Physical states in quantum electrodynamics*

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(Received 3 March 1976)

Because of difficulties with the Gupta-Bleuler subsidiary condition in the charged sectors, an alternative scheme for identifying physical states in the indefinite-metric space $\mathcal{G}$ of quantum electrodynamics is proposed: Any vector $\Phi \in \mathcal{G}$ is a physical state if it is positive on the observables, $\langle \mathcal{O}\Phi, \mathcal{O}\Phi\rangle \geq 0$, $\langle \Phi, \Phi\rangle = 1$, for $\mathcal{O}$ any element of the algebra of observables. Observables $\mathcal{O}$, in turn, are selected by the requirement that they commute with the generators of the restricted gauge transformations of the second kind, $A_\mu \rightarrow A_\mu + \partial_\mu\lambda$, $\psi \rightarrow \psi\exp(ie\lambda)$, with $\lambda(x) = c$–number, $\partial^2\lambda = 0$. This is equivalent to the requirement $[B(x), \mathcal{O}] = 0$, where $B(x) = \partial\cdot A(x)$ in the Feynman gauge. It is proved that the substitute Gupta-Bleuler condition $B^{(-)}(x)\Phi = b^{(-)}(x)\Phi$ provides a subspace $\mathcal{G}_{[b]}$ of physical states, where $b^{(-)}(x)$ is the negative-frequency part of any real c-number solution of the wave equation $\partial^2 b(x) = 0$ satisfying $\int \dot{b}(x)d^3x = q$, with q an eigenvalue of the charge operator. Different functions $b(x)$ characterize different superselection sectors which are eigenspaces of generators $G(\lambda)$ of the restricted gauge transformations of the second kind with eigenvalues $G(\lambda) = \int \lambda(x)\overleftrightarrow{\partial}_0 b(x)d^3x$. In a given superselection sector Maxwell's equations take the form $\partial_\mu F^{\mu\nu} = J^\nu - \partial^\nu b$, where $-\partial^\nu b$ is interpreted as a classical external current which is induced by the quantum-mechanical current J^ν. The proof relies on the axiom of asymptotic completeness $\mathcal{G} = \mathcal{G}^{\text{in}} = \mathcal{G}^{\text{out}}$ and $\mathcal{G}^{\text{in}}$ is specified by the ansatz of infrared coherence, namely, $\lim_{\omega\rightarrow 0} a_\mu^{\text{in}}(k) \sim -(2\pi)^{-3/2}\sum_i e_i p_i/p_i\cdot k$, where $a_\mu^{\text{in}}(k)$ is the photon annihilation operator and p_i is the momentum of an incoming particle of charge e_i, and in$\rightarrow$out. The spectral decomposition of the infrared-coherent space is effected. Its singularity in the neighborhood of the electron mass agrees with the singularity of the electron propagator in the Feynman gauge, which allows an on-shell normalization of the charged field ψ.

I. INTRODUCTION

Quantum electrodynamics as currently formulated[1,2] involves two concepts that are somewhat foreign to each other: the dynamics and the subsidiary condition. The dynamics may be thought of as being specified by a Lagrangian[3]

$$\mathcal{L} = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - B\partial\cdot A + \tfrac{1}{2}B^2 + \bar{\psi}(i\not\partial + e\not A - m)\psi\,, \tag{1.1}$$

which yields canonical commutation relations and the equations of motion

$$\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}\,, \tag{1.2}$$

$$\partial\cdot A = B, \tag{1.3}$$

$$\partial_\mu F^{\mu\nu} + \partial^\nu B = -e\bar{\psi}\gamma^\nu\psi \equiv J^\nu, \tag{1.4}$$

$$(i\not\partial + e\not A - m)\psi = 0, \tag{1.5}$$

which also imply

$$\partial^2 A^\mu = J^\mu, \tag{1.6}$$

$$\partial^2 B = 0. \tag{1.7}$$

By virtue of the canonical commutation relations, the fields A and ψ are promoted to operators on an indefinite-metric space[4] $\mathcal{G}$ with indefinite inner product $\langle\Phi, \Psi\rangle$ for $\Phi, \Psi \in \mathcal{G}$. In addition to the dynamics, there is an independently postulated subsidiary condition for the specification of the physical states which are a subset of the vectors[5] in the indefinite-metric space. These are supposedly provided by the Gupta-Bleuler subsidiary condition[6,7]

$$B^{(-)}(x)\Phi = 0, \tag{1.8}$$

where $B^{(-)}(x)$ is the negative-frequency part of $B(x)$. The purpose of this condition is to provide states with two virtues. First they should be positive on the observables

$$\langle \mathcal{O}\Phi, \mathcal{O}\Phi\rangle \geq 0, \quad \langle\Phi, \Phi\rangle = 1 \tag{1.9}$$

where $\mathcal{O}$ is any element of the algebra of observables. Second the equation of motion (1.4) is manifestly converted by condition (1.8) into

$$\langle\Phi, (\partial_\mu F^{\mu\nu} - J^\nu)\Phi\rangle = 0, \tag{1.10}$$

which, it is asserted, is the correct form of Maxwell's equations.

There are two general questions about this formulation which come to mind. First, are the dynamics and the subsidiary condition compatible or, more precisely, do solutions of the subsidiary condition exist in the indefinite-metric space determined by the dynamics? Second, if we take the dynamics seriously, why not accept as a physical state any vector Φ which satisfies the positivity condition (1.9), and if so might there not be physical states which do not satisfy the Gupta-Bleuler subsidiary condition? These general questions are in fact inspired by specific difficulties with the

Gupta-Bleuler condition which we will come to shortly.

