a breakdown of perturbation theory in the neighborhood of the critical temperature[7] and the critical density, and the expressions for the critical values of these two parameters are valid in the lowest order.

C. Renormalization of meson masses

The self-energy corrections to the pion mass are given by the Feynman diagrams of Fig. 3. With the temperature-dependent propagators the one-loop corrections to the inverse of the pion propagator can be calculated and

$$\begin{aligned}
iZ_\Phi D_\pi(p^2)^{-1}&=Z_\Phi(p^2-m_{\pi,R}^2)\\
&=[p^2-m^2_\pi-\delta m^2_\pi-\Sigma_\pi(p^2)]Z_\Phi\\
&=Z_\Phi p^2-[\delta\mu_0^2+\tfrac{1}{6}\delta\lambda\sigma_0^2+(Z_\Phi^{-1}-1)\tfrac{1}{6}\lambda\sigma_0^2]\\
&\quad-\frac{i\lambda}{6(2\pi)^4}\int d^4k\left(\frac{1}{[\![k^2-m_\sigma^2+i\epsilon]\!]}+\frac{5}{[\![k^2+i\epsilon]\!]}\right)\\
&\quad-\frac{i\lambda^2\sigma_0^2}{9(2\pi)^4}\int d^4k\,\frac{1}{[\![k^2-m_\sigma^2+i\epsilon]\!]}\,\frac{1}{[\![(p+k)^2+i\epsilon]\!]}\\
&\quad+\frac{i2g^2}{(2\pi)^4}\int d^4k\left(\frac{4}{\{\![k^2-M^2+i\epsilon]\!\}}-\frac{2p^2}{\{\![k^2-M^2+i\epsilon]\!\}\{\![(p+k)^2-M^2+i\epsilon]\!\}}\right).
\end{aligned} \tag{40}$$

It is instructive to see how the Goldstone mode survives at finite temperature and density. Use is made of Eq. (23) (with $C=0$) to rewrite the inverse pion propagator at $p^2=0$ in the form

$$\begin{aligned} iZ_\Phi D_\pi(0)^{-1} &= \frac{i\lambda}{3(2\pi)^4}\int d^4k\left(\frac{1}{[\![k^2-m^2_\sigma+i\epsilon]\!]}-\frac{1}{[\![k^2+i\epsilon]\!]}\right)-\frac{i\lambda^2\sigma^2_0}{9(2\pi)^4}\int d^4k\left(\frac{1}{[\![k^2-m_\sigma^2+i\epsilon]\!][\![k^2+i\epsilon]\!]}\right)\\ &= \frac{i\lambda}{3(2\pi)^4}\int d^4k\,\frac{m_\sigma^2}{(k^2-m_\sigma^2)(k^2)}-\frac{\lambda}{3}\int\frac{d^3k}{(2\pi)^3}\left(\frac{n_B(E_k,m_\sigma)}{E_k}-\frac{n_B(|\vec{k}|)}{|\vec{k}|}\right)\\ &\quad-\frac{i\lambda^2\sigma^2_0}{9(2\pi)^4}\int d^4k\,\frac{1}{(k^2-m^2_\sigma)(k^2)}+\frac{\lambda^2\sigma^2_0}{9}\int\frac{d^3k}{(2\pi)^3}\left[\frac{n_B(E_k,m_\sigma)}{E_k(E_k^2-\vec{k}^2)}+\frac{n_B(|\vec{k}|)}{|\vec{k}|(\vec{k}^2-E^2_k)}\right]\\ &=0. \end{aligned} \tag{41}$$

Thus the renormalized pion mass $m_{\pi,R}$ is zero,

$$m_\pi^2(\beta,\mu_F)=0\quad\begin{cases}\beta_C<\beta\,,\\ \mu_F<\mu_{F,c}\,,\end{cases} \tag{42}$$

and the Goldstone mode persists as long as $\sigma_0\neq 0$, verifying Goldstone's theorem for the present relativistic many-body theory for $\mu_F\neq 0$ and $\beta^{-1}\neq 0$.

The self-energy corrections to the σ mass are shown in Fig. 4, and the inverse propagator for the σ is given by

$$\begin{aligned} iZ_\Phi D_\sigma(p^2)^{-1} &= Z_\Phi\left\{p^2-\left[\mu_0^2+\delta\mu_0^2+\frac{\lambda+\delta\lambda}{2}\sigma_0^2+(Z_\Phi^{-1}-1)\frac{\lambda\sigma_0^2}{2}\right]\right\}\\ &\quad-\frac{i\lambda}{2(2\pi)^4}\int d^4k\left(\frac{1}{[\![k^2-m_\sigma^2+i\epsilon]\!]}+\frac{1}{[\![k^2+i\epsilon]\!]}\right)\\ &\quad-\frac{i\lambda^2\sigma_0^2}{6(2\pi)^4}\int d^4k\left(\frac{3}{[\![k^2-m_\sigma^2+i\epsilon]\!][\![(p+k)^2-m_\sigma^2+i\epsilon]\!]}+\frac{1}{[\![k^2+i\epsilon]\!][\![(p+k)^2+i\epsilon]\!]}\right)\\ &\quad+\frac{2ig^2}{(2\pi)^4}\int d^4k\left(\frac{4}{\{\![k^2-M^2+i\epsilon]\!\}}+\frac{(8M^2-2p^2)}{\{\![k^2-M^2+i\epsilon]\!\}\{\![(p+k)^2-M^2+i\epsilon]\!\}}\right). \end{aligned} \tag{43}$$

Using Eq. (23) it is easy to show that the counterterms of the symmetric theory remove the divergences in (43). Since $m_\sigma>2m_\pi$ the σ meson can decay into 2π and the σ mass develops an imaginary part because of the finite width of the σ meson. At finite temperature and density this imaginary part is modified, as we have already seen in Sec. II [Eq. (18)].

