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Mass-shell applications of the renormalization group with arbitrary interpolating fields*

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The use of renormalization-group techniques to study the high-energy, fixed-angle behavior of on-shell S-matrix elements is discussed and reconciled with the fact that the field operator associated with an external on-shell particle is not unique. It is shown that, in a theory with an ultraviolet-stable fixed point, the various choices of interpolating field all give the same effective fixed-point anomalous dimension. The remaining differences in the renormalization-group equations for vertex functions of different interpolating fields are absorbed by the finite renormalization necessary for constructing properly normalized scattering amplitudes.

I. INTRODUCTION

The techniques of renormalization group and Callan-Symanzik equations[1,2] were originally developed to investigate the behavior of Green's functions in the deep Euclidean region where all external particles are far off the mass shell. Subsequently, by incorporating the Wilson operator expansion the analysis was extended to describe inclusive cross sections for deep-inelastic semileptonic processes in which the target hadron is on the mass shell but the mediating vector current is in the deep Euclidean region.[3,4] More recently, the renormalization-group analysis has been used to treat exclusive hadronic processes, in particular, high-energy, fixed-angle elastic and quasielastic scattering in which all the particles are on the mass shell.[5,6] In the last case the scattering amplitude scales asymptotically as a power of the energy, the power being determined by the fixed-point values of the anomalous dimensions of the fields describing the external particles.

This seems puzzling if one recalls the well-known result of Lehmann, Symanzik, and Zimmermann[7] (LSZ) that there is great freedom in choosing the operator to be used as a local interpolating field for a given physical particle in on-mass-shell amplitudes. There is an apparent paradox unless all fields satisfying the same weak asymptotic limit have the same fixed-point anomalous dimension. This should be true whether the particle is elementary (corresponding to the quanta of a canonical field in an underlying Lagrangian) or composite (corresponding to a bound state and representable only by composite fields).

The main point of this paper is to show that the necessary condition is, in fact, satisfied. All equivalent interpolating fields have the same effective fixed-point anomalous dimension and the renormalization-group results are in perfect agreement with the basic theorems of LSZ. In that sense, then, we are discussing only a pseudoproblem here. However, the question posed above has been a source of confusion to us and to colleagues to whom we have posed the apparent paradox, and the question does not seem to have been addressed specifically in the literature. We consider it useful, therefore, to present an explicit answer.

The asymptotic high-energy fixed-angle amplitude is related in these calculations to an amplitude in which the mass parameters approach zero. The high-energy behavior of the physical amplitude is controlled by the zero-mass singularities of the theory. The latter arise from two sources: zero-mass singularities of the unrenormalized (and ultraviolet regularized) Feynman amplitudes and singularities of the wave-function renormalization constants in the zero-mass limit.

It is only the latter which are easily studied by renormalization-group methods and which we wish to discuss here. Therefore, we assume throughout that the unrenormalized theory has a finite zero-mass limit. We must renormalize the theory in a way which can be extended to the zero-mass case by defining our renormalized parameters at Euclidean momenta. A convenient method of accomplishing this is the Weinberg approach to the renormalization group.[8]

In Sec. II we discuss carefully the relation between Green's functions and vertex functions defined via the intermediate renormalization method of Weinberg and the physical T-matrix elements. For particles associated with the quanta of a canonical field we show how to separate the T-matrix elements into factors which have finite zero-mass limits and factors whose zero-mass singularities can be computed by renormalization-group methods. We exhibit explicitly how the LSZ theorem is respected and identical results are obtained when an arbitrary composite operator is used as an interpolating field for the particles in mass-shell amplitudes.

In Sec. III we generalize this last point by considering the renormalization-group equations satisfied by general Green's functions defined for the composite field operator and show that these generally more-complicated equations reduce to the usual form for mass-shell amplitudes. The anomalous dimension thus defined for the composite operator is not the same as that of the canonical field but it has the same value at an ultraviolet-stable fixed point. This is sufficient to give identical predictions for the physical T-matrix elements.

