Numerical study of truncated Green's-function equations

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We develop a Green's-function approximation scheme for the $\lambda\phi^{2N}$ quantum field theory as an alternative to the usual perturbation expansion in powers of λ. The approximation scheme consists of introducing a source term into the Lagrangian, generating a sequence of coupled Green's function equations by repeated differentiation with respect to this source, and forcing the sequence to close by discarding the dependence on higher Green's functions. The validity of this procedure is then checked by using it to calculate the two-point Green's function in $\lambda\phi^4$, $\lambda\phi^6$, $\lambda\phi^8$ theories in one-dimensional space-time. We compare our predictions of the locations of the low-lying poles with previously published tabulations of the eigenvalues of the equivalent quantum-mechanical anharmonic oscillators. The agreement is impressive.

I. INTRODUCTION

In this paper we investigate the accuracy of an iterative truncation scheme for computing the Green's functions in a quantum field theory. Specifically, we use this scheme to find the poles of the two-point Green's functions for the $\lambda\phi^4$, $\lambda\phi^6$, and $\lambda\phi^8$ anharmonic oscillators. Then, by comparing the locations of these poles with previous numerical calculations of the energy levels of the oscillators, we demonstrate in the context of these simple quantum field theories that the truncation scheme we are proposing is rapidly convergent.

The truncation scheme we will use does not give rise to a perturbation series in powers of the coupling constant λ. Rather, it uses functional differentiation to generate systems of coupled Green's-function equations which are then closed by neglecting the contribution of the connected part of the Green's function having the highest number of external legs. A somewhat similar approach was used by Baker, Johnson, and Willey[1] and Cooper, Guralnik, and Kasdan.[2]

We summarize our conclusions as follows. The leading approximation to the two-point Green's function in the $\lambda\phi^{2N}$ theory exhibits just the one-particle pole. In the $\lambda\phi^4$ model the location of this pole is accurate to within 5% of its exact value for *all* values of λ ranging from 0 to ∞. In the $\lambda\phi^6$ and $\lambda\phi^8$ models the maximum error in the location of the one-particle pole for any value of the coupling constant is respectively 15% and 25%. In the next approximation the two-point function develops additional poles. In the $\lambda\phi^4$ theory the error in the prediction of the location of the one-particle pole is improved to 0.6% for all values of the coupling constant. Moreover, the predicted value of the three-particle pole is within 5% of the exact value for all values of λ. The accuracy of these predictions is so impressive that we are led to speculate that our truncation methods may also be accurate in higher-dimensional quantum field theories.

In Sec. II of this paper we describe our iterative truncation procedure and in Sec. III we summarize our numerical calculations. Some further comments are made in Sec. IV.

II. THE ITERATIVE APPROXIMATION SCHEME

The Lagrangian $\mathcal{L}$ for a $\lambda\phi^{2N}$ quantum field theory in the presence of a source $J(x)$ for the field $\phi(x)$ is

$$\mathcal{L} = (\partial\phi)^2/2 - m^2\phi^2/2 - \lambda\phi^{2N} - J\phi\,.$$

Varying $\mathcal{L}$ with respect to $\phi(x)$ gives the field equation

$$\Box\,\phi(x) + m^2\phi(x) + 2N\lambda\phi^{2N-1}(x) + J(x) = 0\,.$$

Next, we take the vacuum expectation value of this equation and normalize by dividing by $\langle 0\,|\,0\rangle_J$. Defining

$$\Phi(x) \equiv \frac{\langle 0\,|\,\phi(x)\,|\,0\rangle_J}{\langle 0\,|\,0\rangle_J}\,,$$

we have

$$\Box\,\Phi(x) + m^2\Phi(x) + 2N\lambda\,\frac{\langle 0\,|\,\phi^{2N-1}(x)\,|\,0\rangle_J}{\langle 0\,|\,0\rangle_J} + J(x) = 0\,. \tag{1}$$

For the moment we specialize to the $N=2$ theory. From the action principle we have

$$i\,\frac{\delta}{\delta J(x)} \to \phi(x)\,.$$

Thus, applying $\delta/\delta J(y)$ to (1) with $N=2$ gives

$$(\Box_x+m^2)G(x,y)+4\lambda\left\{-\left[\frac{\delta}{\delta J(x)}\right]^2 G(x,y)+3i\Phi(x)\frac{\delta}{\delta J(x)}G(x,y)+[3iG(x,x)+3\Phi^2(x)]\,G(x,y)\right\}+\delta(x-y)=0\,, \tag{2}$$

where we define the two-point Green's function $G(x,y)$ by

$$G(x,y)\equiv\frac{\delta\Phi(x)}{\delta J(y)}$$

and have used the identity

$$\frac{\langle 0|\phi^3(x)|0\rangle_J}{\langle 0|0\rangle_J}=-\frac{\delta G(x,y)}{\delta J(x)}+3iG(x,x)\Phi(x)+\Phi^3(x)\,.$$