It is worth mentioning that the coupling constants now acquire finite-temperature-dependent corrections. If the renormalized coupling constant for $\mu_F=0=1/\beta$ is defined at zero external momenta as λ_R, the coupling constant at finite temperature is given by contributions from diagrams of Fig. 2, together with "triangle" and "box" diagrams which are generated by the presence of the cubic meson couplings in Eq. (7) and which have finite Feynman integrals and constitute nonleading contributions. The complete expression for $\lambda_R(\beta,\mu_F)$ is somewhat lengthy and is given in Appendix C. Products of propagators appearing in the diagrams can be simplified using Eqs. (A14) and (A15), or generalizations of these obtained by employing Eq. (A13). At zero temperature $n_B=0$ and $n_F=\theta(\mu_F-p_0)$ and only the fermion contribution leads to a change in λ_R due to many body effects. By returning to the original expression for λ in terms of Feynman integrals over propagators and using Eq. (A15) with $n_F=\theta(\mu_F-p_0)$ we derive

$$\begin{aligned} \lambda_R(\beta^{-1}=0,\ \mu_F) &= \lambda_R-\frac{6g^4}{\pi^2}\ln\left[\frac{\mu_F+(\mu_F^2-M^2)^{1/2}}{M}\right]\\ &\quad+\frac{12g^4M^2}{\pi^2}\left[\frac{1}{M^2}\left(\frac{\mu_F^2-M^2}{\mu_F^2}\right)^{3/2}+\frac{(\mu_F^2-M^2)^{1/2}}{\mu_F^3}+\frac{1}{\mu_F(\mu_F^2-M^2)^{1/2}}\right]\\ &\quad-\frac{30g^4}{\pi^2}\left[\frac{1}{3}\left(\frac{\mu_F^2-M^2}{\mu_F^2}\right)^{3/2}-\frac{1}{5}\left(\frac{\mu_F^2-M^2}{\mu_F^2}\right)^{5/2}+M^4\frac{(\mu_F^2-M^2)^{1/2}}{\mu_F^5}\right]\\ &\quad-\frac{g^4M^4}{\pi^2}\left[24\frac{(\mu_F^2-\mu^2)^{1/2}}{\mu_F^5}-2\frac{1}{\mu_F^3(\mu_F^2-M^2)^{1/2}}+2\frac{1}{\mu_F(\mu_F^2-M^2)^{3/2}}\right], \end{aligned} \tag{44}$$

and for large μ_F

$$\lambda_R(\beta^{-1}=0,\ \mu_F)\approx\lambda_R-\frac{6g^4}{\pi^2}\ln\left[\frac{\mu_F+(\mu_F^2-M^2)^{1/2}}{M}\right]+O(1)\,. \tag{45}$$

At finite temperature and density the σ mass is shifted from its original value of $\lambda\sigma_0^2/3$ by a finite correction term which is obtained by evaluating (43) at $p^2=0$. We obtain

$$\begin{aligned} m_{\sigma,R}^2(\beta,\mu_F)=m_{\sigma,R}^2&-\frac{\lambda^2\sigma_0^2}{4(2\pi)^3}\int d^3k\left[\frac{n_B^2((k^2+m_\sigma^2)^{1/2})\beta e^{\beta(k^2+m_\sigma^2)^{1/2}}}{k^2+m_\sigma^2}+\frac{n_B((k^2+m_\sigma^2)^{1/2})}{(k^2+m_\sigma^2)^{3/2}}\right]\\ &-\frac{\lambda^2\sigma_0^2}{12(2\pi)^3}\int d^3k\left[\frac{n_B^2(|\vec{k}|)\beta e^{\beta|\vec{k}|}}{|\vec{k}|^2}+\frac{n_B(|\vec{k}|)}{|\vec{k}|^3}\right]\\ &-\frac{4g^4\sigma_0^2}{(2\pi)^3}\int d^3p\left[\frac{n_F^2((p^2+M^2)^{1/2},\mu_F)\beta e^{\beta(p^2+M^2)^{1/2}}}{p^2+M^2}+\frac{n_F((p^2+M^2)^{1/2},\mu_F)}{(p^2+M^2)^{3/2}}\right]. \end{aligned} \tag{46}$$

At zero temperature, using $n_F=\theta(\mu_F-p_0)$ and Eq. (A15) we reevaluate m_σ^2 from (43) and obtain

$$m_{\sigma,R}^2(\beta^{-1}=0,\ \mu_F)=m_{\sigma,R}^2-\frac{2g^4\sigma_0^2}{\pi^2}\ln\left[\frac{\mu_F+(\mu_F^2-M^2)^{1/2}}{M}\right].$$

The functional relation between mass and the parameters of the theory, which are all temperature and density dependent, is no longer $m_\sigma^2=\frac{1}{3}\lambda_R\sigma_0^2$ because of the nonleading terms.

D. The mass spectrum in the symmetric phase

Let us first consider the abnormal-nuclear-matter phase at zero temperature, which is reached when $\rho>\rho_c$. Since $\sigma_0=0$, the nucleons remain massless as the density is raised beyond the critical value.[8] The meson masses are now degenerate and

$$\begin{aligned} -iZ_\Phi D_{\sigma,\pi}^{-1}(p^2=0)&=\mu_0^2+\frac{8g^2}{(2\pi)^3}\int d^3k\,\frac{n_F(|\vec{k}|,\mu_F)}{2|\vec{k}|}\\ &=\mu_0^2+\frac{8g^2}{4\pi^2}\int_0^{\mu_F}k\,dk\\ &=\mu_0^2+\frac{g^2}{\pi^2}\mu_F^2,\quad \mu_F\geq\mu_{F,c}\,. \end{aligned} \tag{47}$$

Thus the meson masses which are zero near the critical density continue to increase with μ_F and

$$m_{\sigma,\pi}^2=\frac{g^2}{\pi^2}(\mu_F^2-\mu_{F,c}^2)\theta(\mu_F-\mu_{F,c})\,. \tag{48}$$

Thus above and below the critical density the mass spectrum is positive, as it should be. This is shown in Fig. 5.

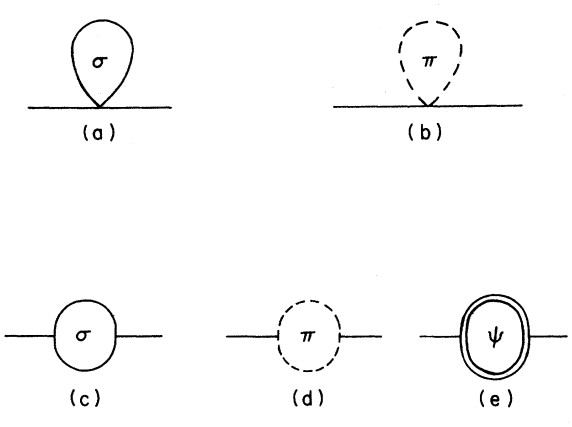

FIG. 4. The one-loop self-energy diagrams for the σ meson.

