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Zero-point energy of fields in a finite volume

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The zero-point energy is studied in the MIT bag model in which the quantum fields are confined, in a covariant way, to a finite region of space. The calculations are performed in the static boundary approximation. The resulting eigenfrequencies are summed using a cutoff. For the three-dimensional problem, new methods are introduced to compute the zero-point energy and to isolate the divergences occurring when the cutoff is removed. Divergences are found which cannot be absorbed by renormalizing the physical parameters in the bag model Lagrangian, as currently formulated. Alternatives are suggested and analyzed.

I. INTRODUCTION

When fields are confined to a finite volume which can change, the zero-point energy $E_0 \equiv \langle 0|H|0\rangle$ cannot be removed by the simple normal-ordering prescription used in conventional field theories. In an MIT bag model, $L = \int d^3x(\mathcal{L}_c - B)$, where $\mathcal{L}_c$ is any conventional Lagrangian density.[1] Two constraint equations on the surface of V ensure covariance; the boundary points are functions of the fields confined to V and consequently the boundary surface can change during physical processes. In this case, a cutoff must be introduced and the zero-point energy explicitly calculated. Divergences which occur as the cutoff is removed are then absorbed into renormalizations of the physical parameters in the theory.

In this paper we calculate the zero-point energy using a dimensional cutoff on the sum over three-dimensional spherical static cavity modes. The leading divergence, which is proportional to the bag volume, is absorbed into a renormalization of the bag constant B. Calculations are performed in the zero-coupling limit of confined fermions and gluons. An important question is then whether the remaining part of the zero-point energy is finite and proportional to $1/R$, the bag radius.[2] If so, the only divergence is proportional to the bag volume and is insensitive to the boundary conditions. The remaining finite part of the zero-point energy is determined by the long-wavelength part of the spectrum where the static boundary approximation is most reliable. On the other hand, divergences proportional to the surface area or to the radius of the bag indicate that there are contributions to the zero-point energy from nonleading but still large frequencies which are sensitive to our treatment of the boundary.

The first zero-point energy calculation was done by Casimir in 1948, for the electromagnetic field confined by two perfectly conducting plates separated by a distance L.[3] The object was to calculate the (attractive) force/area on the plates resulting from the finite term proportional to $1/L$ in the zero-point energy. In 1958 the existence of this force was verified experimentally and its magnitude was found to be consistent with the Casimir prediction.[4]

Later, Casimir speculated that a similar inward force from the vacuum fluctuations of the electromagnetic field, confined to a three-dimensional conducting spherical shell, balances the outward Coulomb force in the semiclassical Abraham-Lorentz model of the electron as charge distributed over a spherical shell.[5] Many years passed before Boyer did this much more complicated calculation and found that the zero-point energy force on the surface of the sphere was outward.[6] Like Casimir before him, Boyer used a subtraction procedure to isolate the finite part of the zero-point energy proportional to $1/R$ because only this piece contributes to a force on the spherical surface. Furthermore, because of the reliance on numerical methods in this work, the divergent zero-point energy of an isolated spherical cavity is never explicitly calculated.

In Sec. II, we develop a method for calculating the zero-point energy when the boundary is a static three-dimensional sphere. The section concludes with explicit formulas for the cutoff zero-point energy of scalar, fermion, and vector fields being given. Details of their derivation are presented in Appendix A. In Appendix B we pause to apply this new method to two zero-point energy problems in which the bag boundary is not three-dimensional.

The formulas derived in Sec. II are analyzed in Sec. III. We isolate terms which diverge as the cutoff is removed and a numerical check on our analytical technique of doing this is provided. In addition to the quartic divergence found in the parallel-plate problem, an additional quadratic divergence in the zero-point energies of the fermion and vector fields occurs as the cutoff τ tends to zero:

$$E_0(\tau) = VO(1/\tau^4) + RO(1/\tau^2) + \cdots .$$

The quadratic divergence cannot be absorbed into a redefinition of any physical parameter and therefore presents a serious obstacle to using the zero-point energy obtained in the static boundary approximation to the bag model.

In the final section we speculate on possible ways to circumvent this difficulty including (i) an alternative method of calculating the zero-point energy in the static boundary approximation, and (ii) allowing the boundary to make small oscillations about its equilibrium shape.

II. GREEN'S FUNCTION METHOD FOR SUMMING MODES IN A SPHERICAL CAVITY

A. Scalar fields

The zero-point energy for a real scalar field confined to a static spherical bag of radius R is[7]

$$E_0^S = \tfrac{1}{2} \sum_{l=0}^{\infty} \sum_{n>0} (zl+1)\omega_{ln} . \tag{2.1}$$

The eigenfrequencies $\{\omega_{ln}\}$ are determined by $j_l(\omega_{ln}R) = 0$. Only the positive solutions are included in the expression for E_0^S. We introduce an exponential cutoff[8]

$$E_0^S(\tau) \equiv \tfrac{1}{2} \sum_{l=0}^{\infty} \sum_{n>0} (2l+1)\omega_{ln} e^{-\omega_{ln}|\tau|} . \tag{2.2}$$

To proceed we define

$$G^S(\vec{r}, \vec{r}\,', \tau) \equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{\phi_{nlm}(\vec{r})\phi^*_{nlm}(\vec{r}\,')}{2\omega_{ln}} e^{-\omega_{ln}|\tau|} , \tag{2.3}$$

where $\phi_{nlm}(\vec{r}) \equiv N_l(\omega_{ln}) j_l(\omega_{ln} r) Y_{lm}(\Omega)$ and $N_l(\omega_{ln})$, the normalization, is chosen so that $\{\phi_{nlm}(\vec{r})\}$ form a complete orthonormal set of eigenfunctions of the wave equation for the spherical cavity. It follows that

$$\int_V d^3r\, G^S(\vec{r}, \vec{r}, \tau) = \sum_{l,n} \frac{(2l+1)}{2\omega_{ln}} e^{-\omega_{ln}|\tau|} , \tag{2.4}$$

and consequently[9] that

$$E_0(\tau) = \frac{\partial^2}{\partial\tau^2} \int_V d^3r\, G^S(\vec{r}, \vec{r}, \tau) . \tag{2.5}$$

It is also easy to check that

$$\left(\frac{\partial^2}{\partial\tau^2} + \vec{\nabla}^2\right) G^S(\vec{r}, \vec{r}\,', \tau) = -\delta(\tau)\delta^3(\vec{r} - \vec{r}\,') . \tag{2.6}$$

This equation, together with the boundary condition $G^S(\vec{r}, \vec{r}\,', \tau)\big|_{|\vec{r}|=R} = 0$, which automatically follows from the definition (2.3), completely specifies the Green's function, $G^S(\vec{r}, \vec{r}\,', \tau)$. The cutoff zero-point energy can now be calculated according to Eq. (2.5).

