Renormalization of the nonlinear σ model in $2+\epsilon$ dimensions

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The nonlinear σ model is renormalizable in two dimensions. It is shown here that there exists a parametrization of this model in which only two renormalization constants are needed. The renormalization of all soft operators is discussed, as well as that of symmetrical operators of dimension 4. The renormalization properties of this model, although simpler, have many features in common with gauge theories.

I. INTRODUCTION

In recent articles[1] two of the authors have discussed the applications of the nonlinear σ model[2] to problems of phase transitions near two dimensions.[3] The corresponding scaling properties follow from the structure of the renormalization of this model. Several aspects of renormalization theory in this case are not completely trivial and may be worth discussing in more detail. In particular, it will be emphasized that the special choice of representation made in Ref. 1 considerably simplifies the analysis. If one chooses another parametrization, as for instance was done in a recent article by Bardeen, Lee, and Shrock,[4] the analysis becomes more intricate and the scaling properties are no longer transparent. From the renormalization standpoint, the nonlinear σ model and the gauge theories have several features in common, though the first model is of course much simpler. We shall base this study on quadratic Ward-Takahashi identities analogous to those derived for gauge theories by one of the authors.[5] Therefore this article may be regarded as a pedagogical introduction to gauge theories.

This article consists of four sections. In the first section we discuss the renormalization of the model, in a particular representation of the fields. In the second section we study the renormalization of the infinite set of soft operators (the "relevant" operators of statistical mechanics). The consequences for the renormalization of the model in other representations of the pion field are outlined. In the third section, we discuss the insertion of some local operators of high dimension, physically interesting for the correction to the leading scaling behavior. In the last section, we briefly summarize a few results which were obtained previously by these methods.

II. RENORMALIZATION AND WARD-TAKAHASHI IDENTITIES

For convenience we shall describe the model corresponding to an O(n)-symmetric interaction (in the vector representation) in a two-dimensional Euclidean space, though the discussion may be extended without difficulty to other symmetry groups and to $2+\epsilon$ dimensions within a double expansion in the coupling constant and in ϵ. The Euclidean action of this model[2] in terms of a σ field and $(n-1)$ $\vec{\pi}$ fields is

$$\mathcal{A}=\int d^2x\left(\tfrac{1}{2}\partial_\mu\pi^i\partial_\mu\pi^i+\tfrac{1}{2}\partial_\mu\sigma\partial_\mu\sigma\right), \tag{1}$$

in which σ stands for

$$\sigma(x)=[1-\vec{\pi}^2(x)]^{1/2} \tag{2}$$

and the corresponding generating functional of the $\vec{\pi}$ Green's functions is defined with an invariant measure as

$$Z(\vec{J})=\int\prod_x\frac{d\vec{\pi}(x)}{[1-\vec{\pi}^2(x)]^{1/2}}\times\exp\left[-\frac{1}{t}\mathcal{A}+\frac{1}{t}\int d^2x\,\vec{J}(x)\cdot\vec{\pi}(x)\right]. \tag{3}$$

In order to write the action (2) we have made the choice of a parametrization of the sphere S_{n-1}. In this representation O(n) group acts linearly on the vector $(\vec{\pi},(1-\vec{\pi}^2)^{1/2})$. In two dimensions the $\vec{\pi}$ field is dimensionless, and all the terms of the Lagrangian are of dimension 2 and the theory is just renormalizable.

Regularization

The model needs in addition an invariant regularization in order to be defined. The first problem arises through the integration measure which is formally an additional interaction of the form

$$-\tfrac{1}{2}\delta^{(2)}(0)\int d^2x \ln[1-\vec{\pi}^2(x)]. \qquad (4)$$

We know of three possibilities in order to solve this problem:

(i) Consider the nonlinear σ model as a formal limit of the linear σ model $[\lambda(\vec{\pi}^2+\sigma^2-1)^2]$ for large coupling constant.[6]

(ii) Ignore the problem altogether by using a dimensional regularization,[7] which would set $\delta^{(2)}(0)$ equal to zero. In this case it is necessary to provide first an infrared cutoff to the theory in order to avoid the singularities due to the pion propagators below two dimensions. A very natural and convenient infrared cutoff is given by a "magnetic field," i.e., a source H linearly coupled to the σ field. Indeed the expansion of

$$H\int d^2x[1-\vec{\pi}^2(x)]^{1/2}$$

in powers of $\vec{\pi}^2$ generates a mass for the pion.