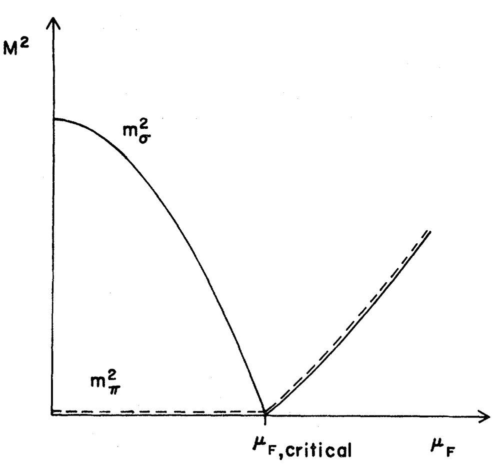

FIG. 5. Schematic graph of the meson spectrum as a function of the nucleon chemical potential μ_F.

Now consider the mass spectrum above the critical temperature. Since $\sigma_0 = 0$ above the critical temperature the boson masses are degenerate and the nucleon mass is zero. The masses increase from zero as the temperature increases above T_c and we have

$$\begin{aligned}-iZ_\Phi D_{\sigma,\pi}{}^{-1}(p^2=0) &= m_{\sigma,\pi}{}^2\\ &= \mu_0{}^2 + \int \frac{d^3k}{(2\pi)^3}\left[\frac{\lambda n_B(|\vec{k}|)}{|\vec{k}|} + \frac{8g^2 n_F(|\vec{k}|)}{2|\vec{k}|}\right]\\ &= \frac{\lambda + 2g^2}{12}(T^2 - T_c{}^2)\theta(T - T_c).\end{aligned} \tag{49}$$

The temperature dependence of the boson spectrum is shown in Fig. 6.

E. Renormalization of the nucleon mass

In the symmetric theory at $\mu_F = 0$ and $1/\beta = 0$ the nucleons are massless and no counterterms for mass renormalization are necessary. The nucleons do require wave-function renormalization. The self-energy diagrams of Fig. 7 can be evaluated to obtain Z_Ψ, the wave-function renormalization constant for the nucleons, and the leading terms are

$$Z_\Psi = 1 + \frac{4g^2}{16\pi^2(n-4)}\ . \tag{50}$$

The same result holds for $\sigma_0 \neq 0$, and no mass renormalization is necessary for the nucleons; also in this renormalization scheme no auxiliary fermion fields are needed. In the presence of nuclear matter the inverse nucleon propagator including the one-loop self-energy corrections is given by

$$\begin{aligned}iZ_\Psi S_F{}^{-1}(p) = Z_\Psi\Bigg[\not p - (g+\delta g)\sigma_0 Z_\Phi{}^{-1/2} - \frac{ig^2}{(2\pi)^4}\int d^4k\left(\frac{\not k + M}{\{[k^2 - M^2 + i\epsilon]\}[[(p+k)^2 - m_\sigma{}^2 + i\epsilon]]}\right)\\ - \frac{i3g^2}{(2\pi)^4}\int d^4k\left(\frac{\not k - M}{\{[k^2 - M^2 + i\epsilon]\}[[(p+k)^2 + i\epsilon]]}\right)\Bigg]\ .\end{aligned} \tag{51}$$

The term $\delta g\sigma_0$ is canceled by $g\sigma_0(Z_\Phi{}^{-1/2} - 1)$, so that $\delta M = 0$. The infinite parts of the two integrals in (51) are eliminated by Z_Ψ.

IV. ABSENCE OF TEMPERATURE-DEPENDENT INFINITIES

For the 't Hooft renormalization procedure[18] to be applicable the higher-order Feynman diagrams should contain no pole terms at $n=4$ with residues which are not polynomials in the external momenta or external masses. At intermediate stages of the calculation, pole terms with residues which have $\log(p^2)$ or temperature-dependent factors do occur in any given Feynman diagram. However, the sum of such terms at any

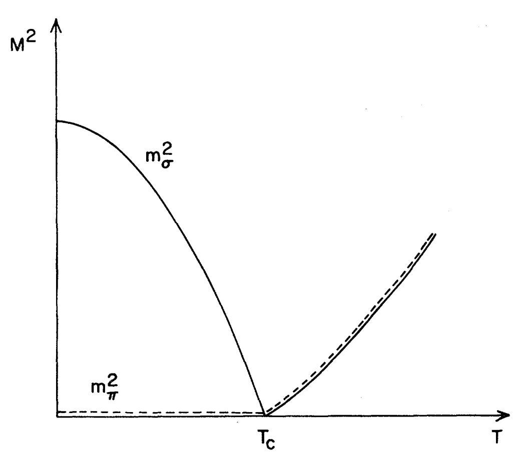

FIG. 6. The temperature dependence of the meson spectrum.

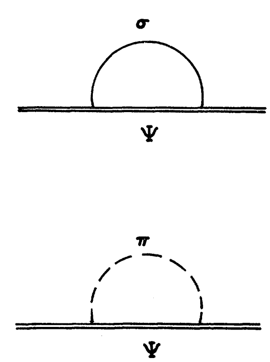

FIG. 7. The nucleon self-energy diagrams with σ and π exchanges.

order is canceled by terms arising from lower-order diagrams with counterterm insertions. Let us consider the two-loop tadpoles of Fig. 8, together with the counterterms of Fig. 9. We now show that these diagrams are free of nonpolynomial or temperature-dependent residues. This is best done by regrouping the terms in each Feynman diagram into (i) the usual Feynman integrals at zero temperature and density, (ii) the temperature- and density-dependent factors multiplying Feynman subintegrals having divergences, and (iii) finite, temperature-dependent integrals. Such a separation of the terms is straightforward because the real-time Green's functions whose use is advocated here have this separation of the usual Feynman poles from the temperature- and density-dependent terms in them.

