Remark on the renormalization group and total cross sections*

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Recently the renormalization-group method has been used to give the asymptotic behavior of the physical fixed-angle scattering amplitude in ϕ^4 field theory. This method does not directly apply to forward (or backward) scattering due to the inevitably singular nature of the zero-mass vertex functions at $\theta_{\text{c.m.}} = 0$ or π. In this note we apply the method used to get the above results to integrals of the vertex functions over the angle variable. The resulting restriction we get on these integrals leads us, with the help of rigorously established inequalities due to Bessis and Singh, to a new strong upper bound on $\sigma_{\text{tot}}(s)$ for the massive ϕ^4 case. The new input we need to get this result is, like many of the assumptions used in connection with the renormalization-group method, valid at least order by order in perturbation theory.

I. INTRODUCTION

Recently it has been shown that the renormalization-group method can be used to study the asymptotic behavior in energy of the fixed-angle elastic scattering amplitude on the mass shell.[1,2] The results hold only for theories with tame infrared behavior such as ϕ^4 or Yukawa theories. In these cases one gets a simple scaling behavior of the elastic amplitude at high energy and fixed center-of-mass scattering angle, $\theta_{\text{c.m.}} \neq 0$ or π:

$$F(s, \theta_{\text{c.m.}}) \sim s^{-2\gamma(g_\infty)} h(\theta_{\text{c.m.}}), \tag{1.1}$$

where $\gamma(g_\infty)$ is the anomalous dimension of the ϕ field at the ultraviolet fixed point $g = g_\infty$ in the ϕ^4 case.

The fact that the renormalization-group method puts restrictions on the high-energy behavior of the fixed-angle amplitude was first pointed out by Huang and Low[3] in 1964. They also correctly pointed out that one cannot use the renormalization-group method at fixed momentum transfer t. The reason for that becomes evident when one recalls that the renormalization-group method gives the asymptotic behavior of a vertex function for a massive theory in terms of expressions which involve the vertex functions of the zero-mass theory. Large s and fixed t gives us essentially an exceptional momentum configuration with $\theta_{\text{c.m.}} \approx 0$. The zero-mass four-point vertex function is certainly singluar at $z = \cos\theta_{\text{c.m.}} = \pm 1$. Thus one cannot use the renormalization-group method directly at fixed t, or at fixed z if $z = \pm 1$.

The nature of the singularity of the zero-mass vertex function at $z = \pm 1$ is not rigorously known. However, in perturbation theory this singularity is at worst logarithmic in the ϕ^4 case. One can also give heuristic though not rigorous arguments that at worst this singularity is a pole. Several authors[4] have made the assumption that this $t = 0$ singularity in the zero-mass case is not worse than what is given by the exchange of two zero-mass particles in perturbation theory, i.e., in the ϕ^4 case one gets a logarithmic singularity. They used this assumption plus the existence of fixed-s dispersion relations to directly obtain bounds for the total cross sections in the *zero-mass* case where the methods of Martin and Froissart do not hold.

In this brief note we shall combine assumptions similar to but weaker than those of Refs. 4 with the renormalization-group techniques of Refs. 1 and 2 to obtain restrictions on the *massive* theory. These restrictions then lead us, via an inequality due to Singh,[5] to a bound on the physical massive total cross section of the form $\sigma_{\text{tot}}(s) \leq \text{const} \times s^{-2\gamma(g_\infty)} \ln^2 s$ for the ϕ^4 case.

The renormalization-group input used in Refs. 1 and 2 has of course not been rigorously justified, but it is based on perturbation theory. Similarly, the assumption of Ref. 4 on the nature of the $z = 1$ singularity is based on perturbation theory. For these reasons one cannot view our result as an improvement of the Froissart bound even for the ϕ^4 case. However, there have been many attempts to determine the asymptotic behavior of $\sigma_{\text{tot}}(s)$ by calculating the leading behavior of a certain subclass of graphs and then summing it up.[6] The route we follow takes account of all diagrams, makes weaker assumptions, and yet leads to different conclusions, namely, the Froissart bound is not saturated. Unfortunately, one cannot make a direct comparison since the methods of Ref. 6 were only used for massive electrodynamics, and ϕ^3 theory, while those of Refs. 1 and 2 can so far only be used for theories with tame infrared behavior such as ϕ^4 or Yukawa theories. At this stage there is no conflict.