The generalization to the case of bound states which must be created by composite operators is briefly discussed. The anomalous dimension of the lowest-dimension composite operator is the relevant quantity here. As noted previously,[6] zero-mass singularities of the bound-state wave functions will usually be present and important in this case.

II. ANOMALOUS DIMENSIONS AND ZERO-MASS SINGULARITIES

We begin by exploring the relation between conventionally normalized on-shell T-matrix elements and the on-shell Green's functions of a field theory defined by an intermediate renormalization at Euclidean momenta and zero mass as proposed by Weinberg. We use the subscript H to denote quantities in the conventional Heisenberg picture where fields have vacuum to one-particle matrix elements of unit norm. Unsubscripted Green's functions, etc., refer to the intermediate renormalized fields.

For simplicity we assume that the theory is characterized by a single coupling constant g and mass parameter m. Another parameter μ with the dimensions of mass specifies the Euclidean renormalization point.

Let $G^{(n)}(\{p\};g,m,\mu)$ denote the Green's functions. Vertex functions (Green's functions with external legs amputated) are defined by

$$\Gamma^{(n)}(\{p\};g,m,\mu)=\left[\prod_{i=1}^{n}\Gamma^{(2)}(p_i^2;g,m,\mu)\right]\times G^{(n)}(\{p\};g,m,\mu). \qquad (2.1)$$

Let $M(g,m,\mu)$ denote the physical mass of the particle. For $n=2$ and p^2 near M^2, we have

$$\Gamma^{(2)}(p^2;g,m,\mu)=\mathfrak{z}^{-1}(g,m,\mu)(p^2-M^2)+O((p^2-M^2)^2). \qquad (2.2)$$

Thus, the physical on-shell T-matrix elements are given by

$$T_H^{(n)}=[\mathfrak{z}(g,m,\mu)]^{n/2}\,\Gamma^{(n)}. \qquad (2.3)$$

The n-point vertex functions satisfy the "improved renormalization-group equation"[8]

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-m\hat{\gamma}(g)\frac{\partial}{\partial m}\right]\Gamma^{(n)}(\{p\};g,m,\mu)=n\gamma(g)\Gamma^{(n)}(\{p\};g,m,\mu). \qquad (2.4)$$

As a convenient shorthand we will use the symbol $\mathfrak{D}$ for the differential operator on the left-hand side of (2.4).

This equation and ordinary dimensional analysis lead to the standard solution

$$\Gamma^{(n)}(\{\lambda p\};g,m,\mu)=\lambda^{4-n}\exp\left[-n\int_1^{\lambda}\gamma(\bar{g}(\lambda'))\,d\lambda'/\lambda'\right]\times\Gamma^{(n)}(\{p\};\bar{g}(\lambda),\bar{m}(\lambda),\mu), \qquad (2.5)$$

where the effective coupling constant and mass are solutions of the differential equations

$$\lambda\frac{\partial\bar{g}}{\partial\lambda}=\beta(\bar{g}),\qquad \lambda\frac{\partial\bar{m}}{\partial\lambda}=-\bar{m}\,(1+\hat{\gamma}(\bar{g})), \qquad (2.6)$$

with initial conditions

$$\bar{g}(1)=g,\qquad \bar{m}(1)=m.$$

We assume that the usual condition $\hat{\gamma}(g)>-1$ is satisfied.

For $n=2$ let $(\lambda p)^2$ approach M^2. Then (2.5) gives

$$\mathfrak{z}^{-1}(g,m,\mu)[(\lambda p)^2-M^2(g,m,\mu)]=\lambda^2\exp\left[-2\int_1^{\lambda}\gamma(\bar{g}(\lambda'))\,d\lambda'/\lambda'\right]\times\Gamma^{(2)}(p;\bar{g}(\lambda),\bar{m}(\lambda),\mu), \qquad (2.7)$$

from which we deduce

$$M(\bar{g}(\lambda),\bar{m}(\lambda),\mu)=M(g,m,\mu)/\lambda \qquad (2.8)$$

and

$$\mathfrak{z}(\bar{g}(\lambda),\bar{m}(\lambda),\mu)=\mathfrak{z}(g,m,\mu)\exp\left[-2\int_1^{\lambda}\gamma(\bar{g}(\lambda'))\,d\lambda'/\lambda'\right]. \qquad (2.9)$$