Consider first the zero-temperature calculation of the set (i) of the two-loop tadpole terms. The terms arising from the diagrams of Figs. 8(h)–8(k) and a pion-mass counterterm in Fig. 9 are

$$T_2^{\mathrm{i}}(h,i,j,k)+T_1^{\mathrm{i}}(\delta m_\pi^2)=\frac{\lambda\sigma_0}{2(2\pi)^4}\int d^4k\frac{1}{(k^2+i\epsilon)^2}[\Sigma_\pi(k^2)+\delta m_\pi^2]\,, \tag{52}$$

where $\Sigma_\pi(k^2)$ is the pion self-energy term appearing in Eq. (40). Now the terms proportional to k^2 in $\Sigma_\pi(k^2)$ give integrals of the form (B1) given in Appendix B, and these terms may be dropped in the dimensional-regularization scheme. Using the mass counterterm, defined by the finiteness of (40), and the integral (B6), we have

$$T_2^{\mathrm{i}}(h,i,j,k)+T_1^{\mathrm{i}}(\delta m_\pi^2)=\frac{i\lambda^3\sigma_0^3}{(16\pi^2)^2(n-4)}\left(-\frac{1}{18}\ln\frac{m_\sigma^2}{\mu^2}\right). \tag{53}$$

The terms of Figs. 8(a)–8(e) are given by the σ self-energy terms inserted in the one-loop tadpole, and we have

$$\begin{aligned}T_2^{\mathrm{i}}(a,b,c,d,e)=&\frac{i\lambda^2\sigma_0 m_\sigma^2}{(16\pi^2)^2}\left[-\frac{3}{n-4}+\frac{3/2}{n-4}-\frac{3\ln(m_\sigma^2/\mu^2)}{n-4}+\frac{3}{2}\ln\left(\frac{m_\sigma^2}{\mu^2}\right)-\frac{3}{2}\ln^2\left(\frac{m_\sigma^2}{\mu^2}\right)\right]\\&+\frac{-i2\lambda\sigma_0 g^2}{(16\pi^2)^2}\left[\frac{-12M^2+4m_\sigma^2}{(n-4)^2}+\frac{-2M^2+2m_\sigma^2}{n-4}+\frac{-12M^2\ln(m_\sigma^2/\mu^2)+4M_\sigma^2\ln(m_\sigma^2/\mu^2)}{n-4}\right.\\&\qquad+m_\sigma^2+(6M^2-4m_\sigma^2)\ln\left(\frac{m_\sigma^2}{\mu^2}\right)+(-4M^2+2m_\sigma^2)\ln^2\left(\frac{m_\sigma^2}{\mu^2}\right)-4M^2\ln\left(\frac{M^2}{\mu^2}\right)\\&\qquad\left.+2M^2\ln^2\left(\frac{M^2}{\mu^2}\right)-M^2\ln^2\left(\frac{M^2m_\sigma^2}{\mu^4}\right)\right].\end{aligned} \tag{54}$$

The diagrams of Figs. 8(f) and 8(g) give

$$T_2^{\mathrm{i}}(f,g)=\frac{i\lambda^2\sigma_0 m_\sigma^2}{(16\pi^2)^2}\left[\frac{-4/3}{(n-4)^2}+\frac{2}{n-4}-\frac{4/3}{n-4}\ln\left(\frac{m_\sigma^2}{\mu^2}\right)-\tfrac{2}{3}+2\ln\left(\frac{m_\sigma^2}{\mu^2}\right)-\frac{2}{3}\ln^2\left(\frac{m_\sigma^2}{\mu^2}\right)\right], \tag{55}$$

and the corrections to the nucleon tadpole represented by diagrams of Figs. 8(l) and 8(m) are

$$\begin{aligned}T_2^{\mathrm{i}}(l,m)=&\frac{-i2g^2M}{(16\pi^2)^2}\left[\frac{-24m_\sigma^2}{(n-4)^2}+\frac{32M^2+28m_\sigma^2}{n-4}+\frac{-24m_\sigma^2\ln(m_\sigma^2/\mu^2)}{n-4}+32M^2\ln\left(\frac{M^2}{\mu^2}\right)\right.\\&\left.+4m_\sigma^2\ln^2\left(\frac{M^2}{\mu^2}\right)+28m_\sigma^2\ln\left(\frac{m_\sigma^2}{\mu^2}\right)-8m_\sigma^2\ln^2\left(\frac{m_\sigma^2}{\mu^2}\right)-12m_\sigma^2-2m_\sigma^2\ln^2\left(\frac{M^2m_\sigma^2}{\mu^4}\right)\right].\end{aligned} \tag{56}$$

The terms in (53)–(56) with the coefficient $\ln(m_\sigma^2/\mu^2)/(n-4)$ are canceled by corresponding terms arising from the counterterm diagrams in Fig. 9. At zero temperature and density we have to this order

$$\begin{aligned}T_1^{\mathrm{i}}=&\frac{i\lambda\sigma_0 m_\sigma^2}{2(16\pi^2)}\left[\frac{2}{n-4}-1+\ln\left(\frac{m_\sigma^2}{\mu^2}\right)\right]\\&-i\left(\frac{1}{32\pi^2}\right)\{\delta m_\sigma^2\lambda\sigma_0+m_\sigma^2[\delta\lambda\sigma_0+(Z_\Phi^{-1}-1)\lambda\sigma_0]\}\left\{\frac{2}{n-4}-1+\ln\left(\frac{m_\sigma^2}{\mu^2}\right)\right.\\&\qquad\left.+\left(\frac{n-4}{2}\right)\left[1-\ln\left(\frac{m_\sigma^2}{\mu^2}\right)+\frac{1}{2}\ln^2\left(\frac{m_\sigma^2}{\mu^2}\right)\right]\right\}\\&-\frac{i\lambda\sigma_0\delta m_\sigma^2}{32\pi^2}\left[1-\frac{n-4}{2}+\frac{n-4}{2}\ln\left(\frac{m_\sigma^2}{\mu^2}\right)\right],\end{aligned} \tag{57}$$

and the following term in (57)

$$\frac{-i\ln(m_\sigma^2/\mu^2)}{32\pi^2}\{\delta m_\sigma^2\lambda\sigma_0+m_\sigma^2[\delta\lambda\sigma_0+(Z_\Phi^{-1}-1)\lambda\sigma_0]\}$$

$$=\frac{-i\ln(m_\sigma^2/\mu^2)}{(16\pi^2)^2(n-4)}\left(-\tfrac{9}{2}m_\sigma^2\lambda^2\sigma_0+24g^2M^2\lambda\sigma_0+48g^4m_\sigma^2\sigma_0-8\lambda g^2m_\sigma^2\sigma_0\right) \tag{58}$$

eliminates the poles with logarithmic residues in the two-loop tadpole. The remaining infinities in T_2 are canceled by terms to order $(n-4)^{-2}$ arising as usual from the mass, coupling-constant, and wave-function renormalization.

