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Bethe-Salpeter equation for fermion–vector-gluon systems

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We investigate the Wick-rotated Bethe-Salpeter equation of a fermion–vector-gluon system of zero total mass by using fermion-like effective interaction kernels. We summarize useful relations among the four-dimensional spinor-vector spherical harmonics in order to derive a system of ordinary differential equations for the radial wave functions. These equations are applied to a study of the short-distance behavior of the solutions satisfying appropriate boundary conditions. As a byproduct, we obtain the indices of the ground-state solution in explicit form. Some comments on the constituent models that involve the radial excitations of fermion-gluon systems are included.

I. INTRODUCTION

The constituent models of matter offer an attractive possibility for describing the low-lying quantum states of hadrons. A standard field-theoretical treatment of the dynamics of composite systems is provided by the Bethe-Salpeter (BS) equation.[1] Important progress has been made on the solution of the BS bound-state problems that involve spin-0 or spin-$\frac{1}{2}$ constituents: Extensive reviews are given by Nakanishi[2] and Böhm, Joos, and Krammer.[3] (For more recent developments see Refs. 4–7.)

In a previous paper,[8] we have discussed a BS equation describing fermion-antifermion systems bound to zero total mass by the exchange of Abelian vector and axial-vector gluons. The BS wave function of a fermion–vector-gluon system (spinor-vector BS wave function) is another ingredient in the constituent models that are based on vector-gluon theories. For this reason, we next extend the investigations to the spinor-vector BS equation involving a ladder-type interaction kernel of fermion quantum number. We only consider a neutral vector gluon V_μ coupled to a fermion field Ψ as given by the effective Lagrangian[9]

$$L = L_0(\Psi) + L_0'(V_\mu) + G_V' :\bar{\Psi}\gamma^\mu \Psi V_\mu: , \tag{1.1}$$

$$L_0(\Psi) = \bar{\Psi}(i\gamma^\mu \partial_\mu - \kappa)\Psi ,$$

$$L_0'(V_\mu) = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m_V^{\,2}}{2} V^\mu V_\mu - \tfrac{1}{2}(1-c)(\partial^\mu V_\mu)^2 ,$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. The gauge-fixing term $\frac{1}{2}(1-c)(\partial^\mu V_\mu)^2$ is included in order to reduce the singular behavior of the conventional propagator of massive vector gluons. (Assuming spontaneously generated masses by the Jackiw-Johnson mechanism,[10] an axial-vector gluon may be introduced in a similar way.) The explicit form of the gluon propagator that belongs to the Lagrangian $L_0'(V_\mu)$ can be written as[11]

$$D^c_{\mu\nu}(q) = \frac{1}{m_V^{\,2} - q^2 - i\epsilon}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 + i\epsilon}\right) + \frac{m_0^{\,2}}{m_V^{\,2}} \frac{1}{m_0^{\,2} - q^2 - i\epsilon} \frac{q_\mu q_\nu}{q^2 + i\epsilon} , \tag{1.2}$$

where the mass m_0 of the Stückelberg ghost is fixed by $m_0^{\,2} = m_V^{\,2}/(1-c)$.

In configuration space, the spinor-vector BS wave function is defined by

$$\tau_{\alpha\nu}(x_1, x_2) = \langle 0 \,|\, T\Psi_\alpha(x_1) V_\nu(x_2) \,|\, \Phi\rangle . \tag{1.3}$$

For Heisenberg states Φ of four-momentum P_μ, translational invariance implies

$$\tau_\nu(x_1, x_2) = \varphi_\nu(z)\exp[-iP_\rho(\mu_1 x_1^\rho + \mu_2 x_2^\rho)], \quad z = x_1 - x_2 \tag{1.4}$$

with the restriction $\mu_1 + \mu_2 = 1$. (The spinor index α is omitted.) The calculations will be carried out in the center-of-mass (c.m.) coordinate frame by choosing

$$P_0 = E, \quad P_j = 0 \quad (j = 1, 2, 3), \tag{1.5}$$

where E is the total c.m. energy of the system.

The present study is based on the homogeneous ladder-type BS equation

$$(-i\gamma^\rho \partial_{1\rho} + \kappa)[(\Box_2 - m_V^{\,2})g^{\mu\nu} + c\partial_2^\mu \partial_2^\nu]\tau_\nu(x_1, x_2) = I^{\mu\nu}(x_1 - x_2)\tau_\nu(x_2, x_1), \tag{1.6}$$

with $\partial_{k\mu} = \partial/\partial x_k^\mu$ $(k = 1, 2)$ and $\Box_2 = -\partial_2^\mu \partial_{2\mu}$. In the strict ladder approximation one has (see Fig. 1)

$$I^{\mu\nu}_{\text{ladder}}(z) = G_V'^{\,2}\gamma^\nu \frac{1}{i} S^c(z;\kappa)\gamma^\mu . \tag{1.7}$$

(S^c is the free fermion propagator.) We shall

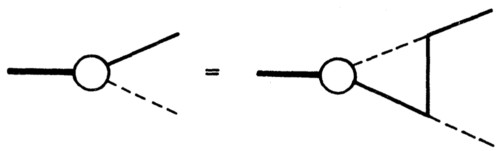

FIG. 1. Graphical representation of the BS equation with the ladder approximation. Dashed lines denote vector gluons.

choose a more general effective interaction kernel which may be written as

$$I^{\mu\nu}(z)=U^{\mu\nu}(z)+O^{\mu\nu}(z), \tag{1.8}$$

$$U^{\mu\nu}(z)=U_1(z)\gamma^\nu i\gamma^\rho \hat{z}_\rho \gamma^\mu - U_0(z) i\gamma^\rho \hat{z}_\rho \hat{z}^\mu \hat{z}^\nu, \tag{1.9}$$

$$O^{\mu\nu}(z)=O_1(z)\gamma^\nu\gamma^\mu, \tag{1.10}$$

where $\hat{z}^\mu=(-z^2)^{-1/2}z^\mu$, and the concrete form of the functions U_1, U_0, and O_1 should be restricted according to the Ward identity (cf. Ref. 12). Near the light cone, the ladder approximation and scaling arguments suggest the following parametrization:

$$U_a(z)i\gamma^\rho\hat{z}_\rho = i\gamma^\rho \frac{\partial}{\partial z^\rho}\int dm^2\rho_a(m^2)\frac{1}{i}D^c(z;m^2), \quad a=0,1 \tag{1.11}$$

$$O_1(z)=\int dm^2 m\sigma(m^2)\frac{1}{i}D^c(z;m^2). \tag{1.12}$$

Here D^c is the free scalar propagator, $\rho_{1,0}(m^2)$ and $\sigma(m^2)$ denote the effective spectral functions, and the integrals

$$\int dm^2\rho_{1,0}(m^2) \text{ and } \int dm^2 m\sigma(m^2)$$

are assumed to be convergent. [$\int dm^2 m\sigma(m^2)$ may vanish.] Notice that the light-cone behavior of the strict ladder approximation (1.7) is radically modified by the second term on the right-hand side of Eq. (1.9). A term of this type is also involved in the fermionlike potential of Ciafaloni and Ferrara.[12]

For simplicity, we next consider the Wick-rotated BS equation of the relative wave function $\varphi_\nu(z)$ at vanishing total c.m. energy E. In analyzing the solutions belonging to a particular sector, we shall use O(4) expansions along the lines suggested in Refs. 13–16. The outline of this paper is as follows. Section II discusses the structure of the Wick-rotated BS equation. To prepare the separation of the angular variables, we define a set of four-dimensional spinor-vector spherical harmonics in Sec. III. The derivation of the radial BS equations is presented in Sec. IV. Section V is devoted to a study of the indicial equations. In addition, we include a brief discussion of the short-distance properties of the BS wave function. Some comments and speculations are left for Sec. VI.