In Sec. II we review the results of Refs. 1 and 2 on the fixed-angle behavior. We then derive our main result in Sec. III and state the new input needed. In Sec. IV we show how the asymptotic behavior of integrals over angle of the scattering amplitude obtained in Sec. III leads to a bound on the total cross section.

II. REVIEW OF FIXED-ANGLE BEHAVIOR

In this section we briefly review the results of Refs. 1 and 2 using the approach of Callan and Gross.[2] The whole discussion is restricted to the ϕ^4 case.

We use the so-called "improved" renormalization-group equations.[7] One considers the n-particle vertex functions

$$\Gamma^{(n)}(p_1,\ldots,p_n;g,m,\mu).$$

The mass parameter μ describes the off-mass-shell point at which subtractions are carried out, and m is a parameter related to, but not the same as, the physical mass, $m_{\text{phys}}=f(g,m,\mu)$. The main point is that as $m\to 0$ one obtains the vertex functions of the zero-mass theory with μ then being the mass needed to define the subtraction point. The renormalization-group equations are given by

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-n\gamma(g)+\hat{\gamma}(g)m\frac{\partial}{\partial m}\right]\Gamma^{(n)}(p_i;g,m,\mu)=0. \quad (2.1)$$

We are only interested here in the case $n=4$, and simple dimensional analysis gives us

$$D_\lambda\Gamma^{(4)}(\lambda p_i;g,m,\mu)=0, \quad (2.2)$$

with

$$D_\lambda\equiv\left\{-\lambda\frac{\partial}{\partial\lambda}+\beta(g)\frac{\partial}{\partial g}-4\gamma(g)+[\hat{\gamma}(g)-1]\,m\frac{\partial}{\partial m}\right\}. \quad (2.3)$$

This last equation has the standard solution

$$\Gamma^{(4)}(\lambda p_i;g,m,\mu)=\Gamma^{(4)}(p_i;\bar{g}(\lambda),m(\lambda),\mu)\exp\left[-4\int_g^{\bar{g}(\lambda)}dx\frac{\gamma(x)}{\beta(x)}\right], \quad (2.4)$$

where

$$\lambda\frac{\partial}{\partial\lambda}\bar{g}(\lambda)=\beta(\bar{g}(\lambda)),$$

$$\bar{g}(1)=g,$$

$$\lambda\frac{\partial}{\partial\lambda}m(\lambda)=[\hat{\gamma}(\bar{g}(\lambda))-1]m(\lambda).$$

If $\beta(g)$ has an ultraviolet-stable fixed point g_∞ and $\hat{\gamma}(g_\infty)<+1$, then $m(\lambda)\to 0$ as $\lambda\to\infty$, and

$$\Gamma^{(4)}(\lambda p_i;g,m,\mu)\to\lambda^{-4\gamma(g_\infty)}\Gamma^{(4)}(p_i;g_\infty,0,\mu). \quad (2.5)$$

Both the existence of the fixed point and the validity of the bound on $\hat{\gamma}$ are necessary for the applicability of the renormalization-group method even for Euclidean p_i. We shall assume that both hold. Note that on the right-hand side of Eq. (2.5) $\Gamma^{(4)}(p_i;g_\infty,0,\mu)$ is a zero-mass vertex function, and if the p_i's are such that $\Gamma^{(4)}$ is singular then Eq. (2.5) is useless.

To apply the renormalization-group method on the mass shell one introduces the momenta

$$p_i(\lambda_1)=q_i+r_i/\lambda_1^2,\quad i=1,\ldots,4, \quad (2.6)$$

where $q_i^2=r_i^2=0$ and $2q_i\cdot r_i=m_0^2$. The parameter λ_1 is at this stage taken to be independent of λ. The mass m_0 is a fixed mass which at a later stage we set equal to the physical mass. We consider the vertex functions $\Gamma^{(4)}(\lambda p_i(\lambda_1);g,m,\mu)$ which now satisfy

$$D_\lambda\Gamma^{(4)}(\lambda p_i(\lambda_1);g,m,\mu)=0. \quad (2.7)$$

The solution is identical to that in Eq. (2.4) with p_i replaced by $p_i(\lambda_1)$. At this point we set $\lambda_1=\lambda$ in the solution and take the limit of large λ to obtain, as $\lambda\to\infty$,

$$\Gamma^{(4)}(\lambda p_i(\lambda);g,m,\mu)\sim\lambda^{-4\gamma(g_\infty)}\Gamma^{(4)}(q_i;g_\infty,0,\mu). \quad (2.8)$$