The advantage of this procedure is manifest when we separate out an inhomogeneous piece of the Green's function

$$G^S(\vec{r}, \vec{r}\,', \tau) \equiv G_0^S(\vec{r} - \vec{r}\,', \tau) + \Gamma^S(\vec{r}, \vec{r}\,', \tau) \tag{2.7}$$

with

$$G_0^S(\vec{r} - \vec{r}\,', \tau) = \frac{1}{4\pi^2[(\vec{r} - \vec{r}\,')^2 + \tau^2]} , \tag{2.8}$$

satisfying

$$\left(\frac{\partial^2}{\partial\tau^2} + \vec{\nabla}^2\right) G_0^S(\vec{r} - \vec{r}\,', \tau) = -\delta(\tau)\delta^3(\vec{r} - \vec{r}\,') \tag{2.9}$$

and

$$\left(\frac{\partial^2}{\partial\tau^2} + \vec{\nabla}^2\right) \Gamma^S(\vec{r}, \vec{r}\,', \tau) = 0 . \tag{2.10}$$

The inhomogeneous term $G_0^S(\vec{r} - \vec{r}\,', \tau)$ represents the direct propagation of the scalar wave from its source at $(\tau' = 0, \vec{r}\,')$ to the observation point at $(\tau, \vec{r})$. It is the solution to Eq. (2.9) in the absence of boundaries and therefore its contribution to $E_0^S(\tau)$,

$$\frac{\partial^2}{\partial\tau^2} \int_V d^3r\, G_0^S(\vec{r}, \vec{r}, \tau) = \frac{6V}{4\pi^2\tau^4} , \tag{2.11}$$

is independent of the boundary conditions. It can also be argued in general terms that this quartic divergence is stronger than any which may appear in $(\partial^2/\partial\tau^2) \int_V d^3r\, \Gamma^S(\vec{r}, \vec{r}, \tau)$. Consequently $(\partial^2/\partial\tau^2) \times \int_V d^3r\, G_0^S(\vec{r}, \vec{r}, \tau)$ is the only contribution to $E_0^S(\tau)$ proportional to the volume, so an important objective is already accomplished when the redefinition

$$B_{\text{ren}} = B + \frac{6}{4\pi^2\tau^4} \tag{2.12}$$

is made to eliminate this divergence. We may now hope that the remaining calculation $(\partial^2/\partial\tau^2) \times \int_V d^3r\, \Gamma^S(\vec{r}, \vec{r}, \tau)$ is facilitated because

(i) the most divergent piece in $E_0(\tau)$ has been isolated, and

(ii) the method makes no reference to the eigenfrequencies $\{\omega_{ln}\}$, so progress can be made using analytical rather than numerical methods.

The construction of $G^S(\vec{r}, \vec{r}\,', \tau)$ may now be completed by expanding $\Gamma^S(\vec{r}, \vec{r}\,', \tau)$ in terms of the solutions to the four-dimensional Laplace equation in cylindrical coordinates. The boundary condition is then applied and the amplitudes involved in the expansion projected out. The calculation is straightforward and the details are given in Appendix A. The result is

$$\tilde{E}_0^S(\tau) \equiv \frac{\partial^2}{\partial\tau^2}\int_V d^3r\, \Gamma^S(\vec{r},\vec{r},\tau)$$
$$= \frac{R^2}{2\pi}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dk\, k^2 \cos k\tau \frac{K_{l+1/2}(kR)}{I_{l+1/2}(kR)}\left[I_{l+1/2}{}^2(kR) - I_{l+3/2}(kR) I_{l-1/2}(kR)\right].$$

Since $\Gamma(\vec{r},\vec{r}',\tau)$ is a solution to Laplace's equation in V, the only divergences in $\lim_{\tau\to 0}(\partial^2/\partial\tau^2) \times \int_V d^3r\, \Gamma^S(\vec{r},\vec{r},\tau)$ occur when the integration variable $\vec{r}$ approaches the boundary. This suggests an alternative cutoff. Define

$$\tilde{E}_0^S(\epsilon) \equiv \left[\frac{\partial^2}{\partial\tau^2}\int_0^{R(1-\epsilon)} r^2 dr \int d\Omega\, \Gamma^S(\vec{r},\vec{r},\tau)\right]_{\tau=0}. \tag{2.14}$$

This expression is finite even when the cutoff $\tau\to 0$ because the range of integration is now restricted to exclude $|\vec{r}|=R$. Using the ϵ-type cutoff, the contribution of the homogeneous Green's function $\Gamma^S(\vec{r},\vec{r}',\tau)$ to the zero-point energy becomes

$$\tilde{E}_0^S(\epsilon) = \frac{1}{2\pi R}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dx\, x^2 \frac{K_{l+1/2}(x)}{I_{l+1/2}(x)}\left[I_{l+1/2}{}^2(x(1-\epsilon)) - I_{l+3/2}(x(1-\epsilon)) I_{l-1/2}(x(1-\epsilon))\right]. \tag{2.15}$$

It will be seen in Sec. III that Eq. (2.13) is more useful for numerical computation while Eq. (2.15) is more amenable to analysis.

B. Fermion fields

The preceding method is easily extended to fermions where the corresponding matrix Green's function $\mathcal{S}_{\alpha\beta}(\vec{r},\vec{r}',\tau)$ is a solution to

$$\left(\gamma^0\frac{\partial}{\partial\tau} + i\vec{\gamma}\cdot\vec{\nabla}\right)_{\alpha\lambda}\mathcal{S}_{\lambda\beta}(\vec{r},\vec{r}',\tau) = -\delta_{\alpha\beta}\delta(\tau)\delta^3(\vec{r}-\vec{r}') \tag{2.16}$$

with boundary condition

$$(1+i\vec{\gamma}\cdot\hat{r})_{\alpha\lambda}\mathcal{S}_{\lambda\beta}(\vec{r},\vec{r}',\tau)=0 \tag{2.17}$$

on $|\vec{r}|=R$. The eigenfrequencies satisfy $j_l(\omega_{ln}R) = \pm j_{l+1}(\omega_{ln}R)$ for $j=l\pm\frac{1}{2}$. $\mathcal{S}_{\alpha\beta}(\vec{r},\vec{r}',\tau)$ is related to the zero-point energy by

$$E_0^F(\tau) = -\frac{\partial}{\partial\tau}\int_V d^3r\, \mathrm{Tr}[\mathcal{S}(\vec{r},\vec{r},\tau)\gamma^0]. \tag{2.18}$$

As before, the inhomogeneous part can be separated out:

$$\mathcal{S}_{\alpha\beta}(\vec{r},\vec{r}',\tau) \equiv \mathcal{S}^{(0)}_{\alpha\beta}(\vec{r}-\vec{r}',\tau) + \mathcal{K}_{\alpha\beta}(\vec{r},\vec{r}',\tau) \tag{2.19}$$

with

$$\mathcal{S}^{(0)}_{\alpha\beta} \equiv \left(\gamma^0\frac{\partial}{\partial\tau} + i\vec{\gamma}\cdot\vec{\nabla}\right)_{\alpha\beta}\frac{1}{4\pi^2[(\vec{r}-\vec{r}')^2+\tau^2]} \tag{2.20}$$

and

$$\left(\gamma^0\frac{\partial}{\partial\tau} + i\vec{\gamma}\cdot\vec{\nabla}\right)_{\alpha\lambda}\mathcal{K}_{\lambda\beta}(\vec{r},\vec{r}',\tau)=0. \tag{2.21}$$

The contribution of $\mathcal{S}^{(0)}_{\alpha\beta}(\vec{r},\vec{r}',\tau)$ to $E_0^F(\tau)$ is $-6V/4\pi^2\tau^4$ and the arguments concerning the contribution of $G_0^S(\vec{r},\vec{r}',\tau)$ to $E_0^S(\tau)$ apply here as well.