II. PRELIMINARIES

Let us start by removing the center-of-mass coordinate. The BS equation of the relative wave function $\varphi_\nu(z)$ becomes

$$(-\gamma^\rho p_{1\rho}+\kappa)[(p_2{}^2-m_V{}^2)g^{\mu\nu}-cp_2^\mu p_2^\nu]\varphi_\nu(z) = e^{i(\mu_1-\mu_2)Ez^0}I^{\mu\nu}(z)\varphi_\nu(-z), \tag{2.1}$$

with Eqs. (1.8)–(1.12) and

$$p_{1\rho}=i\frac{\partial}{\partial z^\rho}+\mu_1 P_\rho, \qquad p_{2\rho}=i\frac{\partial}{\partial z^\rho}-\mu_2 P_\rho. \tag{2.2}$$

The Wick rotation[17] will be carried out by a straightforward generalization of the usual procedures. We define the Wick-rotated relative wave function ψ_ν as

$$\psi_4(x)=i\varphi^0(-ix_4,x_1,x_2,x_3), \qquad \psi_j(x)=\varphi^j(-ix_4,x_1,x_2,x_3), \quad j=1,2,3. \tag{2.3}$$

Here $x_j=z^j$, $x_4=iz^0$ and all the components x_μ are real. In addition, we shall use the following notations:

$$\tilde{\gamma}_4=\gamma^0, \quad \tilde{\gamma}_j=-i\gamma^j, \tag{2.4}$$

$$P_4=E, \quad P_j=0,$$

$$\partial^-_\mu=\partial_\mu-\mu_1 P_\mu, \quad \partial^+_\mu=\partial_\mu+\mu_2 P_\mu, \tag{2.5}$$

and $\partial_\mu=\partial/\partial x_\mu$. The Wick-rotated version of the BS equation (2.1) can be written in the form

$$(\tilde{\gamma}_\rho\partial^-_\rho+\kappa)[(\partial^+_\sigma\partial^+_\sigma-m_V{}^2)\delta_{\mu\nu}-c\partial^+_\mu\partial^+_\nu]\psi_\nu(x) = e^{(\mu_1-\mu_2)Ex_4}\tilde{I}_{\mu\nu}(x)\psi_\nu(-x). \tag{2.6}$$

According to Eqs. (1.8)–(1.12), one obtains

$$\tilde{I}_{\mu\nu}(x)=\tilde{W}_{\mu\nu}(x)+\tilde{M}_{\mu\nu}(x), \tag{2.7}$$

$$\tilde{W}_{\mu\nu}(x)=W_1(R)\tilde{\gamma}_\nu\tilde{\gamma}_\rho\hat{x}_\rho\tilde{\gamma}_\mu+W_0(R)\tilde{\gamma}_\rho\hat{x}_\rho\hat{x}_\mu\hat{x}_\nu, \tag{2.8}$$

$$\tilde{M}_{\mu\nu}(x)=M_1(R)\tilde{\gamma}_\nu\tilde{\gamma}_\mu, \tag{2.9}$$

where $\hat{x}_\mu=x_\mu/R$, $R=(x_\mu x_\mu)^{1/2}$, and

$$W_{1,0}(R)=-\frac{1}{(4\pi)^2}\int dm^2\rho_{1,0}(m^2)\frac{dV(R;m^2)}{dR}, \tag{2.10}$$

$$M_1(R)=\frac{1}{(4\pi)^2}\int dm^2 m\sigma(m^2)V(R;m^2). \tag{2.11}$$

The function $(4\pi)^{-2}V(R;m^2)$ is the Wick-rotated free scalar propagator which involves the first-order modified Bessel function K_1 as given by

$$V(R;m^2)=\frac{4m}{R}K_1(mR). \tag{2.12}$$

The short-distance behavior of the interaction kernel $\tilde{I}_{\mu\nu}$ can be derived by using the well-known expansion

$$V(R;m^2)=4R^{-2}+2m^2\ln R+\cdots \quad \text{for } R\to 0. \tag{2.13}$$

Equations (2.10) and (2.11) yield

$$W_{1,0}(R)=w_{1,0}R^{-3}+\cdots \quad \text{for } R\to 0, \tag{2.14}$$

$$M_1(R)=\tilde{\kappa}R^{-2}+\cdots \quad \text{for } R\to 0, \tag{2.15}$$

with

$$w_{1,0}=\frac{1}{2\pi^2}\int dm^2\rho_{1,0}(m^2), \tag{2.16}$$

$$\tilde{\kappa}=\frac{1}{4\pi^2}\int dm^2 m\sigma(m^2). \tag{2.17}$$

We observe that the interaction kernel $\tilde{I}_{\mu\nu}$ of the third-order BS equation (2.6) is marginally singular at $R\to 0$ if the integrals (2.16) are different from zero. In this case a proper treatment of the boundary conditions is necessary since the short-distance behavior of the wave function depends critically on the coefficients w_1 and w_0.

In the subsequent part of this paper we restrict ourselves to a study of the BS equation (2.6) for $E\to 0$. In this limit one can readily verify that the BS equation is form-invariant under the transformations of the four-dimensional rotation group O(4) extended by three-space reflections Π_3.

III. CONSTRUCTION OF THE BASIS FUNCTIONS

A. Spinor spherical harmonics

For completeness, we next summarize some well-known definitions. The components of the three-dimensional spinor spherical harmonics are

$$Y_{(JLM)1}(\vartheta,\varphi)=C(L\tfrac{1}{2}J;M-\tfrac{1}{2},\tfrac{1}{2})\,Y_{L,M-1/2}(\vartheta,\varphi),$$

$$Y_{(JLM)2}(\vartheta,\varphi)=C(L\tfrac{1}{2}J;M+\tfrac{1}{2},-\tfrac{1}{2})\,Y_{L,M+1/2}(\vartheta,\varphi).$$

Here J is the total angular momentum, $Y_{Lm}(L=J\pm\frac{1}{2})$ denotes the three-dimensional scalar spherical harmonics, and the Clebsch-Gordan coefficients C are listed e.g. by Rose.[18] In addition, we have

$$x_1=r\sin\vartheta\sin\varphi,\quad x_2=r\sin\vartheta\cos\varphi,$$

$$x_3=r\cos\vartheta,\quad r=(x_1{}^2+x_2{}^2+x_3{}^2)^{1/2}.$$

The four-dimensional spinor spherical harmonics[13] $Y_{(NJLM)\beta}$ $(\beta=1,2)$ involve the Gegenbauer polynomials[19] C_n^ν as given by

$$Y_{(NJLM)\beta}(\Omega)=G_N^{(L)}(\theta)Y_{(JLM)\beta}(\vartheta,\varphi),\quad N\geq L$$

$$G_N^{(L)}(\theta)=\left[\frac{2^{2L+1}(N+1)(N-L)!}{\pi(N+L+1)!}\right]^{1/2}\times L!\,C_{N-L}^{L+1}(\cos\theta)\sin^L\theta,$$

where $\cos\theta=x_4/R$, and Ω indicates the angular variables θ,ϑ,φ.