From (2.5) the on-shell vertices satisfy

$$\Gamma^{(n)}(\{\lambda\vec{p}\};g,m,\mu)=\lambda^{4-n}\exp\left[-n\int_1^{\lambda}\gamma(\bar{g}(\lambda'))\,d\lambda'/\lambda'\right]\times\Gamma^{(n)}(\{\vec{p}\};\bar{g}(\lambda),\bar{m}(\lambda),\mu). \qquad (2.5')$$

Note that the vertex functions on both sides of this equation are on the mass shell [see (2.8)]. We assume now and henceforth that there is an ultraviolet-stable fixed point at $g=g_\infty$ $[\beta(g_\infty)=0]$ which controls the asymptotic behavior of the theory. Then

$$\lim_{\lambda\to\infty}\Gamma^{(n)}(\{\lambda\vec{p}\};g,m,\mu)$$

$$=\lambda^{4-n[1+\gamma(g_\infty)]}\exp\left\{n\int_1^\infty[\gamma(g_\infty)-\gamma(\bar{g}(\lambda'))]d\lambda'/\lambda'\right\}\times\Gamma^{(n)}(\{\vec{p}\};g_\infty,0,\mu). \qquad (2.10)$$

Because of the divergence of the integral

$$\lim_{\lambda\to\infty}\int_1^\lambda\gamma(\bar{g}(\lambda'))d\lambda'/\lambda',$$

the residue of the Green's functions at the external mass poles does not have a finite nonvanishing zero-mass limit. However, the zero-mass vertex function Γ does exist in this limit if the unrenormalized theory has no zero-mass singularities.[9] Then combining (2.3) with (2.5) we have

$$\lim_{\lambda\to\infty}T_H(\{\lambda\vec{p}\};g,m,\mu)=\lambda^{4-n[1+\gamma(g_\infty)]}f(\{\theta\}), \qquad (2.11)$$

where

$$f(\{\theta\})=[\mathfrak{z}(g,m,\mu)]^{n/2}\exp\left\{n\int_1^\infty[\gamma(g_\infty)-\gamma(\bar{g}(\lambda'))]d\lambda'/\lambda'\right\}\times\Gamma^{(n)}(\{\vec{p}\};g_\infty,0,\mu).$$

$\{\theta\}$ is a set of angles between the momenta of the particles.

In the case of scattering amplitudes for different particles belonging to multiplets of an approximate symmetry group, the only symmetry-breaking factors in (2.11) are the finite wave-function renormalizations $\mathfrak{z}(g,m,\mu)$. The exponent of λ and the shape of the angular distribution exhibit the exact symmetry. The analysis carried out here refers to the case where only mass terms break the symmetry, but the generalizations to other types of soft-symmetry breaking are straightforward.

To further emphasize the meaning of this calculation, we observe that (2.3), (2.5′), and (2.9) give

$$T_H^{(n)}(\{\lambda\vec{p}\};g,m,\mu)=\lambda^{4-n}T_H^{(n)}((\{\vec{p}\};\bar{g}(\lambda),\bar{m}(\lambda),\mu). \qquad (2.12)$$

This looks like ordinary dimensional scaling and follows directly from

$$\mathfrak{D}T_H=0$$

plus ordinary dimensional analysis. To make this even more transparent, we remove reference to the intermediate coupling constant, g, and Euclidean renormalization point, μ, by defining a physical coupling constant G via

$$\Gamma_H^{(4)}(\{p\};g,m,\mu)|_{\text{sym. pt.}}=-iG(g,m,\mu). \qquad (2.13)$$

The "symmetric point" is

$$p_i\cdot p_j=M^2(4\delta_{ij}-1)/3.$$

It is easy to see that

$$G(g,m,\mu)=G(\bar{g}(\lambda),\bar{m}(\lambda),\mu) \qquad (2.14)$$

and (2.12) can be written as

$$T_H^{(n)}(\{\lambda\vec{p}\};G,M)=\lambda^{4-n}T_H^{(n)}(\{\vec{p}\};G,M/\lambda), \qquad (2.15)$$

which is obviously simple dimensional scaling and useless for predicting the $\lambda\to\infty$ behavior.