$\mathcal{K}_{\alpha\beta}(\vec{r},\vec{r}',\tau)$ is constructed in terms of the solutions to Eq. (2.21). After satisfying the boundary condition Eq. (2.17) the result (see Appendix A) is

$$\tilde{E}_0^F(\tau) \equiv \frac{\partial}{\partial\tau}\int_V d^3r\, \mathrm{Tr}[\mathcal{K}(\vec{r},\vec{r},\tau)\gamma^0]$$
$$= -\frac{R^2}{4\pi}\sum_{j=1/2}^{\infty}(2j+1)\int_0^\infty dk\, k^2 e^{ik\tau}\left[\frac{K_{j+1}(kR)-iK_j(kR)}{I_j(kR)-iI_{j+1}(kR)} - \frac{K_{j+1}(kR)+iK_j(kR)}{I_j(kR)+iI_{j+1}(kR)}\right]$$
$$\times\left[I_{j+1}{}^2(kR) - I_{j+2}(kR)I_j(kR) - I_j{}^2(kR) + I_{j+1}(kR)I_{j-1}(kR)\right], \tag{2.22}$$

or, using the ϵ-type cutoff,

$$\tilde{E}_0^F(\epsilon) = \frac{1}{2\pi R}\sum_{j=1/2}^{\infty}(2j+1)\int_0^\infty dx\, x^2\frac{[K_{j+1}(x)I_{j+1}(x) - K_j(x)I_j(x)]}{[I_j{}^2(x) - I_{j+1}{}^2(x)]}\left[I_{j+1}{}^2 - I_{j+2}I_j - I_j{}^2 + I_{j+1}I_{j-1}\right]\Big|_{x(1-\epsilon)}. \tag{2.23}$$

Formula (2.23) is generalized to massive fermions by making the following change in the limits of integration and integrand:

$$\int_0^\infty dx\, x^2 \to \int_{(mR)}^\infty dx\, x\left[x^2 - (mR)^2\right]^{1/2}. \tag{2.24}$$

C. Vector fields

Finally, the zero-point energy $E_0^V(\tau)$ for vector fields may expeditiously be found by solving two scalar Green's function problems. The transverse electric frequencies $\{\omega_{ln}^{\rm TE}\}$ are solutions to $j_l(\omega_{ln}^{\rm TE}R)=0$. Their cutoff sum is therefore related to the scalar Green's function problem previously described. The transverse magnetic frequencies are solutions to $(d/dR)[Rj_l(\omega_{ln}^{\rm TM}R)]=0$ and their cutoof sum is related to a scalar Green's function problem in which $(d/dR)[RG^{\rm TM}(R,\vec{r}',\tau)]=0$. So we can write the zero-point energy as

$$E_0^V(\tau)=\frac{\partial^2}{\partial\tau^2}\int_V d^3r\,[G_{l\neq 0}^{\rm TM}(\vec{r},\vec{r},\tau)+G_{l\neq 0}^{\rm TE}(\vec{r},\vec{r},\tau)] \tag{2.25}$$

if we are careful to exclude the $l=0$ contributions to $G^{\rm TM}$, $G^{\rm TE}$ as indicated by the subscripts in Eq. (2.25).[10] Then, using the by now familiar methods, the results are (see Appendix A)

$$\begin{aligned}E_0^V(\tau)=&+\frac{12V}{4\pi^2\tau^4}-\left(\frac{R}{\pi\tau^2}-\frac{1}{4\pi R}\right)\\&+\frac{R^2}{2\pi}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty dk\,k^2\cos k\tau\left[\frac{K_{l+1/2}(kR)}{I_{l+1/2}(kR)}+\frac{(d/dk)(\sqrt{k}K_{l+1/2}(kR))}{(d/dk)(\sqrt{k}I_{l+1/2}(kR))}\right]\\&\qquad\times[I_{l+1/2}{}^2(kR)-I_{l+3/2}(kR)I_{l-1/2}(kR)]\ .\end{aligned} \tag{2.26}$$

The second and third terms are the result of subtracting the $l=0$ contribution of the free Green's functions. The complicated piece is the contribution of $\Gamma_{l\neq 0}^{\rm TE}(\vec{r},\vec{r}',\tau)+\Gamma_{l\neq 0}^{\rm TM}(\vec{r},\vec{r}',\tau)$ which we can rewrite, using the ϵ-type cutoff, as

$$\tilde{E}_0^V(\epsilon)=\frac{1}{2\pi R}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty dx\,x^2\left[\frac{K_{l+1/2}(x)}{I_{l+1/2}(x)}+\frac{(d/dx)(\sqrt{x}K_{l+1/2}(x))}{(d/dx)(\sqrt{x}I_{l+1/2}(x))}\right][I_{l+1/2}{}^2-I_{l+3/2}I_{l-1/2}]\bigg|_{x(1-\epsilon)}\ . \tag{2.27}$$

III. EVALUATION OF THE ZERO-POINT ENERGY

We now return to the evaluation of the contribution of the homogeneous Green's functions to the zero-point energy for the three-dimensional spherical bag. The discussion is restricted to the scalar field example; methods will apply without modification to the somewhat more complex problems [see Eqs. (2.23) and (2.27)].

Expression (2.15) is finite so long as $\epsilon\neq 0$. For simplicity we rewrite it as

$$\tilde{E}_0^S(\epsilon)=\frac{1}{2\pi R}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dx\,F_l^S(x,t), \tag{3.1}$$

where the definition of $F_l(x,\epsilon)$ is obvious from Eq. (2.15). The Debye expansions for the Bessel functions $I_\nu(x)$, $K_\nu(x)$ occurring in the definition of $F_l^S(x,\epsilon)$ are expansions in decreasing powers of the argument x.[11] Unlike the corresponding Hankel expansions they are not dependent on any relation between ν and x. The leading terms in the expansion may therefore be substituted into Eq. (3.1) with no approximation made on the sum over the order l. The integrand $\sum_{l=0}^{\infty}(2l+1)F_l^S(x,\epsilon)$ then becomes an expansion in decreasing powers of x which, when the integration is performed for small ϵ, can be used to pick out the divergences in $\tilde{E}_0^S(\epsilon)$ as $\epsilon\to 0$. The substitution of the Debye expansion proceeds in the same way for those Bessel functions with argument $x(1-\epsilon)$ as those with argument x. That is, since the factor $(1-\epsilon)$ is near unity it will not affect the number of terms in the Debye expansion which must be carried in the $x\gg 1$ limit. The $(1-\epsilon)$ factor is simply carried through the process of simplification until the integral is evaluated, the sum over orders is performed, and the divergence is extracted.

To illustrate, the first nonvanishing contribution from the Debye expansion is

$$F_l^S(x,\epsilon)\underset{x\to\infty}{\sim}(\pi e^{-2f_{l+1/2}(x)})\frac{1}{2\pi}\frac{e^{+2f_{l+3/2}(x(1-\epsilon))}}{x^2(1-\epsilon)^2[(l+\frac{1}{2})^2+x^2(1-\epsilon)^2]}\{(l+\tfrac{1}{2})^2+x^2(1-\epsilon)^2+(l+\tfrac{1}{2})[(l+\tfrac{1}{2})^2+x^2(1-\epsilon)^2]^{1/2}\}^2, \tag{3.2}$$

where

$$f_l(x)\equiv(x^2+l^2)^{1/2}-l\sinh^{-1}\left(\frac{l}{x}\right)\ . \tag{3.3}$$

It may be shown that

$$f_l(x(1-\epsilon))=f_l(x)-\epsilon(x^2+l^2)^{1/2}+O(\epsilon^2) \tag{3.4}$$

and that

$$e^{f_l(x)-f_{l+1}(x)}=\left[\frac{l}{x}+\frac{(l^2+x^2)^{1/2}}{x}\right]\times\left[1+\frac{1}{2(l^2+x^2)^{1/2}}-\frac{l}{6(l^2+x^2)^{3/2}}+\frac{1}{8(l^2+x^2)}+\cdots\right]. \tag{3.5}$$

The $O(\epsilon)$ term in this expansion contributes a factor $e^{-2\epsilon[x^2+(l+3/2)^2]^{1/2}}$ to the asymptotic expansion of $F_l^S(x,\epsilon)$. It is easy to show that ϵ may be set equal to zero elsewhere in Eq. (3.2) when calculating the leading divergence in $E_0^S(\epsilon)$. Using Eqs. (3.4) and (3.5) we now obtain

$$F_l^S(x,\epsilon)\underset{\substack{x\to\infty\\ \epsilon\to 0}}{\sim}\frac{x^2}{2[(l+\frac{1}{2})^2+x^2]}e^{-2\epsilon[x^2+(l+3/2)^2]^{1/2}}. \tag{3.6}$$