Let us turn to the normalized basis functions $Z^{(\pm)}_{(NJM)}$ and $\tilde{Z}^{(\pm)}_{(NJM)}$ that belong, respectively, to the irreducible representations $(\frac{1}{2}N\pm\frac{1}{2},\frac{1}{2}N)$ and $(\frac{1}{2}N,\frac{1}{2}N\pm\frac{1}{2})$ of O(4). These functions have been constructed and discussed by Rothe,[13] and the final result can be written as

$$Z^{(\pm)}_{(NJM)\beta}(\Omega)=(2N+2)^{-1/2}[\pm(N+J+\tfrac{3}{2})^{1/2}i^{J\mp1/2}Y_{(NJ,J\mp1/2,M)\beta}(\Omega)+(N-J+\tfrac{1}{2})^{1/2}i^{J\pm1/2}Y_{(NJ,J\pm1/2,M)\beta}(\Omega)], \tag{3.1}$$

$$\tilde{Z}^{(\pm)}_{(NJM)\beta}(\Omega)=(2N+2)^{-1/2}[(N+J+\tfrac{3}{2})^{1/2}i^{J\mp1/2}Y_{(NJ,J\mp1/2,M)\beta}(\Omega)\mp(N-J+\tfrac{1}{2})^{1/2}i^{J\pm1/2}Y_{(NJ,J\pm1/2,M)\beta}(\Omega)]. \tag{3.2}$$

The functions $Z^{(\pm)}$ and $\tilde{Z}^{(\pm)}$ are related by three-space inversion I_3:

$$I_3\tilde{Z}^{(\pm)}_{(NJM)\beta}(\Omega)=(-1)^{J-1/2}Z^{(\pm)}_{(NJM)\beta}(\Omega), \tag{3.3}$$

$$I_3:\ (x_j,x_4)\to(-x_j,x_4)\quad (j=1,2,3).$$

By using the Weyl representation of the $\tilde{\gamma}$ matrices,

$$\tilde{\gamma}_j=\begin{bmatrix}0 & i\sigma_j\\ -i\sigma_j & 0\end{bmatrix},\quad \tilde{\gamma}_4=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \tag{3.4}$$

we now define the four-component spinor spherical harmonics $Y^{(\pm)}_{(NJM)\alpha}(\Omega)$ $(\alpha=1,2,3,4)$ as

$$Y^{(\pm)}_{(NJM)\beta}(\Omega)=2^{-1/2}Z^{(\pm)}_{(NJM)\beta}(\Omega), \tag{3.5}$$

$$Y^{(\pm)}_{(NJM)2+\beta}(\Omega)=2^{-1/2}\tilde{Z}^{(\pm)}_{(NJM)\beta}(\Omega), \tag{3.6}$$

where $N\geq J\mp\frac{1}{2}$. (The spinor index α will be omitted later on.) According to Eq. (3.3), the spherical harmonics $Y^{(\pm)}_{(NJM)}$ are eigenfunctions of the three-space reflection $\Pi_3=\tilde{\gamma}_4 I_3$:

$$\Pi_3 Y^{(\pm)}_{(NJM)}(\Omega)=(-1)^{J-1/2}Y^{(\pm)}_{(NJM)}(\Omega). \tag{3.7}$$

In addition, we have

$$I_4 Y^{(\pm)}_{(NJM)}(\Omega)=(-1)^N Y^{(\pm)}_{(NJM)}(\Omega), \tag{3.8}$$

$$I_4:\ x_\mu\to -x_\mu.$$

The orthonormality properties are given by

$$\int d\Omega\, Y^{(\pm)*}_{(NJM)}(\Omega)Y^{(\pm)}_{(N'J'M')}(\Omega)=\delta_{NN'}\delta_{JJ'}\delta_{MM'}, \tag{3.9a}$$

$$\int d\Omega\, Y^{(\pm)*}_{(NJM)}(\Omega)Y^{(\mp)}_{(N'J'M')}(\Omega)=0, \tag{3.9b}$$

TABLE I. List of the operators d.

$d^{(0)}(\delta;N)=1$
$d^{(1)}(\delta;N)=d(\delta;N,R) \quad (\delta=0,\pm1,\pm2)$
$d(-1;N,R)=\frac{d}{dR}-\frac{N}{R}$
$d(1;N,R)=\frac{d}{dR}+\frac{N+2}{R}$
$d(-2;N,R)=\frac{d^2}{dR^2}-\frac{2N+1}{R}\frac{d}{dR}+\frac{N(N+2)}{R^2}$
$d(0;N,R)=\frac{d^2}{dR^2}+\frac{3}{R}\frac{d}{dR}-\frac{N(N+2)}{R^2}$
$d(2;N,R)=\frac{d^2}{dR^2}+\frac{2N+3}{R}\frac{d}{dR}+\frac{N(N+2)}{R^2}$
$d^\dagger(-2;N,R)=d(2;N+2,R)$
$d^\dagger(0;N,R)=d(0;N,R)$
$d_3^{(0)}(\delta_3;N)=1$
$d_3^{(1)}(\delta_3;N)=d_3(\delta_3;N,R) \quad (\delta_3=\pm1,\pm3)$
$d_3(-3;N,R)=d(-1;N+2,R)\,d(-1;N+1,R)\,d(-1;N,R)$
$d_3(-1;N,R)=d(-1;N,R)\,d(1;N+1,R)\,d(-1;N,R)$
$d_3(1;N,R)=d(1;N,R)\,d(-1;N-1,R)\,d(1;N,R)$
$d_3(3;N,R)=d(1;N-2,R)\,d(1;N-1,R)\,d(1;N,R)$
$d_3^\dagger(-3;N,R)=-d_3(3;N+3,R)$
$d_3^\dagger(-1;N,R)=-d_3(1;N+1,R)$

with

$$\int d\Omega=\int_0^\pi d\theta\,\sin^2\theta\int_0^\pi d\vartheta\,\sin\vartheta\int_0^{2\pi}d\varphi\,. \tag{3.9c}$$

In the subsequent calculations we shall make use of the relations[13]

$$\tilde\gamma_\mu\hat x_\mu Y^{(\pm)}_{(NJM)}(\Omega)=Y^{(\mp)}_{(N\pm1,JM)}(\Omega)\,, \tag{3.10}$$

$$\tilde\gamma_\mu\partial_\mu\phi(R)Y^{(\pm)}_{(NJM)}(\Omega)=Y^{(\mp)}_{(N\pm1,JM)}(\Omega)\times d(\mp1;N,R)\phi(R)\,, \tag{3.11}$$

$$\Box\phi(R)Y^{(\pm)}_{(NJM)}(\Omega)=Y^{(\pm)}_{(NJM)}(\Omega)\times d(0;N,R)\phi(R)\,, \tag{3.12}$$

where $\Box=\partial_\mu\partial_\mu$, and the operators d are included in Table I. Partial integration yields

$$\int_0^\infty dR\,R^3 f_2^*(R)d(-1;N,R)f_1(R)=-\int_0^\infty dR\,R^3 f_1(R)d(1;N+1,R)f_2^*(R)$$

provided that the contributions at $R\to0$ and $R\to\infty$ vanish. It will be convenient to introduce the notation

$$d^\dagger(1;N+1,R)=-d(-1;N,R)\,. \tag{3.13}$$

B. Spinor-vector spherical harmonics

We now construct the spinor-vector spherical harmonics that are derivable from the functions $Y^{(\pm)}_{(NJM)}$ by differentiation or multiplication with the unit vector $\hat x_\mu$:

$$\hat Y^{(1|\pm)\prime}_{(NJM)\mu}(\Omega)=\hat x_\mu Y^{(\pm)}_{(NJM)}(\Omega)\,, \tag{3.14}$$

$$\hat Y^{(2|\pm)\prime}_{(NJM)\mu}(\Omega)=[N(N+2)]^{-1/2}R\partial_\mu Y^{(\pm)}_{(NJM)}(\Omega)\,. \tag{3.15}$$