We emphasize again that the power of the renormalization-group analysis and intermediate renormalization procedure is to express T_H as a product of functions, some of which have well-defined nonzero limits as $\lambda\to\infty$ and others whose asymptotic behavior can be computed. In our case the zero-mass singularity is confined to the finite renormalization constant $\mathfrak{z}(g,m,\mu)$. This becomes more explicit if we consider $\mathfrak{z}$ as a function of G and M instead of g and m. Define

$$\hat{\mathfrak{z}}(G(g,m,\mu),M(g,m,\mu),\mu)=\mathfrak{z}(g,m,\mu). \qquad (2.16)$$

The $M\to 0$ behavior of $\hat{\mathfrak{z}}$ is dictated by (2.9), viz.

$$\hat{\mathfrak{z}}(G,M/\lambda,\mu)=\hat{\mathfrak{z}}(G,M,\mu)\times\exp\left[-2\int_1^\lambda\gamma(\bar{g}(\lambda'))d\lambda'/\lambda'\right]$$

and

$$\lim_{\lambda\to\infty}\hat{\mathfrak{z}}(G,M/\lambda,\mu)\sim\lambda^{-2\gamma(g_\infty)}. \qquad (2.17)$$

The exponential factor which controls the high-energy behavior of T_H can be thought of as the zero-mass singularity of the renormalization constant.

We will use this idea to give a definition of $\gamma(g_\infty)$ which is equivalent to the conventional one for a canonical field and which can be easily extended to define the fixed-point anomalous dimension of a composite field.

Consider any unrenormalized local operator $\Phi(x)$ constructed as a product of canonical fields and their derivatives and carrying the same quantum numbers as the canonical field $\phi(x)$. To construct a properly normalized local interpolating field for on-shell amplitudes we define

$$\Phi_H(x)=\frac{\Phi(x)-\langle 0|\Phi(x)|0\rangle}{\langle 0|\Phi(0)|k\rangle}, \qquad (2.18)$$

where $|k\rangle$ is a one-particle state. There are, of course, delicate limiting procedures which must be defined to give meaning to the quantities above.[10] When necessary we will define divergent quantities as sums of Feynman diagrams regulated by an ultraviolet cutoff parameter Λ.

In particular, we define a wave-function renormalization constant for the field Φ by

$$\langle 0|\Phi(x)|k\rangle = \Lambda^{d_\Phi - 1}\tilde{\mathfrak{z}}_\Phi^{1/2}(G,\Lambda/M)e^{-ikx}\,, \tag{2.19}$$

where d_Φ is the canonical dimension of Φ. For the elementary field ϕ, $\tilde{\mathfrak{z}}_\phi(G,\Lambda/M)$ is the usual on-shell wave-function renormalization and

$$\tilde{\mathfrak{z}}_\phi(G,\Lambda/M) = \hat{\mathfrak{z}}(G,M,\mu)Z(g_0,\Lambda/\mu), \tag{2.20}$$

where g_0 is the bare coupling and Z is the Weinberg intermediate renormalization constant.

Diagrammatically, the on-shell T matrix comes from residues of the poles in the external momenta and it can be seen that demonstrating the renormalization-group result to be independent of the interpolating field is equivalent to demonstrating that the leading zero-mass singularity in $\tilde{\mathfrak{z}}_\Phi$ is the same as that of $\tilde{\mathfrak{z}}_\phi$. To that end we define a quantity $\tilde{\gamma}_\Phi$ associated with the general local operator by

$$\tilde{\gamma}_\Phi = -\tfrac{1}{2}\lim_{\Lambda/M\to\infty}\left[\frac{\ln\tilde{\mathfrak{z}}_\Phi(G,\Lambda/M)}{\ln(\Lambda/M)}\right]. \tag{2.21}$$

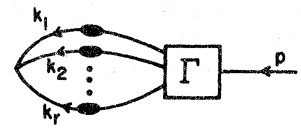

FIG. 1. Graphs contributing to $\tilde{\mathfrak{z}}_\Phi$ in Sec. II and to $\tilde{\Omega}$ in Sec. III.