Finally, since the major contribution to $\tilde{E}_0^S(\epsilon)$ occurs when $l\sim x\gg 1$ no error is made in calculating the leading divergence, when the index l is treated as a continuous variable and $\sum_{l=0}^{\infty}\to\int_0^{\infty}dl$. Therefore,

$$\tilde{E}_0^S(\epsilon)\underset{\epsilon\to 0}{\sim}\frac{1}{2\pi R}\int_0^{\infty}dl\,l\int_0^{\infty}dx\,\frac{x^2}{(x^2+l^2)}\times e^{-2\epsilon(x^2+l^2)^{1/2}}=-\frac{1}{24\pi\epsilon^3}. \tag{3.7}$$

The corresponding expressions for confined fermion and vector fields are

$$F_j^F(x,\epsilon)\underset{\substack{x\to\infty\\ \epsilon\to 0}}{\sim}\frac{(j+\frac{1}{2})^2x^2}{4(j^2+x^2)^{5/2}}e^{-2\epsilon[x^2+(j+1)^2]^{1/2}} \tag{3.8}$$

and

$$F_l^V(x,\epsilon)\underset{\substack{x\to\infty\\ \epsilon\to 0}}{\sim}\frac{(l+\frac{1}{2})^2x^2}{2[(l+\frac{1}{2})^2+x^2]^{5/2}}e^{-2\epsilon[x^2+(l+1/2)^2]^{1/2}}, \tag{3.9}$$

with the result that

$$\tilde{E}_0^F(\epsilon)=\frac{1}{2\pi R}\sum_{j=1/2}^{\infty}(2j+1)\int_0^{\infty}dx\,F_j^F(x,\epsilon)\underset{\substack{x\to\infty\\ \epsilon\to 0}}{\sim}-\frac{1}{120\pi R\epsilon^2} \tag{3.10}$$

and

$$\tilde{E}_0^V(\epsilon)=\frac{1}{2\pi R}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty}dx\,F_l^V(x,\epsilon)\underset{\substack{x\to\infty\\ \epsilon\to 0}}{\sim}-\frac{1}{60\pi R\epsilon^2}. \tag{3.11}$$

In the derivation of Eqs. (3.8) and (3.9) terms which would have contributed cubic divergences canceled between the opposite parity modes so that $\tilde{E}_0^F(\epsilon)$, $\tilde{E}_0^V(\epsilon)$ diverge quadratically $\sim 1/\epsilon^2$ rather than $\sim 1/\epsilon^3$ as in the scalar field problem.[12]

To check the procedure we have used we recall that for fermions $\tilde{E}_0^F(\tau)$ the corresponding expression using the τ-type cutoff may be written

$$\tilde{E}_0^F(\tau)=\frac{1}{2\pi R}\sum_{j=1/2}(2j+1)\int_0^{\infty}dx\,F_j(x)e^{ix\tau/R}, \tag{3.12}$$

where $F_j^F(x)\equiv F_j^F(x,\epsilon)|_{\epsilon=0}$. From Eq. (3.8)

$$\sum_{j=1/2}^{\infty}(2j+1)F_j^F(x)\underset{x\to\infty}{\sim}\int_0^{\infty}dl\,\frac{x^2l^3}{2(l^2+x^2)^{5/2}}=\frac{x}{3}. \tag{3.13}$$

Therefore, we also expect a quadratic divergence in $\tilde{E}_0^F(\tau)$ as $\tau\to 0$:

$$\tilde{E}_0^F(\tau)\underset{\tau\to 0}{\sim}\frac{1}{2\pi R}\int_0^{\infty}dx\left(\frac{x}{3}\right)e^{ix\tau/R}. \tag{3.14}$$

But now the function of x, $\sum_{j=1/2}^{\infty}(2j+1)F_j^F(x)$, may be tabulated numerically and its asymptotic behavior compared with Eq. (3.13). For successive values of j and a fixed value of x we simply evaluate $(2j+1)F_j(x)$. When $j\gg x$ the series $(2j+1)F_j^F(x)$ behaves like $\sim 1/j^2$.[13] The result to high accuracy is that

$$\sum_{j=1/2}^{\infty}(2j+1)F_j^F(x)\underset{x\to\infty}{\sim}x/3$$

and

$$\left[\sum_{j=1/2}^{\infty}(2j+1)F_j^F(x)-x/3\right]\underset{x\to\infty}{\sim}1/x,$$

indicating a logarithmic divergence in $\tilde{E}_0^F(\tau)$ as $\tau\to 0$, in addition to the quadratic divergence already extracted analytically. Unfortunately, the method of using the Debye expansion to calculate the divergences analytically becomes more complicated when we attempt to extract divergences weaker than the quadratic ones previously displayed. But for the bag model the quadratic divergences already present a serious problem. Unlike the quartic divergence, they cannot be absorbed into a redefinition of any physical parameter.

Our results are consistent with the work of Balian and Bloch on smoothed eigenvalue densities for scalar[14] and electromagnetic[15] fields in cavities. For a scalar field satisfying a Dirichlet boundary condition they found that

$$\rho(k)=\frac{Vk^2}{2\pi^2}-\frac{Sk}{8\pi}+\frac{1}{6\pi^2}\int_S d\sigma\,\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\cdots, \tag{3.15}$$

where R_1 and R_2 are the main curvature radii of S. If we write $E_0^S(K)=\int_0^K k\rho(k)dk$ using a large frequency cutoff K, we obtain divergences $O(K^4)$, $O(K^3)$, and $O(K^2)$. From Eq. (3.15) we can understand the absence of quadratic divergences in the parallel plate problem because the curvatures on the boundary surfaces are infinite.

IV. DISCUSSION

It may be useful at this point to recall why Boyer did not encounter any divergences in his zero-point energy calculation.[4] Consider the arrangement of conducting spherical shells in Fig. 1. Boyer computed

$$E_0(a)=\lim_{R\to\infty}\left\{[E_0^{\mathrm{I}}(a)+E_0^{\mathrm{II}}(a,R)] - \left[E_0^{\mathrm{III}}\left(\frac{R}{3}\right)+E_0^{\mathrm{IV}}\left(\frac{R}{3},R\right)\right]\right\}, \qquad (4.1)$$

where the terms in braces are the zero-point energies of the electromagnetic field confined to the regions depicted in the figure. The quartic divergence, proportional to the volume of the region, cancels between $[E_0^{\mathrm{I}}(a)+E_0^{\mathrm{II}}(a,R)]$ and $[E_0^{\mathrm{III}}(R/\eta)+E_0^{\mathrm{IV}}(R/\eta,R)]$ even before the $R\to\infty$ limit is taken. For vector as well as for fermion fields, the cubic divergence cancels between opposite parity modes, region by region. Boyer speculated that the quadratic divergence had the form $c_1(a/\tau^2)$ in region I $[c_1(R/\eta/\tau^2)$ in region III], $c_2(R-a)/\tau^2$ in region II $[c_2(R-R/\eta)/\tau^2$ in region IV] and therefore canceled out like the quartic divergence.[16] However, using our Green's function method we can calculate the zero-point energy of a field external to a spherical region and find, for example, that

$$\tilde{E}_0^V(\text{outside})=-\frac{1}{2\pi R}\sum_{l=0}^{\infty}(2l+1)\int_0^{\infty}dx\,G_l^V(x,\epsilon),$$

where $G_l^V(x,\epsilon)$ may be obtained from $F_l^V(x,\epsilon)$ by merely interchanging I_ν and K_ν Bessel functions. The quadratic divergence in $\tilde{E}_0^V$ (outside) is $+1/60\pi R\epsilon^2$ and therefore the total quadratic divergence for a vector field filling all space and satisfying $\eta_\mu F^{\mu\nu}=0$ on $|\vec{r}|=R$ vanishes. The leading quartic divergences are then canceled in a subtraction between the two configurations in Fig. 1. Boyer used this procedure because only the finite piece in $E_0^{\mathrm{I}}(a)\propto 1/a$ contributes to a force on the surface confining the electromagnetic field. However, it is difficult to see what relevance such a subtraction procedure has to the bag model. Even though it might isolate the finite part of the fermion zero-point energy as it apparently does for the vector zero-point energy, such a calculation would have little significance for