The other basis functions of this sector can be constructed by multiplication with $\tilde\gamma_\mu$. To obtain an orthonormal set of spherical harmonics, we introduce

$$\hat Y^{(0|\pm)\prime}_{(NJM)\mu}(\Omega)=\left[\frac{N+1\pm1}{2(N+1\pm2)}\right]^{1/2}\left\{\tilde\gamma_\mu Y^{(\mp)}_{(N\pm1,JM)}(\Omega)-\hat Y^{(1|\pm)\prime}_{(NJM)\mu}(\Omega)+\frac{[N(N+2)]^{1/2}}{\pm(N+1)+1}\hat Y^{(2|\pm)\prime}_{(NJM)\mu}(\Omega)\right\}. \tag{3.16}$$

The normalized harmonics $\hat Y^{(m|\pm)\prime}_{(NJM)\mu}$ $(m=0,1,2)$ are defined for the quantum numbers $N=N_d$ that satisfy the following requirements: (i) $N_d\ge J\mp\frac12$. (ii) The normalization factors [see Eqs. (3.15) and (3.16)] must be finite at $N=N_d$. Let us apply the relations (3.9)–(3.12) to derive

$$\int d\Omega\,\hat Y^{(m|\pm)\prime *}_{(NJM)\mu}(\Omega)\hat Y^{(m'|\pm)\prime}_{(N'J'M')\mu}(\Omega)=\delta_{mm'}\delta_{NN'}\delta_{JJ'}\delta_{MM'}\,, \tag{3.17a}$$

$$\int d\Omega\,\hat Y^{(m|\pm)\prime *}_{(NJM)\mu}(\Omega)\hat Y^{(m'|\mp)\prime}_{(N'J'M')\mu}(\Omega)=0\,. \tag{3.17b}$$

In addition, we prescribe

$$\hat Y^{(m|\pm)\prime}_{(NJM)\mu}(\Omega)=0 \quad\text{for } N\ne N_d\,. \tag{3.18}$$

The O(4) expansions will take a compact form by using the four-dimensional spinor-vector spherical harmonics $\hat Y^{(m|\pm)}_{(NJM)\mu}$ as defined by the orthogonal transformation

$$\hat Y^{(m|\pm)}_{(NJM)\mu}(\Omega)=\sum_{n=0}^{2}W^T_{mn}(N)\hat Y^{(n|\pm)\prime}_{(NJM)\mu}(\Omega)\ (m=0,1,2), \tag{3.19}$$

where

$$W^T(N)=\begin{bmatrix}1 & 0 & 0\\ 0 & \left[\frac{N}{2(N+1)}\right]^{1/2} & \left[\frac{N+2}{2(N+1)}\right]^{1/2}\\ 0 & -\left[\frac{N+2}{2(N+1)}\right]^{1/2} & \left[\frac{N}{2(N+1)}\right]^{1/2}\end{bmatrix}. \tag{3.20}$$

Space-time inversion yields, according to Eq. (3.8),

$$I_4\hat{Y}^{(m|\pm)}_{(NJM)\mu}(\Omega)=(-1)^{N+1}\hat{Y}^{(m|\pm)}_{(NJM)\mu}(\Omega). \quad (3.21)$$

C. Relations

Before treating the separation of angular variables, we derive some useful relations involving the spherical harmonics (3.19). (To simplify the formalism, we shall suppress the quantum numbers J and M.) Let us first evaluate the products $\mathfrak{X}_\mu\hat{Y}^{(m|\pm)}_{(N)\mu}$, $\partial_\mu\phi(R)\hat{Y}^{(m|\pm)}_{(N)\mu}$, and $\tilde{\gamma}_\mu\hat{Y}^{(m|\pm)}_{(N)\mu}$. It will be convenient to define

$$\hat{Q}^{(0)}_\mu=\mathfrak{X}_\mu,\quad \hat{Q}^{(1)}_\mu=\partial_\mu. \quad (3.22)$$

By using Eqs. (3.10)–(3.12) and the notations of Table I, one obtains ($a=0,1$)

$$\hat{Q}^{(a)}_\mu\phi(R)\hat{Y}^{(0|\pm)}_{(N)\mu}(\Omega)=0, \quad (3.23)$$

$$\hat{Q}^{(a)}_\mu\phi(R)\hat{Y}^{(1|\pm)}_{(N)\mu}(\Omega)=Y^{(\pm)}_{(N)}(\Omega)\left[\frac{N}{2(N+1)}\right]^{1/2}d^{(a)}(-1;N-1)\phi(R), \quad (3.24)$$

$$\hat{Q}^{(a)}_\mu\phi(R)\hat{Y}^{(2|\pm)}_{(N)\mu}(\Omega)=-Y^{(\pm)}_{(N)}(\Omega)\left[\frac{N+2}{2(N+1)}\right]^{1/2}d^{(a)}(1;N+1)\phi(R), \quad (3.25)$$

and

$$\tilde{\gamma}_\mu\hat{Y}^{(0|\pm)}_{(N)\mu}(\Omega)=\left[\frac{2(N+1\pm2)}{N+1\pm1}\right]^{1/2}Y^{(\mp)}_{(N\pm1)}(\Omega), \quad (3.26)$$

$$\tilde{\gamma}_\mu\hat{Y}^{(1|\pm)}_{(N)\mu}(\Omega)=(1\mp1)\left(\frac{N+1}{2N}\right)^{1/2}Y^{(\mp)}_{(N\pm1)}(\Omega), \quad (3.27)$$

$$\tilde{\gamma}_\mu\hat{Y}^{(2|\pm)}_{(N)\mu}(\Omega)=(-1\mp1)\left[\frac{N+1}{2(N+2)}\right]^{1/2}Y^{(\mp)}_{(N\pm1)}(\Omega). \quad (3.28)$$

We now turn to the operator $\Box$. Straightforward calculation yields

$$\Box\phi(R)\hat{Y}^{(m|\pm)}_{(N)\mu}(\Omega)=\hat{Y}^{(n|\pm)}_{(N)\mu}(\Omega)F^{(1|\pm)}_{nm}(N)\phi(R). \quad (3.29)$$