For the elementary field ϕ, $\tilde{\gamma}_\phi = \gamma(g_\infty)$. Our task is to show that $\tilde{\gamma}_\Phi = \tilde{\gamma}_\phi = \gamma(g_\infty)$.

For simplicity we consider the case of Φ as the product of unrenormalized scalar fields, $\sigma^u(x)$, which we can take to be the same as or different from $\phi(x)$. The generalization to a product of several different fields or terms with derivatives is not difficult. Let

$$\Phi(x) = :[\sigma^u(x)]^r:\,. \tag{2.22}$$

$\tilde{\mathfrak{z}}_\Phi$ can be calculated from the set of unrenormalized graphs of Fig. 1. Writing them in terms of the basic Green's functions of the theory, we have

$$\Lambda^{r-1}\tilde{\mathfrak{z}}_\Phi^{1/2}(G,\Lambda/M) = \tilde{\mathfrak{z}}_\phi^{1/2}\left\{\int_{R(\Lambda)}\cdots\int(2\pi)^4\delta^4\left(p-\sum_{i=1}^{r}k_i\right)\left[\prod_{i=1}^{r}\frac{d^4k_i}{(2\pi)^4}\,i\Delta_\sigma^u(k_i;g_0,m_0,\Lambda)\right]\Gamma^u_{(r,1)}(\{k\},p;g_0,m_0,\Lambda)\right\}_{p^2=M^2}, \tag{2.23}$$

where Δ_σ^u is the unrenormalized propagator for the field σ and $\Gamma^u_{(r,1)}$ is the unrenormalized amputated vertex function for $r\,\sigma$ fields and one ϕ field. $R(\Lambda)$ indicates that the integration region has an ultraviolet cutoff characterized by the parameter Λ. To determine the leading $\Lambda\to\infty$ behavior of (2.23) we use renormalization-group methods to find the large-momentum behavior of the integrand.

Renormalizing the Green's functions in (2.23) in the manner of Weinberg, we have

$$\Lambda^{r-1}\tilde{\mathfrak{z}}_\Phi^{1/2}(G,\Lambda/M) = \hat{\mathfrak{z}}^{1/2}(G,M,\mu)[Z_\sigma(g_0,\Lambda/\mu)]^{r/2} \times\left\{\int_{R(\Lambda)}\cdots\int(2\pi)^4\delta^4\left(p-\sum_{i=1}^{r}k_i\right)\left[\prod_{i=1}^{r}\frac{d^4k_i}{(2\pi)^4}\,i\Delta_\sigma(k_i;g,m,\mu)\right]\Gamma_{(r,1)}(\{k\},p;g,m,\mu)\right\}_{p^2=M^2}, \tag{2.24}$$

where Z_σ is the intermediate wave-function renormalization constant for the σ field. The leading cutoff dependence comes from the region where all integration momenta k are $O(\Lambda)$. Therefore, we scale the integration momenta by

$$k^i = (\Lambda/M)l^i\,.$$

As $\Lambda/M\to\infty$ the l^i are restricted to a finite integration region $R(M)$. The leading power of (Λ/M) of the integrand can be obtained by applying renormalization-group results to the functions in the integrand. The result is

$$\tilde{\mathfrak{z}}_\Phi^{1/2}(G,\Lambda/M) = (\Lambda/M)^{r\,\gamma_\sigma(g_\infty)-\gamma(g_\infty)}[Z_\sigma(g_0,\Lambda/\mu)]^{r/2} \times f(G,M,\mu), \tag{2.25}$$

where $f(G,M,\mu)$ is a cutoff-independent function which need not concern us further. To extract the cutoff dependence of (2.25), we define

$$\hat{Z}_\sigma(g,\Lambda/\mu) = Z_\sigma(g_0,\Lambda/\mu) \tag{2.26}$$

and recall that $\gamma_\sigma(g)$ is defined by

$$\mathfrak{D}\hat{Z}_\sigma = 2\gamma_\sigma(g)\hat{Z}_\sigma. \tag{2.27}$$

It is then easy to see that

$$\hat{Z}_\sigma(g,\Lambda/\mu)=(\Lambda/\mu)^{-2\gamma_\sigma(g_\infty)}\times\text{const.} \tag{2.28}$$