Now we set the source $J=0$ and require that

$$\Phi(x)\big|_{J=0}=0\,.$$

(This in turn implies that all Green's functions having an odd number of external legs vanish.)

Equation (2) now reduces to

$$[\Box_x+m^2+12\lambda iG(x,x)]\,G(x,y)-4\lambda\left[\frac{\delta}{\delta J(x)}\right]^2 G(x,y)+\delta(x-y)=0\,. \tag{3}$$

This equation, which is the first of an infinite sequence of coupled Green's-function equations of increasing complexity, is an exact relation between the four-point and two-point Green's functions.

The next equation of the sequence is derived by differentiating (2) with respect to $J(z)$ and $J(w)$ and setting $J=0$. The result is

$$[\Box_x+m^2+12\lambda iG(x,x)]\,G(x,y,z,w)+12\lambda i[G(x,x,z,w)G(x,y)+G(x,x,y,z)G(x,w)+G(x,x,y,w)G(x,z)]+24\lambda G(x,y)G(x,z)G(x,w)-4\lambda\left[\frac{\delta}{\delta J(x)}\right]^2 G(x,y,z,w)=0\,, \tag{4}$$

where we define

$$G(x,y,z,w)\equiv\frac{\delta^2}{\delta J(z)\,\delta J(w)}G(x,y)\,.$$

Equation (4) is an exact relation between the two-point, four-point, and six-point Green's functions. The remaining equations of the sequence are generated by taking functional derivatives two at a time with respect to J.

Now we describe the truncation procedure. The first-truncation approximation consists of dropping the term $[\delta/\delta J(x)]^2G(x,y)$ in (3). This gives a soluble equation for $G(x,y)$ in the $\lambda\phi^4$ model:

$$[\Box_x+m^2+12\lambda iG(x,x)]\,G(x,y)+\delta(x-y)=0\,. \tag{5}$$

Repeating the above derivation of (5) for the $\lambda\phi^{2N}$ theory gives a more general equation which reduces to (5) when $N=2$:

$$(\Box_x+m^2)G(x,y)+2N(2N-1)!!\lambda[iG(x,x)]^{N-1}G(x,y)+\delta(x-y)=0\,. \tag{6}$$

The second-truncation approximation consists of neglecting the term $[\delta/\delta J(x)]^2G(x,y,z,w)$ in (4) but retaining the term $[\delta/\delta J(x)]^2G(x,y)$ in (3). This gives a *coupled system* of equations for the two-point and four-point Green's functions in the $\lambda\phi^4$ theory:

$$[\Box_x+m^2+12\lambda iG(x,x)]\,G(x,y)-4\lambda G(x,x,x,y)+\delta(x-y)=0\,, \tag{7a}$$

$$[\Box_x+m^2+12\lambda iG(x,x)]\,G(x,y,z,w)+12\lambda i[G(x,x,z,w)G(x,y)+G(x,x,y,z)G(x,w)+G(x,x,y,w)G(x,z)]+24\lambda G(x,y)G(x,z)G(x,w)=0\,. \tag{7b}$$

The higher-order truncation approximations are derived in a similar manner.

III. RESULTS IN ONE DIMENSION

The formal derivation in Sec. II of the truncated equations is valid in any number of dimensions. However, in four-dimensional space-time, expressions such as $G(x,x)$ are divergent and it is necessary to develop a consistent renormalization procedure to make sense of the truncated equations. In this paper we choose to avoid this com-

plication by working in one-dimensional space-time where $G(x,x)$ is finite. Field theories in one space-time dimension, known as anharmonic oscillators, prove to be an excellent laboratory in which to study the accuracy of the truncation approximation because there are no divergences.