Here, and in the following part of this paper, the repeated index n implies a summation over n from 0 to 2. The matrix elements $F^{(1|\pm)}_{nm}$ can be written as

$$F^{(1|\pm)}_{00}(N)=d(0;N\pm1,R), \quad (3.30)$$

$$F^{(1|\pm)}_{11}(N)=d(0;N-1,R), \quad (3.31)$$

$$F^{(1|\pm)}_{22}(N)=d(0;N+1,R), \quad (3.32)$$

$$F^{(1|\pm)}_{mn}(N)=0\quad\text{for } m\neq n. \quad (3.33)$$

We need similar relations for all the operators occurring in the BS equation (2.6) ($E\to0$). By combining Eqs. (3.10)–(3.12) and (3.22)–(3.29), we derive, after some algebra,

$$\tilde{\gamma}_\rho\partial_\rho\phi(R)\hat{Y}^{(m|\pm)}_{(N)\mu}(\Omega)=\hat{Y}^{(n|\mp)}_{(N\pm1)\mu}(\Omega)D^{(1|\pm)}_{nm}(N)\phi(R), \quad (3.34)$$

$$\hat{Q}^{(a)}_\mu\hat{Q}^{(a)}_\nu\phi(R)\hat{Y}^{(m|\pm)}_{(N)\nu}(\Omega)=\hat{Y}^{(n|\pm)}_{(N)\mu}(\Omega)G^{(a)}_{nm}(N)\phi(R), \quad (3.35)$$

$$\tilde{\gamma}_\nu\tilde{\gamma}_\mu\hat{Y}^{(m|\pm)}_{(N)\nu}(\Omega)=\hat{Y}^{(n|\pm)}_{(N)\mu}(\Omega)S^{(\pm)}_{nm}(N), \quad (3.36)$$

$$\tilde{\gamma}_\rho\hat{Q}^{(a)}_\rho\hat{Q}^{(a)}_\sigma\hat{Q}^{(a)}_\sigma\phi(R)\hat{Y}^{(m|\pm)}_{(N)\mu}(\Omega)=\hat{Y}^{(n|\mp)}_{(N\pm1)\mu}(\Omega)\hat{F}^{(a|\pm)}_{nm}(N)\phi(R), \quad (3.37)$$

$$\tilde{\gamma}_\rho\hat{Q}^{(a)}_\rho\hat{Q}^{(a)}_\mu\hat{Q}^{(a)}_\nu\phi(R)\hat{Y}^{(m|\pm)}_{(N)\nu}(\Omega)=\hat{Y}^{(n|\mp)}_{(N\pm1)\mu}(\Omega)\hat{G}^{(a|\pm)}_{nm}(N)\phi(R), \quad (3.38)$$

$$\tilde{\gamma}_\nu\tilde{\gamma}_\rho\mathfrak{X}_\rho\tilde{\gamma}_\mu\hat{Y}^{(m|\pm)}_{(N)\nu}(\Omega)=\hat{Y}^{(n|\mp)}_{(N\pm1)\mu}(\Omega)\hat{K}^{(\pm)}_{nm}, \quad (3.39)$$

where the matrices $D^{(1|\pm)}$, $G^{(a)}$, $S^{(\pm)}$ and $\hat{F}^{(a|\pm)}$, $\hat{G}^{(a|\pm)}$, $\hat{K}^{(\pm)}$ are listed in Table II and Table III, respectively. Notice that the matrices with upper index $a=1$ involve the operators d and d_3 defined in Table I. Both Table II and Table III include expressions of the type $[\hat{F}^{(1|+)}_{nm}(N-1)]^\dagger$, which can be evaluated by using the definition of the operators $d^\dagger$ and $d^\dagger_3$ [see Eq. (3.13) and Table I]. For example, we have

$$[\hat{F}^{(1|+)}_{20}(N)]^\dagger=-\frac{1}{N+2}d^\dagger_3(-1;N+1,R)=\frac{1}{N+2}d_3(1;N+2,R).$$

IV. SEPARATION OF THE ANGULAR VARIABLES

Our starting point is the expansion of the BS wave function $\psi_\nu(x)$ in terms of the four-dimensional spinor-vector spherical harmonics (3.19),

$$\psi_\nu(x)=\psi^{(+)}_{(N)\nu}(x)+\psi^{(-)}_{(N)\nu}(x), \quad (4.1)$$

$$\psi^{(+)}_{(N)\nu}(x)=f^{(+)}_{(N)n}(R)\hat{Y}^{(n|+)}_{(N)\nu}(\Omega), \quad (4.2)$$

$$\psi^{(-)}_{(N)\nu}(x)=f^{(-)}_{(N)n}(R)\hat{Y}^{(n|-)}_{(N+1)\nu}(\Omega), \quad (4.3)$$

where $f^{(+)}_{(N)n}$ and $f^{(-)}_{(N)n}$ are radial wave functions. One can also consider the BS wave function $\psi^{(5)}_\nu(x)$ which is given by

$$\psi^{(5)}_\nu(x)=\tilde{\gamma}_5\psi_\nu(-x),\quad \tilde{\gamma}_5=\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_4 \quad (4.4)$$

with Eqs. (4.1)–(4.3). Taking into account that the normalized harmonics $\hat{Y}^{(1|+)}_{(N)\nu}$ and $\hat{Y}^{(0|-)}_{(N+1)\nu}$ are defined only for $N>0$, we shall prescribe

$$f^{(+)}_{(0)1}(R)=0,\quad f^{(-)}_{(0)0}(R)=0. \quad (4.5)$$

We now are in a position to derive the radial BS equation of the functions $f^{(\pm)}_{(N)n}$ by substituting the expansion (4.1)–(4.3) into the Wick-rotated BS equation (2.6) at $E\to0$; the separation of the angular variables can be carried out with the help of the relations (3.21) and (3.29)–(3.39). The final

TABLE II. The matrix elements $D^{(1|\pm)}_{mn}$, $G^{(a)}_{mn}$, and $S^{(\pm)}_{mn}$.

m,n	$D^{(1\|+)}_{mn}(N)$	$G^{(a)}_{mn}(N)$	$S^{(\pm)}_{mn}(N)$
0, 0	$-\left[\frac{N(N+3)}{(N+1)(N+2)}\right]^{1/2} d^{(1)}(1;N+1)$	0	$\frac{2}{-1\mp(N+1)}$
0, 1	$-\frac{1}{N+1}\, d^{(1)}(-1;N-1)$	0	$(-1\pm1)\frac{[(N-1)(N+1)]^{1/2}}{N}$
0, 2	$-\left(\frac{N}{N+2}\right)^{1/2}\frac{1}{N+1}\, d^{(1)}(1;N+1)$	0	$(1\pm1)\frac{[(N+1)(N+3)]^{1/2}}{N+2}$
1, 0	$-\left(\frac{N+3}{N+1}\right)^{1/2}\frac{1}{N+2}\, d^{(1)}(1;N+1)$	0	$(-1\pm1)\frac{[(N-1)(N+1)]^{1/2}}{N}$
1, 1	$\frac{[N(N+2)]^{1/2}}{N+1}\, d^{(1)}(-1;N-1)$	$\frac{N}{2(N+1)}\, d^{(a)}(0;N-1)$	$2-(1\mp1)\frac{N+1}{N}$
1, 2	$-\frac{1}{(N+1)(N+2)}\, d^{(1)}(1;N+1)$	$-\frac{[N(N+2)]^{1/2}}{2(N+1)}\, d^{(a)}(2;N+1)$	0
2, 0	$-\frac{1}{N+2}\, d^{(1)}(-1;N+1)$	0	$(1\pm1)\frac{[(N+1)(N+3)]^{1/2}}{N+2}$
2, 1	0	$-\frac{[N(N+2)]^{1/2}}{2(N+1)}\, d^{(a)}(-2;N-1)$	0
2, 2	$\frac{[(N+1)(N+3)]^{1/2}}{N+2}\, d^{(1)}(-1;N+1)$	$\frac{N+2}{2(N+1)}\, d^{(a)}(0;N+1)$	$2-(1\pm1)\frac{N+1}{N+2}$
	$D^{(1\|-)}_{mn}(N+1) = -[D^{(1\|+)}_{nm}(N)]^\dagger$		

TABLE III. The matrix elements $\hat{F}^{(a|\pm)}_{mn}$, $\hat{G}^{(a|\pm)}_{mn}$, and $\hat{K}^{(\pm)}_{mn}$.