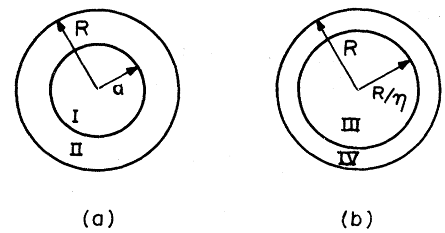

FIG. 1. The electromagnetic field fills regions I, II, III, and IV. $E_{\tan}$ vanishes on the boundaries. $a \ll R/\eta \ll R$ and η is some constant greater than unity.

the bag model in which the field is confined to a finite region of space and the boundary equations arise as the Euler-Lagrange equations of a covariant Lagrangian which we can take seriously as a fundamental theory.

However, there is one hint we can salvage from this discussion of the Boyer problem. We refer the following discussion to the example of confined fermion fields. From Eq. (2.24) it is apparent that the leading, quadratic divergence in $E_0^F(\epsilon)$ is independent of the mass of the fermion field confined to V. Therefore, the cancellation of quadratic divergences previously noted in the Boyer problem should persist when the field either inside or outside the sphere is massive. Now the bag model confinement can be derived from the Lagrangian

$$L=\int_V d^3x\left(\tfrac{1}{2}i\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi-B\right)+\int_{\bar{V}} d^3x\left(\tfrac{1}{2}i\bar{\Psi}\overleftrightarrow{\partial\!\!\!/}\Psi-M\bar{\Psi}\Psi\right) \qquad (4.2)$$

for a massless field ψ inside V and a field Ψ with mass M outside V.[1] The Euler-Lagrange equations are

$$i\partial^\mu\partial_\mu\psi=0 \text{ inside } V, \qquad (4.3a)$$

$$i\gamma^\mu\partial_\mu\Psi=M\Psi \text{ outside } V, \qquad (4.3b)$$

$$\psi=\Psi \text{ on the surface of } V, \qquad (4.3c)$$

$$M\bar{\Psi}\Psi=B \text{ on the surface of } V. \qquad (4.3d)$$

In the $M\to\infty$ limit of (4.3), ψ vanishes and the boundary condition $i\not{n}\psi=\psi$ is recovered from Eq. (4.3). Before $M\to\infty$, waves propagate across the surface of V and the corresponding Green's function from which the zero-point energy can be derived satisfies a continuity condition on the surface, determined by (4.3). We might hope (but not too much) that the quadratic divergence cancels for finite M and is not restored in the $M\to\infty$ limit even though the calculation based directly on the $M\to\infty$ limit of the boundary condition contains this unwelcome divergence.

This zero-point energy problem is substantially more complicated than for the strictly confined field (satisfying $i\not{n}\psi=\psi$ and has not yet been solved. Unfortunately, the parallel plate geometry is not a good laboratory to study Eq. (4.3). We can introduce a quadratic divergence into that problem by allowing the strictly confined field (satisfying $i\not{n}\psi=\psi$) to have a mass m. But with the parallel plate geometry, these divergences do not cancel even in the case when the external and internal fields both satisfy the limiting boundary condition.

Should no method of eliminating the quadratic divergence be found, the impact on the bag model itself would be limited by the particular assumption on which this work is based—that the boundary is static: In this case the classical cavity modes are known and the nonlinear boundary condition is particularly simple. The eigenfrequency equations have also been found for boundaries making small oscillations about a static spherical equilibrium, $\vec{R}(t,\Omega)=R_0\hat{r}+\epsilon\vec{R}_1(t,\Omega)$. In general, these equations are nonlinear because the nonlinear boundary condition must be used to eliminate $\vec{R}_1(t,\Omega)$ from the linear boundary. These equations are interesting because, as we saw in Sec. III, the quadratic divergence occurs when the source point moves indefinitely close to the boundary; it is therefore sensitive to the nature of the boundary.[17] However, while the Green's function method is powerful enough to work for a boundary condition of the form $L_{\text{op}}G(\vec{r},\vec{r}',\tau)=0$, where L_{op} is any linear operator, we have not yet been able to apply it to the nonlinear eigenfrequency equations needed to sum the oscillating boundary modes.[18]

While there is still much work to be done to resolve the questions we have raised, it may be that the zero-point energy can only be understood in a full quantum-mechanical treatment of boundary fluctuations.

ACKNOWLEDGMENTS

We thank Professor Kenneth A. Johnson for suggesting this problem and for his help with the calculations; we also thank Dr. Alan Chodos for many helpful discussions.

APPENDIX A: DERIVATION OF THE ZERO-POINT ENERGY FORMULAS

In this section we derive complete expressions for the Green's functions defined in Sec. II. For the scalar problem

$$G^S(\vec{r},\vec{r}',\tau)=\frac{1}{4\pi^2[(\vec{r}-\vec{r}')^2+\tau^2]}+\sum_{l=0}^{\infty}\int_0^{\infty}dkA_l\,\frac{I_{l+1/2}(kr)}{\sqrt{r}}\,\frac{I_{l+1/2}(kr')}{\sqrt{r'}}\,P_l(\cos\alpha)\cos k\tau. \tag{A1}$$

$\cos\alpha\equiv\hat{r}\cdot\hat{r}'$ is the only angle in this problem. Setting $G^S(\vec{r},\vec{r}',\tau)|_{|\vec{r}|=R}=0$ and projecting out the τ dependence,

$$\frac{1}{4\pi^2}K_{1/2}(k\beta)\left(\frac{\pi k}{2\beta}\right)^{1/2}+\frac{\pi}{2}\sum_{l=0}^{\infty}A_l\,\frac{I_{l+1/2}(kR)}{\sqrt{R}}\,\frac{I_{l+1/2}(kr')}{\sqrt{r'}}\,P_l(\cos\alpha)=0, \tag{A2}$$

where $\beta^2\equiv R^2+r^2-2Rr'\cos\alpha$. Using $K_{1/2}(x)=(\pi/2x)^{1/2}e^{-x}$ and the Gegenbauer expansion,

$$\frac{e^{-k\beta}}{\beta}=\sum_{l=0}^{\infty}(2l+1)\,\frac{K_{l+1/2}(kR)}{\sqrt{R}}\,\frac{I_{l+1/2}(kr')}{\sqrt{r'}}\,P_l(\cos\alpha), \tag{A3}$$

we obtain

$$A_l=-\frac{(2l+1)}{4\pi^2}\,\frac{K_{l+1/2}(kR)}{I_{l+1/2}(kR)}, \tag{A4}$$

and, therefore,

$$\Gamma^S(\vec{r},\vec{r}',\tau)=\frac{1}{4\pi^2}\sum_{l=0}^{\infty}(2l+1)\int_0^{\infty}dk\,\frac{K_{l+1/2}(kR)}{I_{l+1/2}(kR)}\,\frac{I_{l+1/2}(kr)}{\sqrt{r}}\,\frac{I_{l+1/2}(kr')}{\sqrt{r'}}\,P_l(\cos\alpha)\cos k\tau. \tag{A5}$$

The contribution of $\Gamma^S(\vec{r},\vec{r}',\tau)$ to $E_0^S(\tau)$ or to $E_0^S(\epsilon)$ may easily be obtained by performing the operations indicated in Eqs. (2.13) and (2.14).