It is easiest to solve the truncated Green's-function equations in momentum space. Therefore, we specialize to one dimension by replacing $\Box$ by d^2/dt^2 and take Fourier transforms of (6) and (7). The first-truncation approximation for the $\lambda\phi^{2N}$ theory in momentum space is an algebraic equation for $G(E)$:

$$G(E)=\frac{1}{E^2-[m^2+2N(2N-1)!!\lambda i^{N-1}G^{N-1}(x,x)]}, \tag{8}$$

where we have recognized that $G(x,x)$ is a constant.

The second-truncation approximation for the $\lambda\phi^4$ theory is a complicated nonlinear system of integral equations:

$$(-E^2+\mu^2)G(E)=-1+4\lambda J(E), \tag{9a}$$

$$[+(E_1+E_2+E_3)^2-\mu^2]G(E_1,E_2,E_3)=12\lambda i[G(E_1)H(E_2,E_3)+G(E_2)H(E_1,E_3)+G(E_3)H(E_1,E_2)]+24\lambda G(E_1)G(E_2)G(E_3), \tag{9b}$$

where

$$\mu^2=m^2+12\lambda iG(x,x),$$

$$H(E_1,E_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(E_1,E_2,E_3)dE_3,$$

$$J(E_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(E_1,E_2)dE_2.$$

It is easy to solve (8). In momentum space the solution may be written as

$$G(p)=\frac{1}{p^2-M^2},$$

and in coordinate space the solution has the form

$$G(x,y)=\frac{1}{2iM}e^{-iM|x-y|},$$

where M, the pole of $G(p)$, satisfies the relation

$$M^2=m^2+2N\lambda(2N-1)!!(2M)^{1-N}. \tag{10}$$

It is convenient to introduce two dimensionless parameters:

$$\epsilon=\lambda m^{-N-1},$$

which represents the strength of the coupling, and

$$\mu=M/m,$$

which measures the location of the single-particle pole. In terms of these parameters (10) becomes

$$\mu^2=1+2N\epsilon(2N-1)!!(2\mu)^{1-N}. \tag{11}$$

Equation (11) implies that

$$\mu\sim[2N(2N-1)!!\epsilon]^{1/(N+1)}\quad(\epsilon\to+\infty).$$

This relation is consistent with an exact scaling law discovered by Symanzik which states that as $\epsilon\to+\infty$ the energy levels of the

$$H=p^2/2+\mu^2x^2/2+\epsilon x^{2N} \tag{12}$$

anharmonic oscillator grow as $\epsilon^{1/(N+1)}$. Ordinary perturbative treatments of anharmonic oscillators which rely on constructing a series in powers of ϵ do not respect Symanzik scaling to any order in perturbation theory. Because our nonperturbative treatment incorporates Symanzik scaling even in the leading truncation, we anticipate that the numerical values of μ as predicted by (11) should agree well with the exact values for large as well as for small ϵ.

Indeed, in Table I, where we compare our predicted values of μ with the exact values, we see that the agreement is good to 5%, 15%, and 25% in the $\lambda\phi^4$, $\lambda\phi^6$, and $\lambda\phi^8$ models for all positive values of ϵ. It is remarkable that the simple algebraic relation in (11) provides such a uniformly accurate approximation to μ.[3] To make the comparison between our predicted values of μ with previous numerical calculations of energy levels, we note that μ, the mass of the one-particle state, must be identified as the difference between the first excited level and the ground-state energy of the corresponding quantum mechanical anharmonic oscillator. Numerical calculation of the energy levels of the $\lambda\phi^4$ oscillator was done by Biswas *et al.*[4] and numerical calculation of the energy levels of $\lambda\phi^6$ and $\lambda\phi^8$ oscillators was done by Hioe *et al.*[5]

Although in the first-truncation approximation the two-point Green's function has only one pole, in higher-truncation approximations additional poles appear. We regard our truncation scheme as a *low-energy approximation* because, as we will see, the higher poles in the two-point Green's function develop in successively higher-truncation approximations.[6]

We have not been able to obtain an analytical solution to the coupled integral equations (9) of the second truncation. However, we have numerically iterated the integral equations on a computer and have obtained predictions for the values of the one-

TABLE I. Comparison of the exact value of μ (the difference between the first excited level and the ground-state energy) and the value of μ predicted by the leading truncation approximation in (11) for the $\lambda\phi^4$, $\lambda\phi^6$, and $\lambda\phi^8$ oscillators. The relative error = (predicted μ − exact μ)/(exact μ).