Inserting this result in (2.25) we get

$$\lim_{\Lambda/M\to\infty} \mathfrak{z}_\Phi{}^{1/2}(G,\Lambda/M)\sim(\Lambda/M)^{-\gamma(g_\infty)}\times\text{const,} \tag{2.29}$$

which, substituted in (2.21), gives the desired result

$$\tilde{\gamma}_\Phi=\gamma(g_\infty).$$

III. RENORMALIZATION-GROUP EQUATIONS FOR COMPOSITE OPERATORS

To see the structure of the renormalization-group equations in more detail, we consider what happens to the formalism if we actually define renormalized Green's functions for the composite field

$$\Phi(x)=:[\sigma(x)]^r:. \tag{3.1}$$

σ is an intermediate renormalized scalar field.

If we want to define a renormalized Green's function for n external $\Phi(x)$ fields, we need to introduce renormalized proper vertices for $j=1$ to n Φ fields and any number of canonical $\phi(x)$ fields and construct the complete set of renormalized Feynman diagrams by combining these proper diagrams with the appropriate Green's functions for the canonical fields in all possible ways.

The proper vertices for Φ and ϕ fields can be renormalized by subtraction of the necessary number of terms in the Taylor series in the external momenta about $-\mu^2$.[11] Applying the differential operatcr of the renormalization-group equation to these Green's functions will give, in general, inhomogeneous equations. If, however, we are interested only in the mass-shell T matrix, the equations are much simpler. Only graphs with a pole in each external momentum contribute. The only new vertex which we need to introduce is

$$\Omega(k)=\Gamma_\Phi^{(2)}(k)\int e^{-ikx}\langle 0|T(\Phi(0)\phi(x))|0\rangle d^4x. \tag{3.2}$$

The superficial degree of divergence of this vertex is $d_\Omega=r-1>0$ so $\Omega(k)$ is divergent and should be considered as cutoff dependent. We renormalize by subtractions at $-\mu^2$. Let N be the largest integer $\leq\frac{1}{2}d_\Omega$. Then

$$\tilde{\Omega}(k;g,m,\mu)=\Omega(k)-\sum_{j=0}^{N}\frac{(k^2+\mu^2)^j}{j!}\left[\frac{d^j}{d(p^2)^j}\Omega(p^2)\right]_{p^2=-\mu^2} \tag{3.3}$$

defines our cutoff-independent vertex.[12] This renormalized vertex can be conveniently expressed as

$$\tilde{\Omega}(k;g,m,\mu)=\frac{1}{N!}\int_{-\mu^2}^{k^2}(k^2-p^2)^N\left[\frac{d^{N+1}}{d(p^2)^{N+1}}\Omega(p^2)\right]dp^2. \tag{3.3'}$$

The pole terms in the Green's function for n external fields are given by

$$G_\Phi^{(n)}(\{k_i\})|_{\text{pole}}=\left[\prod_{i=1}^{n}\tilde{\Omega}(k_i)\right]G_\Phi^{(n)}(\{k_i\}). \tag{3.4}$$

As before, the residue of the n-tuple poles in the external momenta differs from $T_H^{(n)}$ by a finite renormalization factor. Applying the operator $\mathfrak{D}$ to this Green's function we get

$$\mathfrak{D}G_\Phi^{(n)}|_{\text{pole}}=-n\gamma_\Phi G_\Phi^{(n)}|_{\text{pole}}, \tag{3.5}$$

with

$$\gamma_\Phi=-[\mathfrak{D}\tilde{\Omega}(M^2)]/\tilde{\Omega}(M^2)+\gamma(g). \tag{3.6}$$