For fermions,

$$\begin{aligned}S_{\alpha\beta}(\vec{r},\vec{r}',\tau)=&\left(\gamma^0\frac{\partial}{\partial\tau}+i\vec{\gamma}\cdot\vec{\nabla}\right)_{\alpha\beta}\frac{1}{4\pi^2[(\vec{r}-\vec{r}')^2+\tau^2]}\\&+\sum_{j,m}\int_0^{\infty}dk\,e^{ik\tau}[A_j^{(+)}\psi_{j,m}^{(+)\alpha}(\vec{r})\bar{\psi}_{jm}^{(+)\beta}(\vec{r}')+A_j^{(-)}\psi_{j,m}^{(-)\alpha}(\vec{r})\bar{\psi}_{jm}^{(-)\beta}(\vec{r}')].\end{aligned} \tag{A6}$$

The superscripts + and − refer to $j=l+\frac{1}{2}$, $j=l-\frac{1}{2}$, respectively. The boundary condition is

$(1+i\vec{\gamma}\cdot\vec{r})_{\alpha\lambda}\mathcal{S}_{\lambda\beta}=0$ on $|\vec{r}|=R$. But notice that

$$-i\vec{\sigma}\cdot\hat{r}[(1+i\vec{\gamma}\cdot\hat{r})M]_{1\alpha}=[(1+i\vec{\gamma}\cdot\hat{r})M]_{2\alpha} \tag{A7}$$

for any 2×2 matrix M. Therefore, the boundary condition gives only two independent 2×2 matrix equations. We choose $(1+i\vec{\gamma}\cdot\hat{r})_{1\lambda}\mathcal{S}_{\lambda 1}=(1+i\vec{\gamma}\cdot\hat{r})_{2\lambda}\mathcal{S}_{\lambda 2}=0$:

$$-(1+i\vec{\gamma}\cdot\hat{r})_{1\lambda}\mathcal{S}^{(0)}_{\lambda 1}=-\left(\frac{\partial}{\partial\tau}+\vec{\sigma}\cdot\hat{r}\vec{\sigma}\cdot\vec{\nabla}\right)\frac{1}{4\pi^2[(\vec{r}-\vec{r}\,')^2+\tau^2]}, \tag{A8a}$$

$$-(1+i\vec{\gamma}\cdot\hat{r})_{2\lambda}\mathcal{S}^{(0)}_{\lambda 2}=-\left(\vec{\sigma}\cdot\hat{r}\vec{\sigma}\cdot\vec{\nabla}-\frac{\partial}{\partial\tau}\right)\frac{1}{4\pi^2[(\vec{r}-\vec{r}\,')^2+\tau^2]}, \tag{A8b}$$

and

$$\begin{aligned}(1+i\vec{\gamma}\cdot\hat{r})_{1\lambda}\mathcal{K}_{\lambda 1}=\sum_{j,m}\int_0^\infty dq\,e^{iq\tau}[&A^{(+)}_{jm}F^{(+)}_j(qR)(-)^{j+1/2}f_{j-1/2}(iqr')\phi^{(+)}_{jm}(\Omega)\phi^{(+)*}_{jm}(\Omega')\\ &+A^{(-)}_{jm}F^{(-)}_j(qR)f_{j+1/2}(iqr')\phi^{(-)}_{jm}(\Omega)\phi^{(-)*}_{jm}(\Omega')],\end{aligned} \tag{A9a}$$

$$\begin{aligned}(1+i\vec{\gamma}\cdot\hat{r})_{2\lambda}\mathcal{K}_{\lambda 2}=\sum_{jm}\int_0^\infty dq\,e^{iq\tau}[&A^{(+)}_{jm}F^{(+)}_j(qR)(-)^{j+1/2}f_{j+1/2}(iqr')\phi^{(-)}_{jm}(\Omega)\phi^{(-)*}_{jm}(\Omega')\\ &+A^{(-)}_{jm}F^{(-)}_j(qR)f_{j-1/2}(iqr')\phi^{(+)}_{jm}(\Omega)\phi^{(+)*}_{jm}(\Omega')],\end{aligned} \tag{A9b}$$

where

$$F^{(+)}_j(qR)\equiv[(-)^{j+1/2}f_{j-1/2}(iqR)+(-)^{j-1/2}f_{j+1/2}(iqR)],$$

$$F^{(-)}_j(qR)\equiv[f_{j+1/2}(iqR)+f_{j-1/2}(iqR)],$$

$$f_s(ikr)=\left(\frac{\pi}{2kR}\right)^{1/2}e^{(\pi/2)si}I_{s+1/2}(kR).$$

In obtaining (A9b)

$$\phi^{(\pm)}_{jm}(\Omega)\phi^{(\pm)*}_{jm}(\Omega')=(\vec{\sigma}\cdot\hat{r})(\vec{\sigma}\cdot\hat{r}')\phi^{(\mp)}_{jm}(\Omega)\phi^{(\mp)*}_{jm}(\Omega')$$

has been used.

We continue by projecting out the τ dependence in (A8) and (A9), using the integral encountered in deriving (A1). Then after using the Gegenbauer expansion and the definition of $\phi^{(\pm)}_{j,m}(\Omega)$ we can project out the angular dependence by operating with $\int d\Omega\, Y^*_{st}(\Omega)$. Equation (A8) becomes

$$-\begin{pmatrix} ik\pm\frac{\partial}{\partial R}\mp\frac{t}{R} & \mp[s(s+1)-t(t+1)]^{1/2}\frac{1}{R}\\ \mp[s(s+1)-t(t-1)]^{1/2}\frac{1}{R} & ik\pm\frac{\partial}{\partial R}\pm\frac{t}{R}\end{pmatrix}\frac{K_{s+1/2}(kR)}{\sqrt{R}}\,\frac{I_{s+1/2}(kr')}{\sqrt{r'}}\,Y^*_{st}(\Omega). \tag{A10}$$

Applying the operations to (A9) we find that

$$\int d\Omega\,Y^*_{st}(\Omega)\phi^{(+)}_{jm}(\Omega)\phi^{(+)*}_{jm}(\Omega')=\frac{1}{(2s+1)}\begin{pmatrix}(s+t+1)Y^*_{st}(\Omega') & [s(s+1)-t(t+1)]^{1/2}Y^*_{s,t+1}(\Omega')\\ [s(s+1)-t(t-1)]^{1/2}Y^*_{s,t-1}(\Omega') & (s-t+1)Y^*_{st}(\Omega')\end{pmatrix}, \tag{A11a}$$

$$\int d\Omega\,Y^*_{st}(\Omega)\phi^{(-)}_{jm}(\Omega)\phi^{(-)*}_{jm}(\Omega')=\frac{1}{(2s+1)}\begin{pmatrix}(s-t)Y^*_{st}(\Omega') & -[s(s+1)-t(t+1)]^{1/2}Y^*_{s,t+1}(\Omega')\\ -[s(s+1)-t(t-1)]^{1/2}Y^*_{s,t-1}(\Omega') & (s+t)Y^*_{st}(\Omega')\end{pmatrix}, \tag{A11b}$$

and $j=s+\frac{1}{2}$ in functions multiplying (A11a), $j=s-\frac{1}{2}$ in functions multiplying (A11b).

According to (A10) and (A11) the boundary conditions now become two 2×2 matrix equations. Of these eight equations four are redundant, leaving four equations for $A^{(\pm)}_{s+1/2}$ and $A^{(\pm)}_{s-1/2}$ (which, of course, can be obtained from $A^{(\pm)}_{s+1/2}$ by $s\to s-1$). The solution is

$$A^{(\pm)}_{s+1/2}=\pm\frac{k^2}{\pi^2}e^{-\pi si}\frac{[K_{s+3/2}(kR)\mp iK_{s+1/2}(kR)]}{[I_{s+1/2}(kR)\mp iI_{s+3/2}(kR)]}\,. \tag{A12}$$

Recursion formulas were used to simplify the result. It is now simple to find $\tilde{E}_0^F(\tau)$ or $\tilde{E}_0^F(\epsilon)$ using (A6), (A12), and the prescription given in Sec. II.