ϵ	Predicted μ	Exact μ	Relative error
	$\lambda\phi^4$ model		
0.1	1.2212	1.2104	0.89%
0.5	1.6717	1.6282	2.67%
1	2.0000	1.9341	3.41%
4	3.0000	2.8728	4.43%
50	6.7441	6.4154	5.12%
$\epsilon \to \infty$	$(6\epsilon)^{1/3} \approx 1.8171\epsilon^{1/3}$	$1.7278\epsilon^{1/3}$	5.17%
	$\lambda\phi^6$ model		
0.001	1.0109	1.0107	0.02%
0.01	1.0905	1.0800	0.97%
0.1	1.4426	1.3635	5.80%
1	2.2956	2.0697	10.91%
10	3.9381	3.4737	13.37%
100	6.9236	6.0598	14.25%
1 000	12.2679	10.7097	14.55%
20 000	25.9099	22.5980	14.66%
	$\lambda\phi^8$ model		
0.001	1.0450	1.0370	0.77%
0.01	1.2434	1.1726	6.04%
0.1	1.735	1.517	14.37%
1	2.618	2.179	20.15%
10	4.070	3.309	23.00%
100	6.402	5.153	24.24%
1 000	10.118	8.110	24.76%
20 000	18.395	14.716	25.00%

particle and three-particle poles.[7] The results are given in Table II. Observe that the one-particle pole is now predicted to an accuracy of 0.6% (this is an order-of-magnitude improvement over the result of the first-truncation approximation), and the three-particle pole is predicted correctly to 6% for all values of ϵ. In the second-truncation approximation the two-point Green's function appears to have an infinite number of poles. However, we have only calculated the locations of the lowest two poles.

IV. FURTHER COMMENTS

It is appropriate to make several additional observations.

(a) It was not essential to use the one-field source J to develop a truncation scheme. Indeed, a two-field source $S(x)$, which is introduced by inserting a term of the form $S(x)\phi^2(x)$ in the Lagrangian, may also be used (see Ref. 2). Taking variational derivatives with respect to S and setting $S=0$ gives the following exact Green's-function equation in $\lambda\phi^4$ theory:

$$[\Box_x + m^2 + 4i\lambda G(x,x)]G(x,y) + 4\lambda i \frac{\delta G(x,y)}{\delta S(x)} + \delta(x-y) = 0 .$$

The first-truncation approximation consists of dropping the four-point Green's-function term $\delta G(x,y)/\delta S(x)$. For the $\lambda\phi^4$ anharmonic oscillator the equation corresponding with (11) is

$$\mu^2 = 1 + 2\epsilon/\mu . \tag{13}$$

However, we find that this first-truncated S approximation is not as accurate as the first-truncated J approximation in (11) (see Table III). Apparently the S approximation is not as good as the J approximation because a single variational derivative with respect to $S(x)$ does not induce as much internal structure as two derivatives with respect to $J(x)$ and $J(y)$.

(b) It is interesting that the J approximation overestimates the value of μ while the S approximation underestimates the value of μ (see Tables I and III). There is, of course, a mixed approxi-

TABLE II. Comparison of the exact locations of the one-particle and three-particle poles of the two-point Green's function in the $\lambda\phi^4$ model with their approximate locations as predicted by the second-truncation approximation in (9).

ϵ	Exact one-particle pole	Exact three-particle pole	Predicted one-particle pole and relative error	Predicted three-particle pole and relative error
0.1	1.2104	4.0698	1.2086	3.8663
			−0.15%	−5.00%
0.5	1.6282	5.8822	1.6205	5.5683
			−0.47%	−5.34%
1	1.9341	7.1387	1.9232	6.7786
			−0.56%	−5.04%
4	2.8728	10.8825	2.8550	10.3956
			−0.62%	−4.47%
50	6.4154	24.6929	6.3762	23.6991
			−0.61%	−4.02%

TABLE III. Comparison of the exact values of μ in the $\lambda\phi^4$ model with the approximate value as predicted in the leading S truncation approximation in (13).

ϵ	Predicted μ	Exact μ	Relative error
0.1	1.0880	1.2104	−10%
1	1.5214	1.9341	−21%
50	4.7134	6.4154	−26.5%

mation that is more accurate than either of the two alone. However, we have not tried to formulate a reliable criterion for specifying, *a priori*, the correct mixture of J and S.

(c) In this paper we have restricted our attention to numerical approximations to the two-point Green's function. However, it is clear that even the lowest-truncation approximation may be used to elucidate the pole structure of n-point Green's functions. We do not bother here to examine their structure.