(iii) Express the problem on a lattice,[1] for which the continuous limit gives back the action (2). The problem is then well defined and reads

$$Z_{\text{latt}}(\vec{J})=\int\prod_i \frac{d\vec{\pi}_i}{(1-\vec{\pi}_i^{\,2})^{1/2}} \times\exp\left(-\frac{1}{t}\mathcal{A}_{\text{latt}}+\frac{1}{t}\sum_i \vec{J}_i\cdot\vec{\pi}_i\right), \qquad (5)$$

in which

$$\mathcal{A}_{\text{latt}}=\tfrac{1}{2}\sum{}'\{(\vec{\pi}_i-\vec{\pi}_j)^2+[(1-\vec{\pi}_i^{\,2})^{1/2}-(1-\vec{\pi}_j^{\,2})^{1/2}]^2\}, \qquad (6)$$

$\sum'$ meaning some short-range coupling, for instance a nearest-neighbor interaction.

The method of renormalization used in this work applies both to the dimensional and the lattice regularizations. However, the last one only allows a thorough discussion of the contact terms (4). The role of these terms is of course to eliminate the quadratic divergences of the perturbation series.[8] It is thus justified to use dimensional regularization in practical calculations, as we have done in our previous articles.

We shall now assume that the model has been regularized, either dimensionally (in the presence of the source H) or by a lattice, and we shall discuss the Ward identities and their implications for the structure of the counterterms.

Ward-Takahashi identities and renormalization

The general method has been exposed in Ref. 5. In the functional integral (3) one performs a change of variable on the fields which corresponds to an infinitesimal rotation of the group. In such transformations new operators may be generated: Here it is simply $(1-\vec{\pi}^2)^{1/2}$. It is then necessary to add sources for this new operator in the Lagrangian. One has then to examine what is generated by this new operator under a group transformation, and, if new operators arise, add the corresponding new sources. This has to be repeated until the system is closed under the group transformations. With the parametrization $(\sigma,\vec{\pi})$ chosen here, $(1-\vec{\pi}^2)^{1/2}$ generates nothing but the $\vec{\pi}$ and the system is closed after one step. An arbitrary choice of fields will in general generate an infinite number of successive operators, making the discussion of renormalization more tedious.

Our problem is thus to discuss

$$Z(\vec{J},H)=\int\prod_x \frac{d\vec{\pi}}{(1-\vec{\pi}^2)^{1/2}}\exp\left\{\frac{1}{t}\left[-\mathcal{S}+\int\vec{J}(x)\cdot\vec{\pi}(x)d^2x\right]\right\} \qquad (7a)$$

with

$$\mathcal{S}(\vec{\pi},H)=\int d^2x\left[\tfrac{1}{2}(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi})+\frac{1}{2}\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})(\vec{\pi}\cdot\partial_\mu\vec{\pi})}{(1-\vec{\pi}^2)}-H(x)(1-\vec{\pi}^2)^{1/2}\right]. \qquad (7b)$$

Let us perform the infinitesimal rotation [in the only interesting class $O(n)/O(n-1)$ which mixes $\vec{\pi}$ and σ]

$$\delta\vec{\pi}(x)=[1-\vec{\pi}^2(x)]^{1/2}\vec{\omega},$$
$$\delta[1-\vec{\pi}^2(x)]^{1/2}=-\vec{\omega}\cdot\vec{\pi}(x). \qquad (8)$$