m,n	$\hat{F}^{(a\|+)}_{mn}(N)$	$\hat{G}^{(a\|+)}_{mn}(N)$	$\hat{K}^{(+)}_{mn}$
0, 0	$-\left[\frac{N(N+3)}{(N+1)(N+2)}\right]^{1/2} d_3^{(a)}(1;N+1)$	0	0
0, 1	$-\frac{1}{N+1} d_3^{(a)}(-1;N-1)$	0	2
0, 2	$-\left(\frac{N}{N+2}\right)^{1/2}\frac{1}{N+1}\, d_3^{(a)}(1;N+1)$	0	0
1, 0	$-\left(\frac{N+3}{N+1}\right)^{1/2}\frac{1}{N+2} d_3^{(a)}(1;N+1)$	0	0
1, 1	$\frac{[N(N+2)]^{1/2}}{N+1}\, d_3^{(a)}(-1;N-1)$	$\frac{1}{2}\left(\frac{N}{N+2}\right)^{1/2} d_3^{(a)}(-1;N-1)$	0
1, 2	$-\frac{1}{(N+1)(N+2)}\, d_3^{(a)}(1;N+1)$	$-\frac{1}{2} d_3^{(a)}(1;N+1)$	-2
2, 0	$-\frac{1}{N+2}\, d_3^{(a)}(-1;N+1)$	0	-2
2, 1	0	$-\frac{1}{2}\left[\frac{N(N+3)}{(N+1)(N+2)}\right]^{1/2} d_3^{(a)}(-3;N-1)$	0
2, 2	$\frac{[(N+1)(N+3)]^{1/2}}{N+2}\, d_3^{(a)}(-1;N+1)$	$\frac{1}{2}\left(\frac{N+3}{N+1}\right)^{1/2} d_3^{(a)}(-1;N+1)$	0
	$\hat{F}^{(0\|-)}_{mn}(N+1) = \hat{F}^{(0\|+)}_{nm}(N)$	$\hat{G}^{(0\|-)}_{mn}(N+1) = \hat{G}^{(0\|+)}_{nm}(N)$	$\hat{K}^{(-)}_{mn} = \hat{K}^{(+)}_{nm}$
	$\hat{F}^{(1\|-)}_{mn}(N+1) = -[\hat{F}^{(1\|+)}_{nm}(N)]^\dagger$	$\hat{G}^{(1\|-)}_{mn}(N+1) = -[\hat{G}^{(1\|+)}_{nm}(N)]^\dagger$	

result may be written in the form

$$\hat{B}^{(++)}_{mn}(N)f^{(+)}_{(N)n}(R)+\hat{B}^{(+-)}_{mn}(N)f^{(-)}_{(N)n}(R)=0, \quad (4.6)$$

$$m=0,1,2 \text{ for } N>0, \quad m=0,2 \text{ for } N=0$$

and

$$\hat{B}^{(-+)}_{mn}(N)f^{(+)}_{(N)n}(R)+\hat{B}^{(--)}_{mn}(N)f^{(-)}_{(N)n}(R)=0, \quad (4.7)$$

$$m=0,1,2 \text{ for } N>0, \quad m=1,2 \text{ for } N=0,$$

where

$$\hat{B}^{(++)}_{mn}(N)=\kappa[F^{(1|+)}_{mn}(N)-cG^{(1)}_{mn}(N)-m_V{}^2\delta_{mn}] + (-1)^N M_1(R)S^{(+)}_{mn}(N), \quad (4.8)$$

$$\hat{B}^{(+-)}_{mn}(N)=\hat{F}^{(1|-)}_{mn}(N+1)-c\hat{G}^{(1|-)}_{mn}(N+1)-m_V{}^2D^{(1|-)}_{mn}(N+1) -(-1)^N W_1(R)\hat{K}^{(-)}_{mn}-(-1)^N W_0(R)\hat{G}^{(0|-)}_{mn}(N+1), \quad (4.9)$$

$$\hat{B}^{(-+)}_{mn}(N)=-\hat{F}^{(1|+)}_{mn}(N)+c\hat{G}^{(1|+)}_{mn}(N)+m_V{}^2D^{(1|+)}_{mn}(N) -(-1)^N W_1(R)\hat{K}^{(+)}_{mn}-(-1)^N W_0(R)\hat{G}^{(0|+)}_{mn}(N), \quad (4.10)$$

$$\hat{B}^{(--)}_{mn}(N)=\kappa[-F^{(1|-)}_{mn}(N+1)+cG^{(1)}_{mn}(N+1)+m_V{}^2\delta_{mn}] +(-1)^N M_1(R)S^{(-)}_{mn}(N+1). \quad (4.11)$$

Notice that the matrix elements $F^{(1|\pm)}_{mn}$, $G^{(1)}_{mn}, \ldots$ are explicitly given in Eqs. (3.30)–(3.33) and Tables I–III. In the real region, these formulas yield

$$\hat{B}^{(++)}_{mn}(N)=[\hat{B}^{(++)}_{nm}(N)]^\dagger, \quad (4.12)$$

$$\hat{B}^{(-+)}_{mn}(N)=[\hat{B}^{(+-)}_{nm}(N)]^\dagger, \quad (4.13)$$

$$\hat{B}^{(--)}_{mn}(N)=[\hat{B}^{(--)}_{nm}(N)]^\dagger. \quad (4.14)$$

One may require that the formal relations (4.12)–(4.14) must lead to a manifestly Hermitian operator $\hat{B}$; however, this condition implies a restriction on the choice of the parameters w_1, w_0, and c (cf. Sec. V).

In summary, the radial BS wave functions satisfy a system of ordinary differential equations of third order. We have fixed the short-distance behavior of the interaction kernel by Eqs. (2.14) and (2.15), thus; according to the standard theory of linear differential equations, there exist power-like asymptotic solutions at $R\to 0$,

$$f^{(\pm)}_{(N)n}(R)=a^{(\pm)}_{(N)n}R^\rho+\cdots \text{ for } R\to 0, \quad (4.15)$$

where, according to the prescriptions (4.4) and (4.5),

$$a^{(+)}_{(0)1}=0, \quad a^{(-)}_{(0)0}=0. \quad (4.16)$$

Inserting the leading terms (2.14), (2.15), and (4.15) into the radial BS equations, we obtain the system of homogeneous linear equations for the coefficients $a^{(\pm)}_{(N)n}$

$$\tilde{B}^{(+-)}_{mn}(N,\rho)a^{(-)}_{(N)n}=0 \quad (m=0,2 \text{ for } N=0), \quad (4.17)$$

$$\tilde{B}^{(-+)}_{mn}(N,\rho)a^{(+)}_{(N)n}=0 \quad (m=1,2 \text{ for } N=0), \quad (4.18)$$

where $m=0,1,2$ for $N\neq 0$, and

$$\tilde{B}^{(+-)}_{mn}(N,\rho)=\hat{F}^{(2|-)}_{mn}(N+1)-c\hat{G}^{(2|-)}_{mn}(N+1) -(-1)^N w_1\hat{K}^{(-)}_{mn}-(-1)^N w_0\hat{G}^{(0|-)}_{mn}(N+1), \quad (4.19)$$

$$\tilde{B}^{(-+)}_{mn}(N,\rho)=-\hat{F}^{(2|+)}_{mn}(N)+c\hat{G}^{(2|+)}_{mn}(N) -(-1)^N w_1\hat{K}^{(+)}_{mn}-(-1)^N w_0\hat{G}^{(0|+)}_{mn}(N). \quad (4.20)$$