(d) We have seen that, in general, the second-truncation approximation gives rise to beautiful but analytically intractable integral equations. The simplest integral equation of this type arises in a $\lambda\phi^3$ theory. In this model the first two coupled Green's function equations are

$$(\Box_x + M^2)G(x,y) + 3\lambda i G(x,x,y) + \delta(x-y) = 0$$

and

$$(\Box_x + M^2)G(x,y,z) + 6\lambda G(x,y)G(x,z) + 3\lambda i \frac{\delta G(x,x,y)}{\delta J(z)} = 0,$$

where

$$M^2 = m^2 + 6\lambda\Phi(x)\big|_{J=0}$$

and

$$G(x,y,z) = \frac{\delta G(x,y)}{\delta J(z)}.$$

The second-truncation approximation consists of dropping the $\delta G(x,x,y)/\delta J(z)$ term. If we transform to momentum space and eliminate the three-point Green's function we obtain an integral equation for $G(E)$:

$$G(E) = \frac{1}{E^2 - M^2 - 18\lambda^2 \int_{-\infty}^{\infty} (i/2\pi) dE' G(E')/[(E+E') - m^2]}. \tag{14}$$

We include this lovely equation here in the hope that perhaps a solution will be found.[8]

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[1]See M. Baker, K. Johnson, and R. S. Willey, Phys. Rev. Lett. 11, 518 (1963) and subsequent papers. We thank K. Johnson for a discussion of this point.

[2]F. Cooper, G. S. Guralnik, and S. Kasdan, Phys. Rev. D 14, 1607 (1976). In this paper higher-order corrections are found in the form of a perturbation series generated by the four-point Green's function. In contrast, the truncation method described in the current paper is nonperturbative, much richer in content to any order in the truncation approximation, and presumably more rapidly convergent than the four-point Green's function expansion of Cooper, Guralnik, and Kasdan.

[3]A related calculation using Hartree-Fock factorization techniques was done by S.-J. Chang, Phys. Rev. D 12, 1071 (1975), for the $\lambda\phi^4$ oscillator in lowest order. However, a prescription for obtaining higher-order corrections was not given.

[4]S. N. Biswas, K. Datta, R. P. Saxena, P. K. Srivastava, and V. S. Varma, J. Math. Phys. 14, 1190 (1973). To translate the tables in Biswas *et al.* to conform with our notation in (12), it is necessary to divide their coupling constant and energy eigenvalues by 2.

[5]F. T. Hioe, D. MacMillen, and E. W. Montroll, J. Math. Phys. 17, 1320 (1976). It is not necessary to alter the entries in the tables in this paper because the notation is consistent with that in (12).

[6]We view our infinite sequence of truncated equations as being similar in structure to a truncated matrix approximation. If we expand the wave function of the anharmonic oscillator as an infinite linear combination of harmonic-oscillator wave functions we obtain an infinite-dimensional matrix; setting the determinant of this matrix to zero gives the eigenvalues [see C. M. Bender, H. J. Happ, and B. Svetitsky, Phys. Rev. D 9, 2324 (1974)]. To actually compute the eigenvalues numerically, one truncates this matrix to one of size $n \times n$ starting in the upper left-hand corner. As $n \to \infty$, the eigenvalues of the truncated matrix approach their correct values. Because the lowest-lying eigenvalues approach their correct values fastest, this also is a low-energy approximation.

[7]The solution of (9) by computer iteration is a rapidly convergent procedure for all values of ϵ because ϵ/μ^3 and *not* ϵ is the effective iteration parameter. If we specialize (11) to the case $N=2$, we see that the relation between ϵ and μ is given by $\mu^2 = 1 + 6\epsilon/\mu$. Note that as ϵ

ranges from 0 to ∞, ϵ/μ^3 ranges from 0 to $\frac{1}{6}$. In the $\lambda\phi^{2N}$ theory the effective iteration parameter would be ϵ/μ^{N+1}, which ranges from 0 to $2^N/[4N(2N-1)!!]$ as ϵ ranges from 0 to ∞. It follows that computer iteration of the coupled integral equations becomes more efficient as N increases.

[8]It is an interesting result that if one assumes that $G(E)$ in (14) has the form $G(E)=\sum_{k=0}^{\infty} a_k/(E^2-E_k^2)$, then one can show that as $k\to\infty$ the residues a_k rapidly tend to 0 and the pole locations E_k rapidly tend to their harmonic-oscillator values $2k+1$.