The invariance of the action $\mathcal{A}(\pi)$, of the measure, and of the regularization, by the transformation (8) gives the equations

$$\int d^2x\left[J_i(x)\frac{\delta Z}{\delta H(x)}-H(x)\frac{\delta Z}{\delta J_i(x)}\right]=0 \qquad (9)$$

$$(i=1,2,\ldots,n-1).$$

The generating functional of the connected Green's functions of the $\vec{\pi}$,

$$W(\vec{J},H)=t\ln Z, \qquad (10)$$

satisfies the same equation (9). The Legendre transformation, which generates the generating functional of the one-particle irreducible functions of the $\vec{\pi}$ field, is defined by

$$\Gamma(\vec{\pi},H)=\int d^2x\,\vec{\pi}(x)\cdot\vec{J}(x)-W(\vec{J},H), \tag{11a}$$

$$\pi_i(x)=\frac{\delta W}{\delta J_i(x)}. \tag{11b}$$

Equation (9) is easily translated into an equation for Γ:

$$\int d^2x\left[\frac{\delta\Gamma}{\delta\pi_i(x)}\,\frac{\delta\Gamma}{\delta H(x)}+H(x)\pi_i(x)\right]=0. \tag{12}$$

The main feature of this equation is its quadratic nature in Γ. This will allow, as in gauge theories, the transformation law (8) to become renormalized. The discussion given in Ref. 5 for the implications of Eq. (12) on the structure of the renormalized theory can be easily adapted to our case. The aim is to show that Eq. (12) is stable under renormalization. We shall perform a loop expansion of the functional Γ:

$$\Gamma=\sum_{n=0}^{\infty}\Gamma^{(n)}t^n. \tag{13}$$

The corresponding Feynman diagrams are obtained by expanding in powers of $\vec{\pi}^2$ to the appropriate order the action and the integration measure. At lowest order Γ is the action (7b) itself,

$$\Gamma^{(0)}=\mathcal{S}(\vec{\pi},H), \tag{14}$$

which indeed satisfies Eq. (12).

At one-loop order the equation implies

$$\int d^2x\left(\frac{\delta\Gamma^{(0)}}{\delta\pi_i}\,\frac{\delta\Gamma^{(1)}}{\delta H}+\frac{\delta\Gamma^{(1)}}{\delta\pi_i}\,\frac{\delta\Gamma^{(0)}}{\delta H}\right)=0, \tag{15}$$

which will be denoted simply for brevity

$$\Gamma^{(0)}*\Gamma^{(1)}=0. \tag{16}$$

It is easy to verify that the differential operator

$$\Gamma^{(0)}*\equiv\int d^2x\left(\frac{\delta\Gamma^{(0)}}{\delta\vec{\pi}}\,\frac{\delta}{\delta H}+\frac{\delta\Gamma^{(0)}}{\delta H}\,\frac{\delta}{\delta\vec{\pi}}\right)$$

is an element of the Lie algebra of $O(n)/O(n-1)$.

When the cutoff increases, the divergent part of $\Gamma^{(1)}$ is singled out and satisfies the same equation,

$$\Gamma^{(0)}*\Gamma^{(1)\,\mathrm{div}}=0. \tag{17}$$

This equation (17) is just the condition which allows us to remove all the one-loop divergences by adding to the action (7b) $\mathcal{S}(\vec{\pi},H)$ a counterterm $t\mathcal{S}_1(\vec{\pi},H)$,

$$\mathcal{S}_1=-\Gamma^{(1)\,\mathrm{div}}+O(t), \tag{18}$$

such that the new action $(\mathcal{S}+t\mathcal{S}_1)$ satisfies exactly Eq. (12) to all orders in t. This means that we have constructed a one-loop renormalized action in which the transformation law for the pion field has changed and is now given by

$$\delta\vec{\pi}=\frac{\delta(\mathcal{S}+t\mathcal{S}_1)}{\delta H(x)}\,\vec{\omega}. \tag{19}$$