In terms of the variables $s(\lambda)\equiv\lambda^2[p_1(\lambda)+p_2(\lambda)]^2$, $z(\lambda)=\cos\theta_{\text{c.m.}}$, and $m_e^2=[\lambda p_i(\lambda)]^2$ one has

$$\Gamma^{(4)}(s(\lambda),z(\lambda),m_e=m_0;g,m,\mu)\sim\lambda^{-4\gamma(g_\infty)}\Gamma^{(4)}(s_0,z_0,m_e=0;g_\infty,0,\mu), \quad (2.9)$$

where $s_0=(q_1+q_2)^2$, $t_0=(q_1+q_3)^2$, and $z_0=1+2t_0/s_0$. Also, for large λ

$$s(\lambda)=\lambda^2[s_0+O(1/\lambda^2)],\qquad z(\lambda)=z_0+O(1/\lambda^2). \quad (2.10)$$

Thus in Eq. (2.9) one is taking the limit of large s at fixed $z=\cos\theta_{\text{c.m.}}\equiv z_0$. The vertex function $\Gamma^{(4)}(s_0,z_0,m_e=0;g_\infty,0,\mu)$ is a zero-mass vertex function with both internal and external masses vanishing. It is argued in Ref. 2 that for $-1<z_0<+1$ this zero-mass vertex function exists at least for ϕ^4 theories, and hence Eq. (2.9) gives the large-energy fixed-angle behavior of the massive vertex function on the left. But the massive vertex function $\Gamma^{(4)}(s,z,m_e=m_{\text{phys}};g,m,\mu)$ differs only by a finite factor from the physical scattering amplitude:

$$F(s,z)\equiv C(g,m,\mu)\Gamma^{(4)}(s,z,m_e=m_{\text{phys}};g,m,\mu), \tag{2.11}$$

and hence Eq. (2.9) leads to the scaling relation (1.1).

For forward (or backward) scattering, $z_0=\pm 1$, one cannot use Eq. (2.9) directly since $\Gamma^{(4)}(s_0,z_0,0;g_\infty,0,\mu)$ is singular at $z_0=\pm 1$. However, we shall see in the next section that if this singularity is integrable (as suggested by perturbation theory) one can still get useful results.

One should note that we use the relation (2.11) only for the massive case. For our purposes we do not need to relate the zero-mass vertex function to the zero-mass scattering amplitude. The function $h(\theta)$ in Eq. (1.1) is thus proportional to the zero-mass vertex function $\Gamma^{(4)}(s_0,z,m_e=0;g_\infty,0,\mu)$ and not simply related to the zero-mass scattering amplitude.

III. ASYMPTOTIC BEHAVIOR OF INTEGRALS OVER THE ANGLE VARIABLE

In perturbation theory the singularity of $\Gamma^{(4)}(s_0,z_0,m_e=0;g_\infty,0,\mu)$ at $z_0=+1$ is logarithmic for ϕ^4 theory, and at least order by order in perturbation theory the integral

$$\int_{1-\delta}^{1}\Gamma^{(4)}(s_0,z_0,m_e=0;g_\infty,0,\mu)dz_0,\quad 1>\delta>0$$

exists and is finite. So the first assumption we make in addition to those of Refs. 1 and 2 is the following.

Assumption A. The above integral is finite for the full zero-mass $\Gamma^{(4)}$. This allows us to deal with singularities at $z_0=1$ that are stronger than logarithmic as long as they are integrable at $z_0=1$. This is our main assumption. It is similar to and weaker than those made in Ref. 4. In addition we shall need a technical assumption about interchanging the $\lambda\to\infty$ limit with integrals over z_0 which we shall specify below.

To proceed we now consider

$$\Gamma^{(4)}(\lambda p_i(\lambda_1);g,m,\mu)\equiv\Gamma^{(4)}\left(s(\lambda,\lambda_1),z(\lambda,\lambda_1),m_e=\frac{\lambda m_0}{\lambda_1};g,m,\mu\right). \tag{3.1}$$

For large λ_1, $s(\lambda,\lambda_1)=\lambda^2[s_0+O(1/\lambda_1^2)]$, $z(\lambda,\lambda_1)=z_0+O(1/\lambda_1^2)$, and we get

$$\Gamma^{(4)}(\lambda p_i(\lambda_1);g,m,\mu)=\Gamma^{(4)}\left(\lambda^2 s_0,z_0,m_e=\frac{\lambda m_0}{\lambda_1};g,m,\mu\right)+O\left(\frac{1}{\lambda_1^2}\right). \tag{3.2}$$