To generalize these results to massive fermions we need only begin with the free-space Green's function,

$$\mathcal{S}^{(0)}_{\alpha\beta}(\vec{r}-\vec{r}',\tau)=\left(\gamma^0\frac{\partial}{\partial\tau}+i\vec{\gamma}\cdot\vec{\nabla}-m\right)_{\alpha\beta}D(\vec{r}-\vec{r}',\tau), \tag{A13a}$$

$$D(\vec{r}-\vec{r}',\tau)\equiv\frac{m}{4\pi^2[(\vec{r}-\vec{r}')^2+\tau^2]^{1/2}}\times K_1(m[(\vec{r}-\vec{r}')^2+\tau^2]^{1/2}), \tag{A13b}$$

$$\left(\frac{\partial^2}{\partial\tau^2}+\vec{\nabla}^2-m^2\right)D(\vec{r}-\vec{r}',\tau)=-\delta(\tau)\delta^3(\vec{r}-\vec{r}'). \tag{A13c}$$

After constructing $G^S(\vec{r},\vec{r}',\tau)$ the vector problem is almost trivial. $G^{\rm TE}$, of course, satisfies the same boundary condition as G^S, so the amplitudes are the same. It is also apparent from the derivation of G^S that the amplitude for $G^{\rm TM}$ may be obtained by applying the operator $d/dr^{\sqrt{r}}$ to the numerator and denominator of the G^S amplitude. We then average out the $l=0$ part of the free Green's function,

$$G^{(0)}_{l=0}(\vec{r}-\vec{r}',\tau)=\frac{1}{4\pi}\int d\Omega\,\frac{1}{4\pi^2[(\vec{r}-\vec{r}')^2+\tau^2]}=-\frac{1}{16\pi^2rr'}\ln\left[\frac{(r-r')^2+\tau^2}{(r+r')^2+\tau^2}\right], \tag{A14}$$

and subtract out its contribution to the zero-point energy,

$$2\frac{\partial^2}{\partial\tau^2}\int_V d^3r\,G^{(0)}_{l=0}(\vec{r},\vec{r},\tau)\bigg|_{\tau\to 0}=\frac{R}{\pi\tau^2}-\frac{1}{4\pi R}\,. \tag{A15}$$

The $l=0$ contribution from $\Gamma^{\rm TE}(\vec{r},\vec{r}',\tau)+\Gamma^{\rm TM}(\vec{r},\vec{r}',\tau)$ is eliminated merely by not including it in the sums in Eqs. (2.26) and (2.27).

APPENDIX B: TWO EXAMPLES: THE MASSIVE FERMION SLAB AND THE ONE-DIMENSIONAL MASSIVE SCALAR FIELD

First, we repeat a calculation of the zero-point energy of the fermion field confined by parallel plates at $z=0, L$. To make things more interesting this time we allow the fermion field to have mass m so the Casimir method will no longer work at all.

The Green's function is

$$\mathcal{S}_{\alpha\beta}(\vec{x}_T-\vec{x}_T',z,z',\tau)=\mathcal{S}^{(0)}_{\alpha\beta}(\vec{r}-\vec{r}',\tau)+\mathcal{K}_{\alpha\beta}(\vec{x}_T-\vec{x}_T',z,z',\tau), \tag{B1}$$

where

$$\left(\gamma^0\frac{\partial}{\partial\tau}+i\vec{\gamma}_T\cdot\vec{\nabla}_T+i\gamma_z\frac{\partial}{\partial z}-m\right)_{\alpha\lambda}\mathcal{S}^{(0)}_{\lambda\beta}=-\mathcal{S}_{\alpha\beta}\delta(\tau)\delta^2(\vec{x}_T-\vec{x}_T')\delta(z-z') \tag{B2}$$

is the Green's function in the absence of boundaries and for this geometry may be represented by

$$\mathcal{S}^{(0)}_{\alpha\beta}(\vec{x}_T-\vec{x}_T',z-z',\tau)=\frac{1}{(2\pi)^3}\int d\omega\,dk_T{}^2\,\frac{e^{-i\omega t+i\vec{k}_T\cdot(\vec{x}_T-\vec{x}_T')}}{2(\omega^2+k_T{}^2+m^2)^{1/2}}\times\left(m-\vec{\gamma}_T\cdot\vec{k}_T-i\omega\gamma^0+i\gamma_z\frac{\partial}{\partial z}\right)e^{-(\omega^2+k_T{}^2+m^2)^{1/2}|z-z'|}\,. \tag{B3}$$

$\mathcal{K}_{\alpha\beta}$ is a solution to the homogeneous equation, chosen so that

$$(1+i\gamma_z)\mathcal{S}=0 \text{ on } z=L, \tag{B4a}$$

$$(1-i\gamma_z)\mathcal{S}=0 \text{ on } z=0. \tag{B4b}$$

It is straightforward to apply these conditions and find that

$$\mathcal{K}_{\alpha\beta}(\vec{x}_T-\vec{x}'_T,z,z',\tau)=\frac{1}{(2\pi)^3}\int\frac{d\omega\,dk_T{}^2e^{-i\omega t+i\vec{k}_T\cdot(\vec{x}_T-\vec{x}'_T)}}{4\{\cosh[(\omega^2+k_T{}^2+m^2)^{1/2}L]+[m/(\omega^2+k_T{}^2+m^2)^{1/2}]\sinh[(\omega^2+k_T{}^2+m^2)^{1/2}L]\}}$$
$$\times\left\{e^{i\gamma_z(\vec{\gamma}_T\cdot\vec{k}_T+i\omega\gamma^0+m)z}(1+i\gamma_z)\left[1-\frac{(m+\vec{\gamma}_T\cdot\vec{k}_T+i\omega\gamma^0)}{(\omega^2+k_T{}^2+m^2)^{1/2}}i\gamma_z\right]e^{(m+\vec{\gamma}_T\cdot\vec{k}_T+i\omega\gamma^0)i\gamma_z(L-z')}\right.$$
$$+e^{i\gamma_z(\vec{\gamma}_T\cdot\vec{k}_T+i\omega\gamma^0+m)(z-L)}(1+i\gamma_z)\left[1+\frac{(m+\vec{\gamma}_T\cdot\vec{k}_T+i\omega\gamma^0)}{(\omega^2+k_T{}^2+m^2)^{1/2}}i\gamma_z\right]$$
$$\left.\times e^{-z'(m+\vec{\gamma}_T\cdot\vec{k}_T+i\omega\gamma^0)i\gamma_z}\right\}. \qquad \text{(B5)}$$

The contribution of the inhomogeneous Green's function (B3) to the zero-point energy is

$$\frac{\partial}{\partial\tau}\int_V d^3x\,\mathrm{Tr}(\mathcal{S}^{(0)}\gamma^0)=\frac{LA}{\pi^2}\frac{\partial^2}{\partial\tau^2}\left[\frac{m}{\tau}K_1(m\tau)\right] \qquad \text{(B6)}$$

in a section of the parallel plates with area, A.