It is possible to continue abstractly along these lines,[5] show that the renormalized action satisfies Eq. (12), and finally solve this equation to exhibit the structure of the renormalized theory. An alternative way, less general but more explicit, consists in solving directly (17). From power counting we know that $\Gamma^{(1)\,\mathrm{div}}$ is a local function of dimension 2 of the π field. Noting that $H(x)$ is also of dimension 2, $\Gamma^{(1)\,\mathrm{div}}$ is at most of first degree in H. Thus the most general form of $\Gamma^{(1)\,\mathrm{div}}$ is

$$\Gamma^{(1)\,\mathrm{div}}=\int d^2x[B\{\pi\}+H(x)C\{\pi\}], \tag{20}$$

in which B contains at most two derivatives and C is derivative-free. Now Eq. (17) yields the following two conditions:

$$\frac{\vec{\pi}(x)}{(1-\vec{\pi}^2)}C=\frac{\delta C}{\delta\vec{\pi}(x)}, \tag{21a}$$

$$\int d^2x\left\{\left[-\Delta\vec{\pi}+\vec{\pi}\,\frac{\Delta(1-\pi^2)^{1/2}}{(1-\pi^2)^{1/2}}\right]C-(1-\pi^2)^{1/2}\,\frac{\delta B}{\delta\vec{\pi}(x)}\right\}=0. \tag{21b}$$

The most general solution of this system is

$$\Gamma^{(1)\,\mathrm{div}}=\lambda\mathcal{A}+\mu\int d^2x\left[\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{(1-\pi^2)^2}+\frac{H}{(1-\pi^2)^{1/2}}\right], \tag{22}$$

in which $\mathcal{A}$ is the action (1).

It is easy to verify that Eq. (18) is now satisfied if one writes the rescaled action in terms of a rescaled field as

$$\mathcal{S}=\int d^2x\left\{\frac{Z}{Z_1}\left[\tfrac{1}{2}(\partial_\mu\vec{\pi})^2+\frac{1}{2}\,\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{(1/Z-\pi^2)}\right]-H\left(\frac{1}{Z}-\pi^2\right)^{1/2}\right\}, \tag{23}$$

with

$$\frac{Z}{Z_1}=1-\lambda t+O(t^2), \tag{24}$$

$$Z=1-2\mu t+O(t^2), \tag{25}$$

which achieves the renormalization at one-loop order.

We have noticed that after this renormalization the transformation law of the π field has been modified since the new action is invariant under

$$\delta\vec{\pi} = (1/Z - \pi^2)^{1/2}\vec{\omega}.$$

Therefore the integration measure $d\vec{\pi}/(1-\pi^2)^{1/2}$ is no longer invariant. It is thus necessary to modify it and replace it by $d\vec{\pi}/(1/Z-\pi^2)^{1/2}$. This modification of the interaction will only affect the two-loop order [since the expression (4) comes into the interaction without the factor $1/t$]. Thus the new one-particle irreducible (1PI) functional satisfies again Eq. (12).

We are now going to proceed inductively: Let us assume that we have constructed a renormalized action up to order $(n-1)$ which fulfills Eq. (12). With this action we construct the loop expansion up to order n. From Eq. (12) we find that

$$\Gamma^{(0)} * \Gamma^{(n)} = -(\Gamma^{(1)} * \Gamma^{(n-1)} + \Gamma^{(2)} * \Gamma^{(n-2)} + \cdots), \tag{26}$$

i.e., the right-hand side contains only finite renormalized terms.

Thus taking the large-cutoff limit, we obtain

$$\Gamma^{(0)} * \Gamma^{(n)\,\text{div}} = 0. \tag{27}$$

The integration of this equation for $\Gamma^{(n)\,\text{div}}$ has already been discussed, and we see that at order n the effect of the renormalizations may be again absorbed into a rescaling of the π field and of the action.

This completes the induction and shows that for this parametrization of the model there are only two renormalization constants: one of the coupling constant t, the other one of the field strength.