The temperature-dependent infinities arise whenever some internal lines in any given Feynman diagram are put on the mass shell by the temperature-dependent terms in the propagators while other lines form subdiagrams with loops which have their associated infinities. At the two-loop level this corresponds to applying the "cutting" rule to any one internal line, as described in Appendix A. When two internal lines are cut the Feynman integrals are limited to finite regions of momentum space and are finite. These constitute the finite terms of set (iii). We now show that the temperature-dependent infinities are absent. In order to make the expressions compact let us define

$$J_1=\int\frac{d^3k}{(2\pi)^3}\,\frac{n_B((k^2+m_\sigma^2)^{1/2})}{(k^2+m_\sigma^2)^{1/2}}, \tag{59}$$

$$J_2=\int\frac{d^3k}{(2\pi)^3}\,\frac{n_B(|\vec{k}|)}{|\vec{k}|}, \tag{60}$$

$$J_3=\int\frac{d^3k}{(2\pi)^3}\left[\frac{n_B^2((k^2+m_\sigma^2)^{1/2})\beta e^{\beta(k^2+m_\sigma^2)^{1/2}}}{2(k^2+m_\sigma^2)}+\frac{n_B((k^2+m_\sigma^2)^{1/2})}{2(k^2+m_\sigma^2)^{3/2}}\right], \tag{61}$$

and

$$J_4=\int\frac{d^3k}{(2\pi)^3}\left[\frac{n_B^2(|\vec{k}|)\beta e^{\beta|\vec{k}|}}{2|\vec{k}|^2}+\frac{n_B(|\vec{k}|)}{2|\vec{k}|^3}\right]. \tag{62}$$

The two-loop terms of Figs. 8(h)–8(k) together with the pion-mass counterterm in the one-loop tadpole have the temperature-dependent infinities

$$T_2^{ii}(h,i,j,k)+T_1^{ii}(\delta m_\pi^2)=\frac{-i\lambda^2\sigma_0}{16\pi^2(n-4)}\left(\tfrac{5}{6}J_2+\tfrac{1}{6}J_1\right)-\frac{i\lambda\sigma_0g^2}{16\pi^2(n-4)}(4J_2). \tag{63}$$

The two-loop terms of Figs. 8(a)–8(e) together with the σ-mass counterterm in Fig. 9 give the temperature-dependent infinity

$$T_2^{ii}(a,b,c,d,e)+T_1^{ii}(\delta m_\sigma^2)=\frac{-i\lambda^2\sigma_0}{16\pi^2(n-4)}\left[\tfrac{1}{2}(J_1+J_2)\right]-\frac{i\lambda g^2\sigma_0}{16\pi^2(n-4)}\,4J_1. \tag{64}$$

The diagrams of Figs. 8(f), 8(g) and 8(l), 8(m)

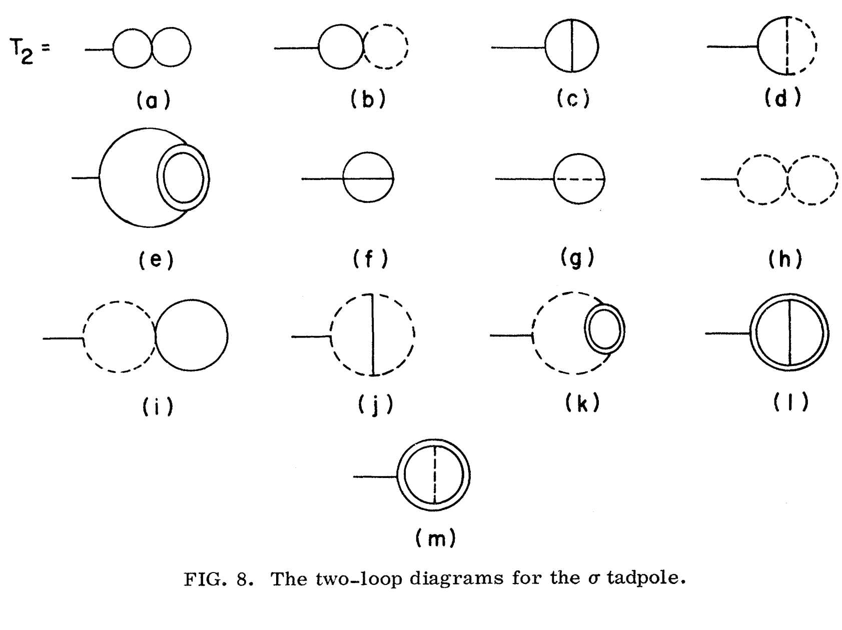

FIG. 8. The two-loop diagrams for the σ tadpole.

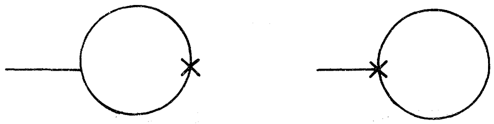

FIG. 9. The counterterm diagrams for the σ tadpole at the one-loop level.

have the terms

$$T_2^{ii}(f,g)=\frac{-i\lambda^2\sigma_0}{16\pi^2(n-4)}\left(\tfrac{4}{3}J_1+\tfrac{2}{3}J_2\right) \tag{65}$$

and

$$T_2^{ii}(l,m)=\frac{+ig^3M}{16\pi^2(n-4)}(48J_1+48J_2). \tag{66}$$

The infinite terms (63)–(66) are canceled by the coupling-constant counterterms

$$T_1^{ii}(\delta\lambda,Z_\Phi)=-\frac{i}{2}\left[\delta\lambda\sigma_0+(Z_\Phi^{-1}-1)\lambda\sigma_0\right](J_1+J_2). \tag{67}$$

We have thus shown that there are no pole terms, in Feynman integrals as $n\to 4$, with $\log(m^2)$ or temperature-dependent residues. The finite, temperature-dependent terms for the two-loop tadpole are readily evaluated by the repeated use of the cutting rule of Appendix A. Details will be presented elsewhere.

V. CONCLUDING REMARKS

It has been shown that the statistical-mechanical aspects modify the usual results of the field-theoretic σ model. The scattering amplitudes develop contributions to their imaginary parts appropriate to the emission and absorption process in the presence of a thermal background of various particles and resonances.