Next we define G_δ as

$$G_\delta\left(\lambda^2 s_0,m_e=\frac{\lambda m_0}{\lambda_1};g,m,\mu\right)\equiv\int_{1-\delta}^{1}dz_0\Gamma^{(4)}\left(\lambda^2 s_0,z_0,m_e=\frac{\lambda m_0}{\lambda_1};g,m,\mu\right). \tag{3.3}$$

It follows from the previous section that

$$D_\lambda G_\delta\left(\lambda^2 s_0,m_e=\frac{\lambda}{\lambda_1}m_0;g,m,\mu\right)=O\left(\frac{1}{\lambda_1^2}\right). \tag{3.4}$$

The solution of this equation is the same as before, and after setting $\lambda_1=\lambda$ we obtain

$$G_\delta(\lambda^2 s_0,m_e=m_0;g,m,\mu)=G_\delta\left(s_0,m_e=\frac{m_0}{\lambda};\bar g(\lambda),m(\lambda),\mu\right)\times\exp\left[-4\int_g^{\bar g(\lambda)}\frac{\gamma(x)}{\beta(x)}dx\right]+O\left(\frac{1}{\lambda^2}\right). \tag{3.5}$$

As $\lambda\to\infty$ we get

$$G_\delta(\lambda^2 s_0,m_e=m_0;g,m,\mu)\sim\lambda^{-4\gamma(g_\infty)}G_\delta(s_0,m_e=0;g_\infty,0,\mu). \tag{3.6}$$

Given our assumption (A) the right-hand side of (3.6) will exist if we are allowed to interchange the $\lambda\to\infty$ limit with the integration over z_0. Namely, we have to assume the following.

Assumption B.

$$\lim_{\lambda\to\infty}\int_{1-\delta}^{1}\Gamma^{(4)}\left(s_0,z_0,m_e=\frac{m_0}{\lambda};\bar g(\lambda),m(\lambda),\mu\right)dz_0=\int_{1-\delta}^{1}\Gamma^{(4)}(s_0,z_0,m_e=0;g_\infty,0,\mu)dz_0. \tag{3.7}$$

Simply, we are assuming that the zero-mass limit of the integral over angle is the same as the integral of the zero-mass vertex function. One can easily make simple mathematical counterexamples with no uniform convergence in λ where Eq. (3.7) is not true. However, these counterexamples do not seem to be consistent with what one expects to get in each order of perturbation theory. Following the general philosophy on which all use of the renormalization-group method rests, namely, the extension of some property of perturbation theory to the full theory, we shall assume that the above interchange holds.

As an aside it is perhaps worthwhile to note that assumptions similar to Eq. (3.7) concerning uniform convergence in the high-momentum (or zero-mass) limits are often implicitly made in the literature when one is relating the scaling of the structure functions of electroproduction to that of the Wilson functions.[8]

Given the validity of (A) and (B) the right-hand side of Eq. (3.6) will exist and will lead us to the following asymptotic behavior for integrals over $\Gamma^{(4)}$:

$$\int_{1-\delta}^{1} dz\, \Gamma^{(4)}(\lambda^2 s_0, z, m_e = m_0; g; m, \mu) \sim \text{const} \times \lambda^{-4\gamma(g_\infty)}. \tag{3.8}$$

This holds for any small $\delta > 0$, and any nonvanishing external mass including $m_e = m_0 = m_{\text{phys}}$. For $m_0 = m_{\text{phys}}$, the integrand on the left involves the massive physical vertex function, which differs from the scattering amplitude only by a finite constant. We finally obtain, as $s \to \infty$,

$$\int_{1-\delta}^{1} F(s,z)dz \sim \text{const} \times s^{-2\gamma(g_\infty)}, \quad 1 > \delta > 0. \tag{3.9}$$

The differential equation (3.4) holds for fixed δ which is independent of λ. However, the solution (3.5) of this equation is an identity for any δ. So in the solution, Eq. (3.5), we can set $\delta = \delta(\lambda)$ and then take the limit $\lambda \to \infty$ even if $\delta(\lambda) \to 0$ as $\lambda \to \infty$. We shall do this in applying Eq. (3.9) in the next section and we only need the right-hand side of (3.9) to give an upper bound.