The renormalized action, divided by the renormalized coupling constant t, reads

$$\frac{S}{t} = \int d^2x \left\{ \frac{Z}{2Z_1 t}\left[(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}) + \frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{(1/Z-\pi^2)}\right] - \frac{H}{t}(1/Z-\pi^2)^{1/2}\right\}. \tag{28}$$

III. RENORMALIZATION OF SOFT OPERATORS INSERTIONS

In this theory, since the π field is dimensionless, any local function without derivative of the π field is a soft (relevant) operator. This means that the Green's functions with one insertion of such operators will have a superficial degree of divergence reduced by two units; two such insertions are superficially convergent.

In order to discuss the renormalization of these soft operators, we shall assume that we have added one such operator to the action and that we derive the new WT identities as in the preceding section. It is clear from the previous analysis that if we have chosen a set of operators which correspond to the basis of an irreducible representation of the O(n) group, they will not be mixed to other operators under renormalization. Furthermore, it is easy to verify that there is only one renormalization constant for a given irreducible representation. It is well known that the basis of these irreducible representations of O(n) is given by the traceless tensor product constructed from the n vector $[\vec{\pi}, (1/Z-\vec{\pi}^2)^{1/2}]$.

For instance, the spin-two tensor corresponds to the operators[9]

$$\left[\pi_i\pi_j - \frac{1}{Z}\frac{\delta_{ij}}{n};\ \pi_i(1/Z-\vec{\pi}^2)^{1/2};\ \frac{1}{Z}\frac{n-1}{n} - \vec{\pi}^2\right].$$

For each irreducible representation there is only one operator which is a scalar for the unbroken subgroup O($n-1$). It is obtained by taking every index of the tensor corresponding to the σ component. The result is for the spin-l representation

$$\mathcal{O}_l(\pi) = C_l^{n/2-1}[(1-Z\vec{\pi}^2)^{1/2}], \tag{29}$$

in which $C_l^{n/2-1}(x)$ is the classical Gegenbauer polynomial [i.e., orthogonal for the measure $(1-x^2)^{(n-3)/2}$ on the interval $(-1,1)$]. Thus in order to renormalize the insertion of an arbitrary soft operator, one has first to project it on the set of Gegenbauer polynomials. Each component obtained in this way is renormalized multiplicatively. An explicit one-loop calculation gives the renormalization constant of $\mathcal{O}_l$ which is

$$Z_l = 1 - \frac{l(l+n-2)}{2(d-2)}\left(\frac{t}{2\pi}\right). \tag{30}$$

The consequences for statistical mechanics will be published elsewhere.[11]

The σ model in other parametrizations

We are now in position to discuss the renormalization of this model for an arbitrary parametrization of the sphere S_{n-1}. If the Lagrangian is expressed in terms of fields $\vec{\phi}$ related to our $\vec{\pi}$ by

$$\vec{\phi} = \vec{\pi} f(\pi^2), \tag{31}$$

in order to generate the Green's functions of the ϕ fields we may add in the above $\vec{\pi}$ representation a source coupled to $\vec{\pi}f(\pi^2)$. Therefore, in order to renormalize these new Green's functions, it is necessary to expand $\vec{\pi}f(\pi^2)$ on the successive irreducible representations discussed previously. This is achieved by expanding $f(\pi^2)$ on the first derivatives of the Gegenbauer polynomials. Consequently the renormalization of $\vec{\phi}$ is not simply multiplicative, since each component is independently renormalized. In other terms the renormalized parametrization of the sphere is different from the bare one (31). Under these conditions the renormalization-group equations for an arbitrary parametrization involves the diagonalization of an infinite-dimensional matrix. This justifies again our particular choice of parametrization.