Equation (3.5) can be solved in the same form as (2.5) except that γ_Φ depends on m and μ as well as g. The integral in the exponent is replaced by

$$\int_1^\lambda \gamma_\Phi(\bar{g}(\lambda'),\bar{m}(\lambda'),\mu)\,d\lambda'/\lambda'. \tag{3.7}$$

The scaling behavior of the T matrix is controlled by the value of γ_Φ at the fixed point. To obtain a function with a finite nonvanishing value in the zero-mass limit we need to introduce again an amputated vertex function $\Gamma_\Phi^{(n)}$ by first defining a two-point function

$$\Delta_\Phi(k)=i\int d^4x\,e^{ikx}\langle 0|T(\Phi(0)\Phi(x))|0\rangle - \text{subtractions}. \tag{3.8}$$

Δ_Φ requires subtractions of its one-particle irreducible part but this is irrelevant when we look at the pole term. Then

$$\Gamma_\Phi^{(n)}\equiv\left[\prod_{i=1}^{n}\Delta_\Phi{}^{-1}(k_i)\right]G_\Phi^{(n)} \tag{3.9}$$

and for $k_i{}^2=M^2$, all i,

$$\mathfrak{D}\Gamma_\Phi^{(n)}=n\gamma_\Phi\Gamma_\Phi^{(n)}. \tag{3.10}$$

It is important that the rescalings of $\bar{g}(\lambda)$ and $\bar{m}(\lambda)$ keep the amplitudes on the mass shell so that inhomogeneous terms from the extra counterterms never enter these equations. The solution of (3.10) is of the standard form. Note that physical dimensional analysis gives

$$\left(\mu\frac{\partial}{\partial\mu}+m\frac{\partial}{\partial m}+\lambda\frac{\partial}{\partial\lambda}\right)\Gamma_\Phi^{(n)}(\{\lambda p\};g,m,\mu)=(4-nr)\Gamma_\Phi^{(n)}. \tag{3.11}$$

It is straightforward to see that agreement with the results of Sec. II requires that

$$\gamma_\Phi(g_\infty,0,\mu)=1-r+\gamma(g_\infty). \tag{3.12}$$

Applying the renormalization-group operator to $\tilde{\Omega}(M^2=0)$ gives

$$\mathfrak{D}\tilde{\Omega}(0;g_\infty,0,\mu)=\frac{1}{N!}\int_{-\mu^2}^{0}(-p^2)^N\frac{d^{N+1}}{d(p^2)^{N+1}}[\mathfrak{D}\Omega(p^2;g_\infty,0,\mu)]dp^2 +2\mu^2\left\{\frac{d}{dk^2}\,\frac{1}{N!}\int_{k^2}^{0}(-p^2)^N\left[\left(\frac{d}{dp^2}\right)^{N+1}\Omega(p^2)\right]dp^2\right\}_{k^2=\mu^2}. \tag{3.13}$$

At the fixed point the scaling properties of the Green's functions drastically simplify the evaluation of these derivatives. The structure of $\Omega(p^2)$ is given in Fig. 1 and can be expressed as

$$\Omega(p^2)=\int\cdots\int\left[\prod_{i=1}^{r}\Delta_\sigma(k_i;g_\infty,0,\mu)\frac{d^4k_i}{(2\pi)^4}\right]\Gamma_{(r,1)}(\{k\},p;g_\infty,0,\mu)(2\pi)^4\delta^4(p-\textstyle\sum k_i). \tag{3.14}$$

The distributive property of the $\mathfrak{D}$ operator gives

$$\mathfrak{D}\Omega(p^2;g_\infty,0,\mu)=[-r\gamma_\sigma(g_\infty)+\gamma(g_\infty)]\Omega(p^2;g_\infty,0,\mu). \tag{3.15}$$

To evaluate the second term in (3.13) we use the scaling properties of the Green's functions at the fixed point to obtain

$$\tilde{\Omega}(0;g_\infty,0,\mu)=\frac{1}{N!}(\mu^2)^{N+1}\left[\left(\frac{d}{dp^2}\right)^{N+1}\Omega(p^2;g_\infty,0,\mu)\right]_{p^2=-\mu^2}2\int_0^1\lambda^{-1+r[1+\gamma_\sigma(g_\infty)]-[1+\gamma(g_\infty)]}d\lambda. \tag{3.16}$$