The renormalization of the model at the one-loop level brings out the feature that the Goldstone mode is maintained even at finite temperature and density till a critical temperature or a critical density is reached. The full symmetry of the σ model is restored above these critical values for the temperature or density.

The ground-state energy of a system of nucleons has been studied in the σ model.[8] The results could be improved by going beyond perturbation-theory arguments using renormalization-group techniques. Since masses and couplings parameters become functions of temperature and density, the differential equations of the renormalization group must now include terms which take this into account. This has been independently proposed by Kislinger and Morley.[14]

In the presence of many-body effects the predictions for couplings based on current-algebra sum rules would be modified. The effect this has on the position and the width of the $\Delta(1236)$ resonance, on the quenching of the axial-vector current coupling,[26] and on the current-algebra spectral-function sum rules are problems for further study.

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APPENDIX A

1. Temperature Green's functions

In the zero-loop or tree-diagram approximation the particle propagators are

$$iS_F(p)=\int d^4x\, e^{ip\cdot x}\langle 0|T(\Psi(x),\bar\Psi(0))|0\rangle$$
$$=i/(\not p-M+i\epsilon) \tag{A1}$$

and

$$iD_F(p)=\int d^4x\, e^{ip\cdot x}\langle 0|T(\phi(x),\phi(0))|0\rangle$$
$$=i/(p^2-m^2+i\epsilon) \tag{A2}$$

for fermions and for bosons, respectively. We shall be interested in the statistical Green's functions defined as the expectation value of time-ordered products of fields averaged over ensembles of particle distributions. We define

$$iS_F(p,\mu,\beta)=\int d^4x\, e^{ip\cdot x}\frac{\mathrm{Tr}\,e^{-\beta(H-\mu N)}T(\Psi(x),\bar\Psi(0))}{\mathrm{Tr}\,e^{-\beta(H-\mu N)}}$$
$$=\frac{i(\not p+M)}{2E_p}\left(\frac{1-n_F}{P_0-E_p+i\epsilon}+\frac{n_F}{p_0-E_P-i\epsilon}-\frac{1-\bar n_F}{p_0+E_p-i\epsilon}-\frac{\bar n_F}{p_0+E_p+i\epsilon}\right), \tag{A3}$$

where $\beta=1/K_BT$, μ is the chemical potential, N is the number of fermions, E_p is the energy $(p^2+M^2)^{1/2}$, and

$$n_F=n_F(p_0,\mu,\beta)$$
$$=1/(e^{\beta(p_0-\mu)}+1) \tag{A4}$$

is the Fermi distribution function for a system of fermions. K_B is the Boltzmann constant, and it is set equal to unity together with c and $\hbar$ in this paper. It should be noted that now the positions of the poles in the propagator (A3) correspond to the presence of particle, particle-hole, antiparticle, and antiparticle-hole excitations. (This is shown on the frequency plane in Fig. 10.) Correspondingly, $\bar{n}_F$ (the Fermi distribution function) is a function of $-p_0$ and of $\bar{\mu}$, the chemical potential for the antiparticle distribution. Consider the case where there are only N fermions and no antifermions, so that $\bar{n}_F$ is zero. The propagator (A3) then reduces to the form

$$iS_F(p,\mu,\beta) = i(\not p + M)\left[\frac{1}{p^2-M^2+i\epsilon} + \frac{2\pi i n_F(p_0,\mu,\beta)\delta(p_0-E_p)}{2E_p}\right]. \tag{A5}$$

At zero temperature the fermions occupy energy levels up to a maximum corresponding to the Fermi-level energy μ, and the Fermi distribution function becomes $n_F = \theta(\mu - p_0)$.

For a system of noninteracting fermions at zero temperature the number density is[13]

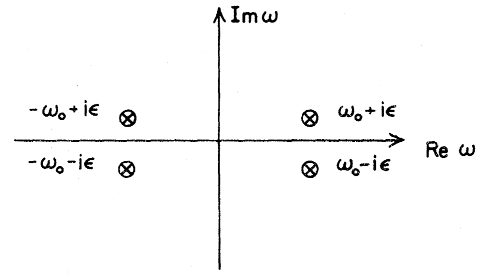

FIG. 10. The poles of the particle propagators on the complex energy plane in many-body theory.

$$\begin{aligned} N/V &= \rho \\ &= -i\lim_{x^0\to 0}\int\frac{d^4p}{(2\pi)^4}\left(\frac{e^{ip_0x^0}+e^{-ip_0x^0}}{2}\right)\mathrm{Tr}[\gamma^0 S_F(p,\mu)] \\ &= 2\int_0^{p_F}\frac{d^3p}{(2\pi)^3}n_F \\ &= \frac{p_F^3}{3\pi^2}. \end{aligned} \tag{A6}$$

If the number of species in a multiplet of fermions is κ the total number of fermions per unit volume is $\kappa p_F^3/3\pi^2$. Here p_F is the Fermi momentum. For nuclear matter $\kappa = 2$.

The energy density of this system of fermions is

$$\begin{aligned} E &= \frac{2\kappa}{(2\pi)^3}\int_0^{p_F} d^3p\,(p^2+M^2)^{1/2} \\ &= \frac{\kappa}{\pi^2}\left\{\frac{p_F}{4}(p_F^2+M^2)^{3/2} - \frac{M^2}{8}p_F(p_F^2+M^2)^{1/2} - \frac{M^4}{8}\ln\left[\frac{p_F+(p_F^2+M^2)^{1/2}}{M}\right]\right\}. \end{aligned} \tag{A7}$$

In an analogous manner the propagator for the bosons is defined as

$$\begin{aligned} iD_F(k,\beta) &= \int d^4x\, e^{ip\cdot x}\,\frac{\mathrm{Tr}e^{-\beta H}T(\phi(x),\phi(0))}{\mathrm{Tr}e^{-\beta H}} \\ &= \frac{i}{2E_k}\left[\frac{1+n_B(k_0,\beta)}{k_0-E_k+i\epsilon} - \frac{n_B(k_0,\beta)}{k_0-E_k-i\epsilon} - \frac{1+\bar{n}_B(-k_0,\beta)}{k_0+E_k-i\epsilon} + \frac{\bar{n}_B(-k_0,\beta)}{k_0+E_k+i\epsilon}\right]. \end{aligned} \tag{A8}$$

Here $n_B(k_0,\beta)$ is the Bose-Einstein distribution function

$$n_B(k_0,\beta) = 1/(e^{\beta k_0}-1).$$

For the present problem it is assumed that the boson chemical potentials are zero. When there is no explicit symmetry breaking in the model considered here, the pions will be massless and their spectral distribution will be that given by Planck's law. On the other hand, setting the chemical potential of the σ particle to zero is the correct procedure for our choice of spontaneous symmetry breaking [Eq. (6)], which does not require a Bose-Einstein condensation of the σ particles in order to trigger off the symmetry breaking. At zero temperature there are then no σ mesons in the ground state.