III. RENORMALIZATION OF INVARIANT OPERATORS OF HIGHER DIMENSION

The problem of invariant operators of higher dimension arises in statistical mechanics, if one describes the approach to the scaling limit.[10] The first nonsoft insertion to be considered corresponds to the Lagrangian itself. By renormalization the Lagrangian is coupled to another operator which is $[(\partial_\mu\sigma)^2/\sigma^2+Z_1H/Z\sigma]$. However, it is clear that since the bare Lagrangian insertion corresponds to a derivative with respect to the coupling constant no new renormalization constant besides Z and Z_1 may arise. The next possibility corresponds to invariant operators with four derivatives. Within the bare theory there are only three such invariant operators, namely

$$\mathcal{O}_1=\tfrac{1}{2}\{\tfrac{1}{2}[(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi})+(\partial_\mu\sigma\partial_\mu\sigma)]\}^2, \tag{32}$$

$$\mathcal{O}_2=\tfrac{1}{8}(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi}+\partial_\mu\sigma\partial_\nu\sigma)(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi}+\partial_\mu\sigma\partial_\nu\sigma), \tag{33}$$

$$\mathcal{O}_3=\tfrac{1}{2}(\Delta\vec{\pi}\cdot\Delta\vec{\pi}+\Delta\sigma\Delta\sigma). \tag{34}$$

We shall now describe how these operators are renormalized. The technique again is to add a source for $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ to the Lagrangian, all calculations being of course limited to the first order in the source. It is clear from the analysis of Sec. II that the equation for the complete action including these operators is again

$$\int d^2x\left[\frac{\delta\mathcal{S}}{\delta H(x)}\,\frac{\delta\mathcal{S}}{\delta\vec{\pi}(x)}+H(x)\vec{\pi}(x)\right]=0. \tag{35}$$

We expand Eq. (35) to first order in the sources of the $\mathcal{O}_i$ with the notation

$$\mathcal{S}=\mathcal{S}^{(0)}+\mathcal{S}^{(1)}. \tag{36}$$

One obtains

$$\int d^2x\left(\frac{\delta\mathcal{S}^{(0)}}{\delta H}\,\frac{\delta\mathcal{S}^{(1)}}{\delta\vec{\pi}}+\frac{\delta\mathcal{S}^{(0)}}{\delta\vec{\pi}}\,\frac{\delta\mathcal{S}^{(1)}}{\delta H}\right)=0. \tag{37}$$

We have already solved the similar equation (15) for operators of dimension 2. The technique for dimension-4 operators is analogous, but now quadratic terms in H are allowed,

$$\mathcal{S}^{(1)}=\int d^2x(B\{\pi\}+C\{\pi\}H+D\{\pi\}H^2), \tag{38}$$

in which B contains at most four derivatives, C contains at most two, and D is derivative-free. The resulting equations are

$$\int d^2x\,\sigma(x)\,\frac{\delta D(y)}{\delta\vec{\pi}(x)}-2D(y)\,\frac{\vec{\pi}(y)}{\sigma(y)}=0,$$

$$\int d^2x\sigma(x)\,\frac{\delta C(y)}{\delta\vec{\pi}(x)}-C(y)\,\frac{\vec{\pi}(y)}{\delta(y)}+2D(y)\,\frac{\delta\mathcal{a}}{\delta\vec{\pi}(y)}=0,$$

$$\int d^2x\left[\sigma(x)\,\frac{\delta B}{\delta\vec{\pi}(x)}+C(x)\,\frac{\delta\mathcal{a}}{\delta\vec{\pi}(x)}\right]=0.$$

The solution of these equations is

$$\mathcal{S}^{(1)}=\int d^2x\left\{\lambda_1\frac{Z^2}{4}[(\partial_\mu\vec{\pi})^2+(\partial_\mu\sigma)^2]^2+\lambda_2\frac{Z^2}{4}(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi}+\partial_\mu\sigma\partial_\nu\sigma)^2+\lambda_3\frac{Z}{2}(\Delta\vec{\pi}\cdot\Delta\vec{\pi}+\Delta\sigma\Delta\sigma)\right.$$
$$\left.+\lambda_4\left(\frac{HZ_1+Z\Delta\sigma}{Z\sigma}\right)Z[(\partial_\mu\vec{\pi})^2+(\partial_\mu\sigma)^2]+\lambda_5\left(\frac{HZ_1+Z\Delta\sigma}{Z\sigma}\right)^2+\text{operators of dimension 2}\right\}. \tag{39}$$