The same scaling properties allow us to write

$$\left[\left(\frac{d}{dp^2}\right)^{N+1}\Omega(p^2;g_\infty,0,\mu)\right]_{p^2=-(\mu^2+\Delta\mu^2)}=\left[\left(\frac{d}{dp^2}\right)^{N+1}\Omega(p^2;g_\infty,0,\mu)\right]_{p^2=-\mu^2}\left(1+\frac{\Delta\mu^2}{\mu^2}\right)^{-(N+1)+r[1+\gamma_\sigma(g_\infty)]/2-[1+\gamma(g_\infty)]/2} \tag{3.17}$$

With these results it is easy to evaluate the second term on the right-hand side of (3.13) and obtain

$$2\mu^2\frac{d}{dk^2}\,\frac{1}{N!}\left\{\int_{-k^2}^{0}(-p^2)^N\left[\left(\frac{d}{dp^2}\right)^{N+1}\Omega(p^2;g_\infty,0,\mu)\right]dp^2\right\}_{k^2=\mu^2}=\{r[1+\gamma_\sigma(g_\infty)]-[1+\gamma(g_\infty)]\}\tilde{\Omega}(0;g_\infty,0,\mu). \tag{3.18}$$

Putting all the pieces together we finally get

$$\gamma_\Phi(g_\infty,0,\mu)=1-r+\gamma(g_\infty), \tag{3.19}$$

which is the desired result. The generalization to the case of a $\Phi(x)$ defined as a product of several different operators is straightforward.

In the case of a bound-state particle for which no canonical interpolating field exists in the theory, the composite field of lowest dimension (lowest twist) can be multiplicatively renormalized and an anomalous dimension can be assigned to this operator.[13] The analysis of the first part of Sec. II can be repeated in this case. However, there will, in general, be additional zero-mass singularities from the bound-state wave function and these must be evaluated to predict the high-energy behavior of the T matrix.[6]

If a higher-dimension operator is used as an interpolating field for a composite particle, we can carry through the analysis in analogy to the treatment of a composite operator as the interpolating field for an elementary particle. In all cases the fixed-point anomalous dimension controlling S-matrix elements is independent of the interpolating field. Because the operator of lowest dimension is multiplicatively renormalizable, it provides the simplest definition of the anomalous dimension.

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[9]This result agrees with those of Gell-Mann and Zachariasen (Ref. 6), and Creutz and Wang (Ref. 5). We differ from Callan and Gross (Ref. 5) in the sign of the exponential term involving the anomalous dimension. The last authors assume a nonzero limit for

$$\lim_{\lambda\to\infty}\left\{\prod_i [k_i^2 - M^2(\lambda)]\right\} G^{(n)}(\{k_i\}, \bar{g}(\lambda), \bar{m}(\lambda), \mu)\Big|_{k_i^2=M^2},$$

which disagrees with our conclusions.

[10]See, e.g., W. Zimmermann, Nuovo Cimento 10, 597 (1958).

[11]We are assuming that the Bogoliubov-Parasiuk-Hepp-Zimmermann method of subtracting a sufficient number of terms in the Taylor series about $p_\mu = 0$ can be extended to expansion about a Euclidean point. For a rigorous treatment of the definition of composite operators by the former method, see W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and M. Pendleton (MIT Press, Cambridge, Mass. 1970); Ann. Phys. (N.Y.) 77, 536 (1973).

[12]To be precise, the Taylor-series expansion should be made for the integrand of the Feynman integral defining Ω (Ref. 11). It is more convenient to use this shorthand notation. The important point to keep in mind in the following manipulations is that the loop integrals over the subtracted integrand are convergent and the momentum integration variables can be freely scaled and translated.

[13]We assume that there is a unique operator of lowest twist and neglect the problem of mixing which occurs in the renormalization of two or more operators of the same twist.