IV. BOUND ON THE TOTAL CROSS SECTION

We show below how Eq. (3.9) restricts the asymptotic behavior of the total cross section. To do that we transform back to the t variable, and get for large s

$$\int_{-\delta s/2}^{0} dt\, F(s,t) \sim \text{const} \times s^{1-2\gamma(g_\infty)}. \tag{4.1}$$

Taking the imaginary part of (4.1) we have

$$\left| \int_{-\delta s/2}^{0} dt\, A(s,t) \right| \le \text{const} \times s^{1-2\gamma(g_\infty)}. \tag{4.2}$$

This bound is true for large s and any $\delta > 0$. We recall the result of Bessis[9] on the zeros of $A(s,t)$, which states that there exists a constant c_1 such that for $-c_1/\ln^2 s \le t \le 0$, $A(s,t)$ is free of zeros and therefore positive. An improved version of this result given by Singh[5] states that for $t \le 0$,

$$\frac{A(s,t)}{A(s,0)} \ge 1 + \frac{t}{16 m_{\text{phys}}^2} \left[\ln\left(\frac{s}{s_0^2 \sigma_{\text{tot}}} \right) \right]^2. \tag{4.3}$$

We can always choose c_1 small enough such that for t in the interval $-c_1/\ln^2 s \le t \le 0$, the second term on the right-hand side in Eq. (4.3) is less than unity in modulus. The Jin-Martin lower bound[10] on $\sigma_{\text{tot}}(s)$ gives us $\sigma_{\text{tot}}(s) \ge c/s^6$ for large s. Then in the above interval, Eq. (4.3) leads us to

$$\frac{A(s,t)}{A(s,0)} \ge 1 + \frac{49t}{16 m_{\text{phys}}^2} \left(\ln \frac{s}{s_0} \right)^2, \quad -\frac{c_1}{\ln^2 s} \le t \le 0. \tag{4.4}$$

Choosing $\delta = 2c_1/s\ln^2 s$, we can drop the absolute value sign in Eq. (4.2) and get

$$A(s,0) \int_{-c_1/\ln^2 s}^{0} dt \left[\frac{A(s,t)}{A(s,0)} \right] \le \text{const} \times s^{1-2\gamma(g_\infty)}. \tag{4.5}$$

From Eq. (4.4) we have

$$A(s,0) \int_{-c_1/\ln^2 s}^{0} dt \left[1 + \frac{49t}{16 m_{\text{phys}}^2} \left(\ln \frac{s}{s_0} \right)^2 \right] \le \text{const} \times s^{1-2\gamma(g_\infty)}, \tag{4.6}$$

where by judicious choice of c_1 the quantity in brackets is positive. This finally gives us, for large s,

$$A(s,0) \le \text{const} \times s^{1-2\gamma(g_\infty)} \ln^2 s, \tag{4.7}$$

and hence

$$\sigma_{\text{tot}}(s) \le \text{const} \times s^{-2\gamma(g_\infty)} \ln^2 s. \tag{4.8}$$

From the positivity of the propagator one knows that $\gamma(g_\infty) \ge 0$, and hence if the anomalous dimension vanishes, $\gamma(g_\infty) = 0$, we still get the bound $\sigma_{\text{tot}}(s) \le \ln^2 s$ for large s. Note also that combining Eq. (4.8) with the Jin-Martin lower bound gives us an upper bound on $\gamma(g_\infty)$, namely, $\gamma(g_\infty) \le 3$.

V. CONCLUSIONS

In conclusion it is perhaps worthwhile to stress the remarks listed below:

(1) The use of the renormalization-group method for on-mass-shell amplitudes as in Refs. 1 and 2 is on less firm ground than when one is dealing with the deep Euclidean region. Even though z is

physical and in the analyticity domain, the variable s is on the s cut. Terms in the Callan-Symanzik equation which are negligible for momenta inside the analyticity domain might not be so on the boundary of the domain.

(2) Our assumption (A) which is true order by order in perturbation theory can be significantly weakened. One can handle the case where the singularity of the full zero-mass vertex function at $z=1$ is less singular than any finite-order pole. The arguments needed to carry out this are quite involved and lead to a weaker bound on $\sigma_{\text{tot}}(s)$, namely $\sigma_{\text{tot}}(s) \leqslant \text{const} \times (\ln \ln s)^2$. We shall publish these results in a later paper, especially since they might have more general interest independent of the renormalization group.

(3) If one uses the method of Tiktopoulos[1] and applies it directly to a function G_δ defined as an integral over a $\Gamma^{(4)}$ normalized in the same way as in Ref. 1, it should be possible to check our uniform-convergence assumption (B) at least order by order in perturbation theory. For the lowest nontrivial orders it is true. Work along this line is in progress.