One notices that two new operators of dimension 4, which are not invariant in the naive sense, have been induced by the renormalization mechanism. The reason is that the insertion of these operators modifies again the transformation law of the pions, which, as we have already discussed in Sec. II, is not given *a priori* but depends on the renormalized action itself. Hence $(\mathcal{S}^{(0)}+\mathcal{S}^{(1)})$ is indeed invariant, up to first order in the sources, but for a modified transformation law. More precisely the two additional operators $\mathcal{O}_4$ and $\mathcal{O}_5$ can be generated by a change in the definition of the $\vec{\pi}$ field of the form

$$\vec{\pi}'=\left[1+\alpha\mathcal{L}+\beta\left(\frac{HZ_1+Z\Delta\sigma}{Z\sigma}\right)\right]\vec{\pi}, \tag{40}$$

in which the β term is induced by the iteration of the α term. The situation is very similar to the one encountered for the renormalization of gauge-invariant operators in gauge theories.[12] The renormalization of these four-dimensional operators thus introduces a 5×5 matrix given in the Appendix. The consequences of these results for statistical mechanics will be published elsewhere.[11]

IV. SUMMARY OF RESULTS

For completeness, let us briefly summarize a few results previously obtained by this method.[1] First of all the previous analysis applied in d dimensions provided one performs a double expansion in the coupling constant and in $(d-2)$. From the form of the renormalized Lagrangian one derives in the usual way the renormalization-group equation,

$$\left\{\mu \frac{\partial}{\partial \mu}+W(t) \frac{\partial}{\partial t}-\frac{N}{2}\zeta(t)+\left[\tfrac{1}{2}\zeta(t)+\frac{W(t)}{t}-(d-2)\right] H \frac{\partial}{\partial H}\right\}\Gamma^{(N)}(\vec{p}, t, H, \mu)=0,$$

in which μ is the arbitrary length scale which defines the renormalized theory. The function $W(t)$ has the low-t expansion

$$W(t)=(d-2)t-(n-2)\frac{t^2}{2\pi}+O(t^3),$$

which shows that this theory is asymptotically free in two dimensions, and that there is an ultraviolet (UV) fixed point above two dimensions. This has several consequences for the Heisenberg ferromagnets concerning the expansion in $(d-2)$ of critical exponents and for scaling laws.[1] From the point of view of field theory, the situation is the following. The existence of a UV fixed point implies that the theory is renormalizable above two dimensions in contradiction with naive power counting. At this fixed point perturbation theory becomes meaningless; in particular for $d=2$, this occurs for any value of the coupling constant. In statistical-mechanics language, the fixed point corresponds to a continuous phase transition. Above that point another phase appears in which the expectation value of the σ vanishes in zero source. The spectrum of the theory consists now of $(n-1)$ massive π particles, plus a σ bound state degenerate in mass with the pion. The mass, which is proportional to $\exp[\int^t dt'/W(t')]$, behaves as $\exp[-2\pi/t(n-2)]$ in two dimensions. These facts are easily understood within the equivalent statistical-mechanics problem, in which the phase transition corresponds to a restoration of the O(n) symmetry which was spontaneously broken by the σ expectation value in the low-t phase. Furthermore, they can also be checked directly order by order in the $1/n$ expansion[13] by showing that the nonlinear σ model is equivalent to the linear one[14] in which the four-point coupling constant is taken at the infrared-stable fixed point. This constitutes the solution of the infrared-slavery problem in this model.