Now we calculate the contribution of $\mathcal{K}_{\alpha\beta}$,

$$\frac{\partial}{\partial\tau}\int_V d^3x\,\mathrm{Tr}(\mathcal{K}\gamma^0)=\frac{A}{2(2\pi)^3}\frac{\partial}{\partial\tau}\int_0^L dz\int d\omega\,dk_T{}^2e^{-i\omega\tau}$$
$$\times\frac{2i\omega}{(\omega^2+k_T{}^2+m^2)^{1/2}}\frac{1}{\left\{\cosh[(\omega^2+k_T{}^2+m^2)^{1/2}L]+\frac{m}{(\omega^2+k_T{}^2+m^2)^{1/2}}\sinh[(\omega^2+k_T{}^2+m^2)^{1/2}L]\right\}}$$
$$\times\left\{e^{-(\omega^2+k_T{}^2+m^2)^{1/2}L}+\frac{m}{(\omega^2+k_T{}^2+m^2)^{1/2}}\cosh[(2z-L)(\omega^2+k_T{}^2+m^2)^{1/2}]-e^{-(\omega^2+k_T{}^2+m^2)^{1/2}L}\right\}. \qquad \text{(B7)}$$

Shifting to the ϵ-type cutoff described in Sec. II, the contribution of $\mathcal{K}_{\alpha\beta}$ becomes

$$\tilde{E}_0^F(\epsilon,m)=+\frac{1}{(2\pi)^3}\int\frac{d\omega\,dk_T{}^2\omega^2}{(\omega^2+k_T{}^2+m^2)^{1/2}}\frac{1}{\left\{\cosh[(\omega^2+k_T{}^2+m^2)^{1/2}L]+\frac{m}{(\omega^2+k_T{}^2+m^2)^{1/2}}\sinh[(\omega^2+k_T{}^2+m^2)^{1/2}L]\right\}}$$
$$\times\left\{\left[1-\frac{m}{(\omega^2+k_T{}^2+m^2)^{1/2}}\right]e^{-(\omega^2+k_T{}^2+m^2)^{1/2}L}(1-2\epsilon)L\right.$$
$$\left.+\frac{m}{(\omega^2+k_T{}^2+m^2)^{1/2}}\sinh[(\omega^2+k_T{}^2+m^2)^{1/2}L(1-z\epsilon)]\right\}. \qquad \text{(B8)}$$

To make progress it is convenient to define spherical coordinates: $r=(\omega^2+k_T{}^2)^{1/2}L$, $d\omega\,dk_T{}^2=(r^2/L^3)\times d(\cos\theta)\,d\phi$. The angular integrals are trivial, leaving

$$\tilde{E}_0^F(\epsilon,m)=\frac{1}{6\pi^2L^3}I(mL,\epsilon), \qquad \text{(B9)}$$

where

$$I(mL,\epsilon)\equiv\int_{mL}^{\infty}dy\,[y^2-(mL)^2]^{3/2}\frac{1}{\{\cosh y+[(mL)/y]\sinh y\}}\left\{\left(1-\frac{mL}{y}\right)(1-2\epsilon)e^{-y}+\frac{mL}{y^2}\sinh[y(1-2\epsilon)]\right\}. \qquad \text{(B10)}$$

When $m\to0$, $K_1(m\tau)\to1/m\tau$,

$$I(mL,\epsilon)\to\int_0^\infty dy\,y^3(1/\cosh y)e^{-y}$$
$$=\tfrac{3}{4}(2^3-1)\pi^4|B_4|/4!$$

and the massless result is recovered. For $m\neq0$, the additional dimensional parameter permits additional divergence which we extract from $I(mL,\epsilon)$,

$$I(mL,\epsilon)\underset{\epsilon\to0}{\sim}\frac{(mL)}{4\epsilon^2}-\frac{(mL)^2}{2\epsilon}+\tfrac{3}{2}(mL)^3\ln\epsilon. \qquad \text{(B11)}$$

Even in one space dimension the zero-point energy of massive confined fields cannot be calculated with the Casimir method. However, the problem is interesting because a finite expression is obtained; the mass introduces no divergences beyond that which can be absorbed into a redefinition of the bag constant. We readily find that

$$E_0^S(L,m) = BL + \tfrac{1}{2}\sum_{n=0}^{\infty}\left[\left(\frac{n\pi}{L}\right)^2 + m^2\right]^{1/2}$$
$$= B_{\text{ren}}L - \frac{1}{4\pi L}f(2mL)\,, \tag{B12a}$$

where

$$B_{\text{ren}} \equiv B + \frac{1}{2\pi}\frac{\partial^2}{\partial\tau^2}K_0(\tau m)\,, \tag{B12b}$$

and

$$f(x) = \pi\frac{x}{2} + x\sum_{s=1}^{\infty}\frac{K_1(xs)}{s}\,. \tag{B12c}$$

When $m \to 0$, $f \to \pi^2/6$ and the massless result is, of course, recovered. Notice that only the second term in $f(x)$ contributes to the inward force caused by the zero-point energy on the boundaries at $x = 0, L$. This term vanishes when $m \to \infty$ in accordance with our expectation that the zero-point energy is a relativistic effect.

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[8]We can easily generalize the cutoff

$$\int_0^\infty d\lambda\, f_c(\lambda)\sum_{n=0}^{\infty}\frac{\omega_n}{z}e^{-\omega_n\tau\lambda}$$
$$= -\frac{1}{2}\frac{\partial}{\partial\tau}\int_0^\infty d\lambda\,\frac{f_c(\lambda)}{\lambda}\sum_{n=0}^{\infty}e^{-\omega_n\tau\lambda}\,.$$

λ is dimensionless and $\int_0^\infty d\lambda\, f_c(\lambda) = 1$. $F(\tau\lambda) \equiv \sum_{n=0}^{\infty}e^{-\omega_n\tau\lambda}$ can be expanded in $\tau\lambda$. The numerical coefficients of the divergence depend on $f_c(\lambda)$, but the finite piece does not because it is contributed by the linear term in the expansion of $F(\tau\lambda)$ and $\int_0^\infty d\lambda\, f_c(\lambda) = 1$. We must also have $|\int_0^\infty d\lambda\, f_c(\lambda)\lambda^{(n\geq 1)}| < \infty$ so that higher-order contributions of $F(\tau\lambda)$ vanish when $\tau \to 0$.

[9]In deriving Eq. (2.7) we have used $[\epsilon(\tau)]^2 = 1$ and set $d/d\tau[\epsilon(\tau)] = \delta(\tau)$ equal to zero. Possible complications at $\tau = 0$ may be overlooked because the cutoff is never actually removed. Rather, we hope to absorb the cutoff-dependent terms into redefinitions of physical quantities.

[10]A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience, New York, 1965), p. 31.

[11]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. II, p. 86.

[12]The exclusion of $l = 0$ does not affect the leading divergence in $E_0^V(\epsilon)$.

[13]This result follows from a straightforward application of series expansions for the Bessel functions.

[14]R. Balian and C. Bloch, Ann. Phys. (N.Y.) 60, 401 (1970).

[15]R. Balian and C. Bloch, Ann. Phys. (N.Y.) 64, 271 (1971).

[16]Boyer calculated the coeffecients of $1/\tau^4$ and $1/\tau^2$ in region I by fitting orthogonal polynomials to the cutoff sum over the electromagnetic cavity modes. That calculation did not allow for weaker divergences such as $1/\tau$ or $\ln(1/\tau)$ in $E_0(a)$. Boyer assumed that, as in the Casimir problem, the divergences could be derived from an expression of the form $(\partial/\partial\tau)F(\tau)$, where $F(\tau)$ is some function of the cutoff that has a Laurent expansion as $\tau \to 0$.

[17]We can infer from the results of Balian and Bloch that, at least for the scalar and electromagnetic fields, the quadratic divergence persists for any static boundary without sharp edges; but there has been no work on the distribution of modes in cavities with moving boundaries.

[18]For the special case of a real scalar field, the oscillating boundary frequencies $\{x_{ln}\}$ satisfy $(d/dx_{ln})(x_{ln}j_l) + j_l = 0$. The zero-point energy has quadratic as well as cubic and quartic divergences. However, for the real scalar field the only solution to the static bag problem is a periodic step function.