(4) This paper is restricted to ϕ^4 field theory. One should not take the scattering of the mesons in that theory to represent actual physical π-π scattering. Hadrons are certainly much more complicated objects. So the fact that for $\gamma(g_\infty) > 0$ Eq. (4.8) implies $\sigma_{\text{tot}}(s) \to 0$ as $s \to \infty$ for the ϕ^4 mesons has little to do with the data on rising proton-proton total cross sections. Anyway, no experiment has yet ruled out the possibility that in the real world eventually $\sigma_{\text{tot}}(s) \to 0$ as $s \to \infty$.

(5) In Ref. 2 the existence of exotic pinch-type singularities of the zero-mass vertex function in the interval $-1 < z < +1$ was excluded. We are of course here making the same assumptions as in that paper in regard to this point. Furthermore, even though the $z=1$ singularities of the zero-mass vertex function are presumably logarithmic in each order in perturbation theory, it is quite possible that the sum of all orders might lead to a much stronger singularity at $z=1$ for the full $\Gamma^{(4)}$. However, if this happens, it would cast great doubt on the validity of the renormalization-group method itself, where one ignores terms that are small order by order in perturbation theory and hopes that their sum is not large.

Note added in proof. By a minor modification of the method above we can also handle the case where the singularity of the zero-mass vertex function at $z=1$ is a simple pole. The result is the same as in Eq. (4.8) except now one has a factor $\ln^3 s$ instead of $\ln^2 s$ on the right. However, one must stress that a pole at $z=1$, i.e., $t=0$, implies the existence of a zero-mass bound state in the zero-mass theory which can couple to two of the fundamental zero-mass mesons. This leads to a three-particle zero-mass vertex similar to that in ϕ^3 theory and would then cast strong doubt on the assumptions of Refs. 1 and 2. Namely, a ϕ^4 theory with such a zero-mass bound state might not have "tame" infrared behavior.

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[1]G. Tiktopoulos, Phys. Rev. D 11, 2252 (1975).

[2]C. G. Callan, Jr. and D. J. Gross, Phys. Rev. D 11, 2904 (1975); M. Creutz and L.-L. Wang, *ibid.* 10, 3749 (1974).

[3]K. Huang and F. E. Low, Zh. Eksp. Teor. Fiz. 46, 845 (1964) [Sov. Phys.-JETP 19, 579 (1964)].

[4]A. D. Dolgov, V. I. Zakharov, and L. B. Okun, Yad. Fiz. 15, 808 (1972) [Sov. J. Nucl. Phys. 15, 451 (1972)]; Phys. Lett. 37B, 298 (1971); A. D. Dolgov, V. N. Gribov, L. B. Okun, and V. I. Zakharov, Nucl. Phys. B59, 611 (1973); J. B. Healy, Phys. Rev. D 10, 644 (1974).

[5]V. Singh, Phys. Rev. Lett. 26, 530 (1971).

[6]H. Cheng and T. T. Wu, Phys. Rev. Lett. 24, 1456 (1970).

[7]S. Weinberg, Phys. Rev. D 8, 3497 (1973); Z. Jinn-Justin, in *Cargese Lectures in Physics*, edited by E. Schatzman (Gordon and Breach, New York, 1973), Vol. 6. Our approach is based on an unpublished Princeton report by C. G. Callan, Jr., which treats the new improved renormalization group for ϕ^4 theory.

[8]It is often assumed in the literature that the moments of the electroproduction structure functions, $\int_1^\infty dw\, w^{-n-2} F_2(w, q^2)$, scale as $q^2 \to \infty$ if $F_2(w, q^2)$ itself scales as $q^2 \to \infty$. One can easily make counterexamples where $F_2(w, q^2)$ scales as $q^2 \to \infty$, but some moments do not. Without uniform convergence of the integrals for large q^2, one cannot relate the scaling of F_2 to that of the moments. Yet this assumption is often implicitly made without its being recognized. See, for example, A. De Rujula, S. L. Glashow, H. D. Politzer, S. B. Treiman, F. Wilczek, and A. Zee, Phys. Rev. D 10, 1649 (1974); specifically note the sentence following Eq. (1) in that paper.

[9]J. D. Bessis, Nuovo Cimento 45A, 974 (1966).

[10]Y. S. Jin and A. Martin, Phys. Rev. 135, B1369 (1964). Also H. Cornille, Nuovo Cimento 4A, 549 (1971).