For self-conjugate fields $n_B = \bar{n}_B$ and the boson propagator is

$$iD_F(k,\beta) = \left[\frac{i}{k^2-m^2+i\epsilon} + 2\pi\frac{n_B(k_0,\beta)}{2E_k}\delta(k_0-E_k) + 2\pi\frac{n_B(-k_0,\beta)}{2E_k}\delta(k_0+E_k)\right]. \tag{A9}$$

In this paper we designate the propagators by double brackets to denote the presence of temperature-dependent terms in the propagators:

$$i S_F(p) = i(\not{p} + M)/\{\!\{ p^2 - M^2 + i\epsilon \}\!\}, \tag{A10}$$

$$i D_F(k) = i/[\![p^2 - m^2 + i\epsilon]\!]. \tag{A11}$$

2. Products of Green's functions

In Eqs. (A5) and (A9) the Green's functions have been written as distributions using the well-known identity

$$1/(x \pm i\epsilon) = \mathrm{P}(1/x) \mp i\pi\, \delta(x). \tag{A12}$$

Products of Green's functions with coincident poles require the relation[27]

$$\frac{1}{(x \pm i\epsilon)^n} = \mathrm{P}\frac{1}{x^n} \mp i\pi \frac{(-1)^{n-1}}{(n-1)!}\, \delta^{(n-1)}(x). \tag{A13}$$

Thus, for example,

$$\frac{1}{[\![k^2 - m^2 + i\epsilon]\!]^2} = \frac{1}{(k^2 - m^2 + i\epsilon)^2} + 2\pi i \left[\frac{n_B(k_0)}{4E_k^{\,2}} \delta'(k_0 - E_k) + \frac{n_B(k_0)\delta(k_0 - E_k)}{2E_k^{\,2}(k_0 + E_k)} \right]$$
$$- 2\pi i \left[\frac{\bar{n}_B(-k_0)}{4E_k^{\,2}} \delta'(k_0 + E_k) + \frac{\bar{n}_B(-k_0)\delta(k_0 + E_k)}{2E_k^{\,2}(k_0 - E_k)} \right] \tag{A14}$$

and

$$\frac{1}{\{\!\{ p^2 - M^2 + i\epsilon \}\!\}^2} = \frac{1}{(p^2 - M^2 + i\epsilon)^2} - 2\pi i \left[\frac{n_F(p_0, \mu)}{4E_p^{\,2}} \delta'(p_0 - E_p) + \frac{n_F(p_0, \mu)\delta(p_0 - E_p)}{2E_p^{\,2}(p_0 + E_p)} \right]. \tag{A15}$$

As a simple example, a loop diagram with two vertices having external legs with zero 4-momenta can be calculated using (A14) or (A15) and the result can be compared with that obtained using the imaginary-time formalism.[7,9]

When the poles in the product of n Green's functions are not at the same value of k_0 the products can be written as the usual Feynman term plus all terms with $1, 2, 3, \ldots, n$, δ-function factors. These factors select regions of momentum space in which the lines of virtual particles in the diagram correspond to real on-mass-shell particles. This has to be consistent with energy-momentum conservation at each vertex. The parallel with Cutkosky rules[28] of field theory is obvious. For one application of these "cutting" rules see Ref. 29.

APPENDIX B

Dimensional regularization has been done using the analytic continuation

$$\frac{1}{(2\pi)^4} \int d^4k \Rightarrow \frac{(\mu^2)^{2-n/2}\pi^{2-n/2}}{(2\pi)^4\Gamma(3-n/2)} \int d^nk,$$

where μ is a constant mass called the unit of mass.[18] This definition avoids terms with Euler's constant in the intermediate stages of the renormalization and simplifies the algebra. The uniqueness of analytic continuation in dimension has been discussed by Kang.[30] The n-dimensional integrals have their usual values,[15] and some useful relations are the following:

$$1.\ \int d^nk (k^2)^\beta = 0, \quad \beta \geq -1 \tag{B1}$$

$$2.\ \int d^nk\, \delta(k^2) = 0, \tag{B2}$$

$$3.\ \int d^nk (k^2 + 2p\cdot k - m^2)^{-\alpha} = i(-1)^\alpha \pi^{n/2} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)(m^2 + p^2)^{\alpha - n/2}}, \tag{B3}$$

$$4.\ \int d^nk\, k_\mu (k^2 + 2p\cdot k - m^2)^{-\alpha} = i(-1)^\alpha \pi^{n/2}(-p_\mu) \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)(m^2 + P^2)^{\alpha - n/2}}, \tag{B4}$$

5. $$\int d^n k (k_\mu k_\nu)(k^2 + 2p\cdot k - m^2)^{-\alpha} = \frac{i(-1)^\alpha \pi^{n/2}}{\Gamma(\alpha)(m^2+p^2)^{\alpha-n/2}}[\Gamma(\alpha - n/2)p_\mu p_\nu - \tfrac{1}{2} g_{\mu\nu}\Gamma(\alpha - 1 - n/2)(p^2+m^2)], \quad \text{(B5)}$$

6. $$\int d^n k \frac{1}{(k^2+i\epsilon)^2} = i\pi^2 \frac{\Gamma(2-n/2)}{\Gamma(2)}. \quad \text{(B6)}$$