Here $\hat{F}^{(2|\pm)}_{mn}$ and $\hat{G}^{(2|\pm)}_{mn}$ can be obtained from the matrix elements $\hat{F}^{(a|+)}_{mn}$ and $\hat{G}^{(a|+)}_{mn}$ (see Table III) by the formal substitutions $a=2$ and

$$d^{(2)}_3(-3;N)=(\rho-N-4)(\rho-N-2)(\rho-N), \quad (4.21)$$

$$d^{(2)}_3(-1;N)=(\rho-N-2)(\rho-N)(\rho+N+2), \quad (4.22)$$

$$d^{(2)}_3(1;N)=(\rho-N)(\rho+N)(\rho+N+2), \quad (4.23)$$

$$d^{(2)}_3(3;N)=(\rho+N-2)(\rho+N)(\rho+N+2), \quad (4.24)$$

$$\hat{F}^{(2|-)}_{mn}(N+1)=-[\hat{F}^{(2|+)}_{nm}(N)]^\dagger, \quad (4.25)$$

$$\hat{G}^{(2|-)}_{mn}(N+1)=-[\hat{G}^{(2|+)}_{nm}(N)]^\dagger, \quad (4.26)$$

$$d^{(2)\dagger}_3(-3;N)=-d^{(2)}_3(3;N+3), \quad (4.27)$$

$$d^{(2)\dagger}_3(-1;N)=-d^{(2)}_3(1;N+1). \quad (4.28)$$

V. INDICIAL EQUATIONS

A. $N\neq 0$

We begin by investigating Eqs. (4.17) and (4.18) for nonzero values of N. To guarantee nontrivial solutions, the indices ρ have to satisfy the indicial equations

$$\det|\tilde{B}^{(+-)}_{mn}(N,\rho)|=0, \quad (5.1)$$

$$\det|\tilde{B}^{(-+)}_{mn}(N,\rho)|=0. \quad (5.2)$$

Let us first consider regularized interaction kernels for which $w_{1,0}=0$. In this case, the indicial equations can be explicitly solved by rewriting them in the factorized form

$$\det|\tilde{B}^{(+-)}_{mn}(N,\rho)|_{w_{1,0}=0}=A^{(-)}(N)(\rho-N-2)^2(\rho-N)^2(\rho+N)(\rho+N+2)^3(\rho+N+4)=0, \quad (5.3)$$

$$\det|\tilde{B}^{(-+)}_{mn}(N,\rho)|_{w_{1,0}=0}=A^{(+)}(N)(\rho-N-3)(\rho-N-1)^3(\rho-N+1)(\rho+N+1)^2(\rho+N+3)^2=0, \quad (5.4)$$

where

$$A^{(-)}(N)=\det\left|\hat{F}^{(0|-)}_{mn}(N+1)-c\hat{G}^{(0|-)}_{mn}(N+1)\right|, \quad (5.5)$$

$$A^{(+)}(N)=\det\left|-\hat{F}^{(0|+)}_{mn}(N)+c\hat{G}^{(0|+)}_{mn}(N)\right|. \quad (5.6)$$

We now observe that, for $R\to 0$ and $w_{1,0}=0$, the radial BS equations (4.6) and (4.7) have nine independent regular asymptotic solutions and nine irregular ones provided that

$$A^{(\pm)}(N)\neq 0. \quad (5.7)$$

Notice that all the solutions ρ_k ($\rho_k \geq \rho_{k+1}$) of the indicial equations (5.3) and (5.4) are integers.

Let us turn to a brief discussion of the short-distance behavior of the radial wave functions for marginally singular interaction kernels ($w_{1,0}\neq 0$). Equations (4.19) and (4.20) show that, in general, the solutions of the indicial equations depend on the parameters w_1, w_0, and c:

$$\rho_k=\rho_k(N,w_1,w_0,c) \quad (k=1,2,\ldots,18). \quad (5.8)$$

Here we call "good" or "bad" indices which go smoothly into positive (including zero) or negative values, respectively, if one takes the weak-coupling limit $w_{1,0}\to 0$. Neglecting the nonleading terms in the interaction kernel, the 18 independent solutions of the radial BS equations can be expanded in the powerlike series

$$f^{(\pm|k)}_{(N)n}(R)=R^{\rho_k}\sum_{h=0} a^{(\pm|k)}_{(N)nh}R^h, \quad (5.9)$$

provided the differences $\rho_k-\rho_{k'}$ of the indices are nonintegers. Otherwise, according to well-known theorems, the expansion of the solutions may involve logarithmic factors for $k>1$. If $\rho_1-\rho_2=$ integer, for example, then the second solution is given by $b^{(2)}_{(N)}\ln R$ times the powerlike first solution plus the series (5.9) for $k=2$, where the coefficients b and a can be calculated by standard recursion formulas (see, e.g., Ref. 14).

To formulate the boundary conditions at $R\to 0$, we shall follow the procedure of Ref. 8. As a first step, the choice of the parameters w_1, w_0, and c will be restricted by requiring that the nine solutions that involve good indices only ("good" solutions) must be less singular than the other ones. We now may prescribe that, as $R\to 0$, an acceptable solution of the radial BS equations (4.6) and (4.7) must be a linear combination of the nine good solutions.

B. $N=0$

In the particular case $N=0$, the radial BS equations are reduced to four coupled ordinary differential equations of third order. To obtain the short-distance expansion of the solutions, we can apply the standard recursion procedures outlined in Sec. V A. The indices ρ_k are the roots of the indicial equations

$$\Delta^{(-)}(\rho;w_1,w_0)=\begin{vmatrix}\tilde{B}^{(+-)}_{01}(0,\rho) & \tilde{B}^{(+-)}_{02}(0,\rho)\\ \tilde{B}^{(+-)}_{21}(0,\rho) & \tilde{B}^{(+-)}_{22}(0,\rho)\end{vmatrix}=0 \quad (5.10)$$

and

$$\Delta^{(+)}(\rho;w_1,w_0)=\begin{vmatrix}\tilde{B}^{(-+)}_{10}(0,\rho) & \tilde{B}^{(-+)}_{12}(0,\rho)\\ \tilde{B}^{(-+)}_{20}(0,\rho) & \tilde{B}^{(-+)}_{22}(0,\rho)\end{vmatrix}=0, \quad (5.11)$$

where the matrix elements $\tilde{B}$ are given in Eqs. (4.19) and (4.20). In particular, we have

$$\tilde{B}^{(+-)}_{01}(0,\rho)=-\frac{3^{1/2}}{2}(\rho-2)\rho(\rho+2), \quad (5.12)$$

$$\tilde{B}^{(+-)}_{21}(0,\rho)=-\tfrac{1}{2}(1-c)(\rho-2)\rho(\rho+2)+2w_1+\tfrac{1}{2}w_0. \quad (5.13)$$

Proceeding as before, we first consider the solutions for regularized interactions. The indicial equations become

$$\Delta^{(-)}(\rho;0,0)=-(1-c)(\rho-2)^2\rho(\rho+2)^2(\rho+4)=0, \quad (5.14)$$

$$\Delta^{(+)}(\rho;0,0)=-(1-c)(\rho-3)(\rho-1)^2(\rho+1)(\rho+3)^2=0. \quad (5.15)$$