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APPENDIX: ONE-LOOP CALCULATION OF THE INSERTIONS OF FOUR-DIMENSIONAL OPERATORS

Since the renormalization of higher-order composite operators developed in Sec. IV has interesting applications to statistical mechanics,[11] we give here the one-loop calculations. The five coupled operators derived in Sec. IV are

$$\mathcal{O}_1=\frac{Z^2}{8}(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+\partial_\mu\sigma\partial_\mu\sigma)^2,$$

$$\mathcal{O}_2=\frac{Z^2}{8}(\partial_\mu\vec{\pi}\cdot\partial_\nu\vec{\pi}+\partial_\mu\sigma\partial_\nu\sigma)^2,$$

$$\mathcal{O}_3=\frac{Z}{2}(\Delta\vec{\pi}\cdot\Delta\vec{\pi}+\Delta\sigma\Delta\sigma),$$

$$\mathcal{O}_4=\frac{1}{2}\left(\frac{Z_1H+Z\Delta\sigma}{Z\sigma}\right)Z(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+\partial_\mu\sigma\partial_\mu\sigma),$$

$$\mathcal{O}_5=\frac{1}{8}\left(\frac{Z_1H+Z\Delta\sigma}{Z\sigma}\right)^2.$$

An explicit one-loop calculation of the insertion of these operators at zero momentum in the two-point function gives [keeping only the pole part in $1/(d-2)$]

$$\Gamma_1^{(2)}(p)=-\frac{t}{2\pi(d-2)}\left(-\frac{H}{2}p^2\right),$$

$$\Gamma_2^{(2)}(p)=-\frac{t}{2\pi(d-2)}\left[-\frac{H}{4}(n+2)p^2\right],$$

$$\Gamma_3^{(2)}(p)=(p^2)^2-\frac{t}{2\pi(d-2)}[-2Hp^2+(n-1)H+(2-n)(p^2)^2],$$

$$\Gamma_4^{(2)}(p)=Hp^2-\frac{t}{2\pi(d-2)}[(4-n)Hp^2-(n-1)H^2],$$

$$\Gamma_5^{(2)}(p)=\frac{H^2}{4}-\frac{t}{2\pi(d-2)}\left[\frac{H^2}{2}+\frac{(p^2)^2}{4}\right].$$

Similarly we have calculated the one-loop four-point function $\Gamma^{(4)}(p1;-p1;q2;-q2)$ (in which p and q are the momenta and 1 and 2 the isospin components) with $\mathcal{O}_i$ inserted at zero momentum:

$$\Gamma_1^{(4)} = p^2q^2 - \frac{t}{2\pi(d-2)}\left[(4-2n)p^2q^2 - 4(p\cdot q)^2 - \frac{n}{2}H(p^2+q^2)\right],$$

$$\Gamma_2^{(4)} = (p\cdot q)^2 - \frac{t}{2\pi(d-2)}\left[-2p^2q^2 - 2(n-1)(p\cdot q)^2 - \frac{(n+2)}{4}H(p^2+q^2)\right],$$

$$\Gamma_3^{(4)} = -\frac{t}{2\pi(d-2)}[4p^2q^2 - 16(p\cdot q)^2 - 2H(p^2+q^2) + 4(n-1)H^2],$$

$$\Gamma_4^{(4)} = H(p^2+q^2) - \frac{t}{2\pi(d-2)}[-4p^2q^2 + 16(p\cdot q)^2 + (4-n)H(p^2+q^2) - 4(n-1)H^2],$$

$$\Gamma_5^{(4)} = H^2 - \frac{t}{2\pi(d-2)}[p^2q^2 - 4(p\cdot q)^2 + 2H^2].$$

The 5×5 renormalization matrix, defined by the finiteness of $\Gamma^{(N)}_{\mathcal{O}_i\,\text{ren}}$ i.e., $\mathcal{O}_i^{\text{ren}} = Z_{ij}\mathcal{O}_j$ is thus at one-loop order

$$(Z_{ij}) = (\delta_{ij}) + \frac{t}{2\pi(d-2)}\begin{bmatrix} (4-2n) & -4 & 0 & -\frac{n}{2} & 0 \\ -2 & -2(n-1) & 0 & -\frac{(n+2)}{4} & 0 \\ 4 & -16 & (2-n) & -2 & 4(n-1) \\ -4 & 16 & 0 & (4-n) & -4(n-1) \\ 1 & -4 & \frac{1}{4} & 0 & 2 \end{bmatrix}.$$

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