APPENDIX C

The corrections to the meson coupling constant at the one-loop level due to many-body effects can be evaluated by separating out the temperature- and density-dependent terms in the propagators entering Feynman integrals for the meson-meson scattering amplitude. The diagrams of Fig. 2 together with triangle and square diagrams which are generated by the presence of cubic couplings in Eq. (7) contribute at the one-loop level. An efficient way of evaluating these corrections is to use the definition of Coleman and Weinberg[31] for the coupling constant in terms of the effective potential:

$$\lambda_R(\beta, \mu_F) = \lambda_R + \frac{\partial^4}{\partial \sigma_0^{\,4}} \overline{V}_1(\beta, \mu_F, \sigma_0), \quad \text{(C1)}$$

where $\overline{V}_1(\beta, \mu_F, \sigma_0)$ is the temperature- and density-dependent part of the effective potential, or the thermodynamic potential, at the one-loop level. For the σ model a direct extension of the results of Dolan and Jackiw[9] yields

$$\overline{V}_1(\beta, \mu_F, \sigma_0) = -4\int \frac{d^3p}{(2\pi)^3}\frac{1}{\beta}\ln(1+e^{\beta\mu_F - \beta E_M}) + \int \frac{d^3k}{(2\pi)^3}\frac{1}{\beta}[\ln(1-e^{-\beta E_1}) + 3\ln(1-e^{-\beta E_2})], \quad \text{(C2)}$$

where

$$E_M = (p^2 + g^2\sigma_0^{\,2})^{1/2},$$

$$E_1 \equiv E_{m_\sigma} = (k^2 + \mu_0^{\,2} + \tfrac{1}{2}\lambda\sigma_0^{\,2})^{1/2},$$

and

$$E_2 \equiv E_{m_\pi} = (k^2 + \mu_0^{\,2} + \tfrac{1}{6}\lambda\sigma_0^{\,2})^{1/2}.$$

We are discussing a system of nucleons with no antiparticle distribution (i.e., $\overline{n}_F = 0$), and the number of species of fermions is 2; this accounts for the factor of 4 in front of the integral over p in Eq. (C2). As in the work of Coleman and Weinberg,[31] the coupling constant is defined for a value of σ_0 away from the minimum of the total effective potential in order to avoid the infrared problem. Substituting (C3) in (C1) we obtain

$$\begin{aligned}
\lambda_R(\beta, \mu_F) = \lambda_R &- \frac{12 g^4}{(2\pi)^3}\int d^3p \left(\frac{n_F}{E_M^{\,3}} + \frac{n_F^{\,2}\beta e^{\beta E_M}}{E_M^{\,2}}\right) \\
&+ \frac{24 g^6 \sigma_0^{\,2}}{(2\pi)^3}\int d^3p\left(\frac{3n_F}{E_M^{\,5}} + \frac{3n_F^{\,2}\beta e^{\beta E_M}}{E_M^{\,4}} + \frac{2n_F^{\,3}\beta^2 e^{2\beta E_M}}{E_M^{\,3}} - \frac{n_F^{\,2}\beta^2 e^{\beta E_M}}{E_M^{\,3}}\right) \\
&+ \frac{4 g^8\sigma_0^{\,4}}{(2\pi)^3}\int d^3p\left(-\frac{15 n_F}{E_M^{\,7}} - \frac{15 n_F^{\,2}\beta e^{\beta E_M}}{E_M^{\,6}} - \frac{12 n_F^{\,3}\beta^2 e^{2\beta E_M}}{E_M^{\,5}} + \frac{6 n_F^{\,2}\beta^2 e^{\beta E_M}}{E_M^{\,5}}\right. \\
&\qquad\qquad \left. - \frac{6n_F^{\,4}\beta^3 e^{3\beta E_M}}{E_M^{\,4}} + \frac{6n_F^{\,3}\beta^3 e^{2\beta E_M}}{E_M^{\,4}} - \frac{n_F^{\,2}\beta^3 e^{\beta E_M}}{E^4}\right) \\
&- \frac{3\lambda_R^{\,2}}{4(2\pi)^3}\int d^3k\left[\left(\frac{n_1}{E_1^{\,3}} + \frac{n_1^{\,2}\beta e^{\beta E_1}}{E_1^{\,2}}\right) + \frac{1}{3}\begin{pmatrix} n_1 \to n_2 \\ E_1 \to E_2\end{pmatrix}\right] \\
&- \frac{\lambda_R^{\,3}\sigma_0^{\,2}}{2(2\pi)^3}\int d^3k\left[\left(-\frac{3n_1}{E_1^{\,5}} - \frac{3n_1^{\,2}\beta e^{\beta E_1}}{E_1^{\,4}} - \frac{2n_1^{\,3}\beta^2 e^{2\beta E_1}}{E_1^{\,3}} + \frac{n_1^{\,2}\beta^2 e^{\beta E_1}}{E_1^{\,3}}\right) + \frac{1}{9}\begin{pmatrix} n_1 \to n_2 \\ E_1 \to E_2\end{pmatrix}\right] \\
&- \frac{\lambda_R^{\,4}\sigma_0^{\,4}}{16(2\pi)^3}\int d^3k\left[\left(\frac{15n_1}{E_1^{\,7}} + \frac{15n_1^{\,2}\beta e^{\beta E_1}}{E_1^{\,6}} + \frac{12n_1^{\,3}\beta^2 e^{2\beta E_1}}{E_1^{\,5}} - \frac{6n_1^{\,2}\beta^2 e^{\beta E_1}}{E_1^{\,5}}\right.\right. \\
&\qquad\qquad \left. + \frac{6n_1^{\,4}\beta^3 e^{3\beta E_1}}{E_1^{\,4}} - \frac{6n_1^{\,3}\beta^3 e^{2\beta E_1}}{E_1^{\,4}} + \frac{n_1^{\,2}\beta^3 e^{\beta E_1}}{E_1^{\,4}}\right) \\
&\qquad\qquad \left. + \frac{1}{27}\begin{pmatrix} n_1 \to n_2 \\ E_1 \to E_2\end{pmatrix}\right],
\end{aligned} \quad \text{(C3)}$$

where

$$n_1 = n_{1B} = (e^{\beta E_1} - 1)^{-1}$$

and the terms arising from pions are obtained as indicated in (C3) by the replacement of (n_1, E_1) by (n_2, E_2) in the integrands.

At zero temperature, $n_B = 0$ and the fermion terms alone survive. They are evaluated by taking the zero-temperature limit after differentiating the logarithm in (C1), and

$$\lambda_R(\beta^{-1}=0,\ \mu_F) = \lambda_R + \frac{\partial^3}{\partial\sigma_0{}^3}\left[\frac{4g^2\sigma_0}{(2\pi)^3}\int d^3p\ \frac{\theta(\mu - E_M)}{E_M}\right]. \tag{C4}$$

This then defines the appropriate branch of the logarithm. The resultant expression is Eq. (44) of Sec. III.

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