Thus, for $N=0$, $w_{1,0}=0$, and $c\neq 1$, the 12 independent asymptotic solutions of the radial BS equations consist of six regular solutions and six irregular ones in the region $R\to 0$, and the short-distance behavior of the leading regular solution is governed by the sixth index $\rho_6=0$. Substituting $\rho=0$ into Eqs. (4.17) and (4.18) (and denoting the solutions by $a^{(-|6)}_{(0)n}$ and $a^{(+|6)}_{(0)n}$, respectively) one readily verifies

$$a^{(-|6)}_{(0)1}\neq 0, \quad a^{(-|6)}_{(0)2}=0, \quad (5.16)$$

$$a^{(+|6)}_{(0)n}=0. \quad (5.17)$$

We now apply the expansions (4.1)–(4.3) to obtain the following short-distance behavior of the BS wave function:

$$\psi_\nu(x)\big|_{N=0}=a^{(-|6)}_{(0)1}\hat{Y}^{(1|-)}_{(1)\nu}(\Omega)+\cdots \text{ for } R\to 0. \quad (5.18)$$

Taking into account that in the vector-gluon model (1.1) the field equations involve the operator $\gamma^\mu\Psi V_\mu$, it is instructive to consider the function $\tilde{\gamma}_\mu\psi_\mu(x)$ in the limit $R\to 0$. Equations (3.27) and (5.18) yield

$$\tilde{\gamma}_\mu\psi_\mu(x)\big|_{N=0}=2a^{(-|6)}_{(0)1}Y^{(+)}_{(0,1/2,M)}(\Omega)+\cdots \text{ for } R\to 0. \quad (5.19)$$

Let us return to the behavior of the solutions at nonzero coupling parameters $w_{1,0}$. First, we consider the "good" index $\rho_6(0, w_1, w_0, c)$ that goes smoothly into the leading index $\rho_6(0,0,0,c)=0$ by taking the weak-coupling limit $w_{1,0}\to 0$. We observe that, for a particular class of the coupling parameters w_1, w_0, and c, the indicial equation (5.10) yields a negative index ρ_6 characterizing a singular solution of the radial equations (4.6) and (4.7). Such solutions of the fermion-antifermion BS equation have been found by Guth and Soper[5] to correspond to admissible normalizable bound-state wave functions. In addition, it has been shown that the "bad" (abnormal) solutions can be ruled out by means of the normalizability condition proposed by Tiktopoulos[20] if the coupling strength is sufficiently small. Thus, in this way, the normalizability condition is crucial in selecting the admissible solutions of the bound-state problem.

The application of a non-gauge-covariant ladder-type approximation implies that, in general, the root ρ_6 of the indical equation (5.10) is not independent of the gauge parameter c. Callan and Gross[4] have discussed a similar "spurious" gauge dependence in the ladder approximation of the fermion-antifermion wave function which could lead to quite different results for the asymptotic behavior of form factors by changing the gauge.

By a judicious choice of the effective gauge and coupling parameters, a ladder-type model might produce some relevant short-distance properties of the BS wave function calculated in a more realistic higher approximation. However, we have no arguments of this type to select a particular effective gauge parameter c for the fermion–vector-gluon BS equation (2.6).

The ladder model (2.6) provides a study of the spinor-vector BS wave function at a semiphenomenological level. Within the framework of this model, let us restrict the choice of the coupling parameters w by prescribing the gauge independence of the index ρ_6 $(E\to 0)$:

$$\frac{\partial}{\partial c}\rho_6(0, w_1, w_0, c)=0\,. \tag{5.20}$$

Inspection of the explicit form of the indicial equation (5.10) shows that our prescription (5.20) is satisfied if

$$w_0=-4w_1\,. \tag{5.21}$$

On account of this requirement, the indicial equations (5.10) and (5.11) can be written in the following form:

$$\Delta^{(-)}(\rho; w_1, -4w_1)=-(\rho-2)\rho(\rho+2)[(1-c)(\rho-2)(\rho+2)(\rho+4)+(2+c)w_1]=0\,, \tag{5.22}$$

$$\Delta^{(+)}(\rho; w_1, -4w_1)=-(\rho-1)(\rho+1)(\rho+3)[(1-c)(\rho-3)(\rho-1)(\rho+3)-(2+c)w_1]=0\,. \tag{5.23}$$

Let us notice that the fulfillment of the prescription $w_0=-4w_1$ guarantees six gauge-independent indices ($\rho=0$, ±1, ±2, and -3) which survive the unitary gauge limit $c\to 1$, if the interaction kernel is marginally singular ($w_1\neq 0$). Moreover, these indices solve the indicial equations in the nonsingular case $w_0=w_1=0$, $c\neq 1$. At $c=1$ and $w_1\neq 0$, all the negative indices ($\rho=-1,-2,-3$) characterize irregular (abnormal) solutions which can be ruled out by means of the normalizability condition. (The short-distance behavior $\sim R^{-1}$ of a fermion-antifermion BS wave function was first shown by Mandelstam[21] to be unacceptable from the standpoint of the normalizability condition.) On the other hand, the solutions controlled by the indices $\rho=0,1,2$ are obviously compatible with the normalizability condition and, in the present model, we obtain the asymptotic properties (5.18) and (5.19) also for $w_0=-4w_1\neq 0$.

For $c\neq 1$, the 12 solutions of the indicial equations (5.22) and (5.23) include six gauge-dependent indices which are fixed by the roots of two cubic equations. We would like to emphasize that there is a large class of the parameters w_1 ($w_0=-4w_1$) and c, for which three of the gauge-dependent indices are larger than zero and the other ones are involved in non-normalizable singular solutions of the BS equation. Such effective gauge and coupling parameters are particularly suitable for practical calculations because, in this case, we recover the leading short-distance properties of the wave function calculated in the unitary gauge.

VI. COMMENTS

The radial BS equations (4.6) and (4.7) may be regarded as a particular prototype of the description of fermion–vector-gluon systems. So far we only considered a ladder-type approximation where the mass operators of fermions and gluons are replaced by constant external masses. We now look forward to more interesting applications which include the numerical solutions of the BS equation (2.6) for nonzero total c.m. energies. In this case, the four-dimensional rotation symmetry is broken and the O(4) expansions lead to an infinite set of coupled radial equations. On the other hand, one may expect on the basis of some more familiar examples[13] that the short-distance behavior of the wave function is still controlled by the asymptotic solutions of the radial equations at zero c.m. energy.

The scope of this paper should be extended to incorporate a gauge theory based on $U(1)\times SU(2)$. We have in mind a "lepton model" involving a medium-strong coupling to massive $U(1)$ vector gluons. (The basic fermion fields carry unit lepton number.) We note here that, although no higher internal symmetry [e.g. $SU(3)_{\rm color}$] is included in the Lagrangian, the spinor-vector BS equations of the type (2.6) may generate a mass spectrum of the spin-$\frac{1}{2}$ particles F_G that correspond to the solutions $\psi_v(J=\frac{1}{2})$ describing the various radial excitations of a fermion–vector-gluon system.

One of the intriguing possibilities is that the spectrum of the spin-$\frac{1}{2}$ particles may include some particular composite states of the type $F'_G F'_G \overline{F}'_G$ in which each of the excited fermion-gluon systems F'_G $(J=\frac{1}{2})$ must disintegrate in order to undergo a real decay. This triple decay could lead to a "weak" transition in spite of the medium-strong fermion-gluon interaction in the underlying field theory.

The fermions of our model carry no familiar baryon number. In spite of this fact, the strongly suppressed triple decays that are of the type $F'_G F'_G \overline{F}'_G \to$ ground-state fermions might imitate the weak quark $\to$ lepton transitions encountered in the prequark model[22] of Pati, Salam and Strathdee. This possibility suggests a parallel between the radial excitations of the fermion-gluon systems F'_G, on the one hand, and the spin-$\frac{1}{2}$ integer charge prequarks, of which the quarks may be composite, on the other hand.

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