In the present article we adopt as our philosophy the notion that any vector in the indefinite-metric space which is positive on the observables is to be accepted as a physical state. Observables, in turn, are identified in Sec. II by a gauge-invariance criterion. Thus we do not make an independent assumption about what are the physical states. They are determined by the dynamics, augmented by a gauge-invariance principle which identifies observables, and the requirement of positivity.

In order to investigate which vectors satisfy the positivity requirement, we must specify the indefinite-metric space $\mathcal{G}$. We make the usual scattering postulate that it is asymptotically complete, $\mathcal{G}=\mathcal{G}^{\rm in}=\mathcal{G}^{\rm out}$. But the usual assumption that the asymptotic "in" and "out" spaces are Fock spaces is replaced by the ansatz that they are infrared-coherent. The infrared-coherent space is described in detail in Appendix A. In essence it has the property that an "in" scattering vector[5] which is diagonal in the four momenta p_a of the incoming charged particles with charges e_a, $a=1,\ldots,s$, is an eigenvector of the zero-frequency photon annihilation operator $a_\mu^{\rm in}(k)$,

$$[a_\mu^{\rm in}(k), a_\nu^{\rm in\dagger}(k')] = (-g_{\mu\nu})2\omega\delta^3(k-k'), \tag{1.11}$$

with eigenvalue

$$\lim_{\omega\to 0} a_\mu^{\rm in}(k) \sim \frac{-1}{(2\pi)^{3/2}} \sum_{a=1}^{s} \frac{e_a p_a}{p_a\cdot k}, \tag{1.12a}$$

where $k=(\omega,\vec{k})$, $\omega=|\vec{k}|$, and in $\to$ out. This may be stated as the operator equation

$$\lim_{\omega\to 0}\left\{\omega\left[a_\mu^{\rm in}(k)+\frac{1}{(2\pi)^{3/2}}\int d^4p\,\frac{\rho^{\rm in}(p)}{p\cdot k}\,p_\mu\right]\right\}=0, \tag{1.12b}$$

where $\rho^{\rm in}(p)$ is the charge density operator in momentum space

$$\rho^{\rm in}(p)=\theta(p^0)\delta(p^2-m^2)(-e)\times\sum_s[b_s^{\dagger\,\rm in}(p)b_s^{\rm in}(p)-d_s^{\rm in\,\dagger}(p)d_s^{\rm in}(p)],$$

and $b_s^{\rm in}(p)$ and $d_s^{\rm in}(p)$ are annihilation operators for electrons and positrons, respectively. In Appendix B the spectral decomposition of the infrared-coherent space is effected for four-momenta in the neighborhood of the electron mass shell. It is found to have the same singularity at the mass shell as the Wightman two-point function of the electron in the Feynman gauge, namely[8]

$$W(p)=c\,\frac{(\not{p}+m)}{\Gamma(1-\beta)}\,\frac{1}{2m^2}\,p^\mu\,\frac{\partial}{\partial p_\mu}\left[\frac{\theta(p^0)\theta(p^2-m^2)}{(p^2-m^2)^\beta}\right], \tag{1.13}$$

where $\beta=\alpha/\pi$, whereas the spectral decomposition of the Fock space has the singularity $c\theta(p^0)\delta(p^2-m^2)$. [The priority of the Feynman gauge in the present circumstances arises because it is the only one among the covariant gauges for which $A^{\rm in}$ satisfies the wave equation, so the ansatz (1.12) applies only in this gauge.] Besides providing support for the ansatz of infrared coherence, this result allows the normalization of the charged field to be related to the normalization of the asymptotic states, as is shown in Appendix B.

Of course the ansatz of infrared coherence is hardly new; it is supported by model calculations dating back to the 1930's.[9] There is little doubt that it would have found easy acceptance in the form (1.12) long before now, were it not for the fact that it directly contradicts the Gupta-Bleuler condition. To see this observe that because $B(x)$ is a free field, $B(x)=B^{\rm in}(x)=B^{\rm out}(x)$, the Gupta-Bleuler condition, with $B^{\rm in}=\partial\cdot A^{\rm in}$, reads

$$k\cdot a^{\rm in}(k)\Phi=0, \tag{1.14}$$

whereas the ansatz (1.12) gives

$$\lim_{\omega=0} k\cdot a^{\rm in}(k) = -(2\pi)^{-3/2}\sum_{a=1}^{s} e_a = -(2\pi)^{-3/2}Q, \tag{1.15}$$

where Q is the charge operator. Thus we immediately conclude from the ansatz that the Gupta-Bleuler condition has no solution in the charged sectors. This conclusion is in fact supported by recently published proofs that no localized charged state satisfies the Gupta-Bleuler condition.[10]

Because of this contradiction with the Gupta-Bleuler condition, the ansatz (1.12) has generally been avoided by various devices. For example, Kulish and Faddeev[11] solved the dynamical equations of quantum electrodynamics at asymptotic times and obtained a space characterized by the low-frequency limit (1.12). Then in order to satisfy the Gupta-Bleuler condition, they introduced a different space with low-frequency limit

$$\lim_{\omega\to 0} a^\mu(k) \sim \frac{-1}{(2\pi)^{3/2}}\sum_a e_a\left[\frac{p_a^\mu}{p_a\cdot k}-c^\mu(k)\right],$$

where $c^\mu(k)$ is a fixed vector function of k satisfying $k\cdot c(k)=1$. This space does not admit a representation of the Poincaré group. In recent work by this author,[12,13] the Gupta-Bleuler condition was maintained by introducing a retarded representation which is asymmetric between in and out states. In the retarded representation the incoming charged particles with charges and momenta, e_i and p_i, arrive unescorted by a coherent infrared photon wave, $\lim_{\omega\to 0}\omega\, a^{\mu\,\rm in}(k)=0$, whereas the outgoing particles, with charges and momenta, e_f and p_f, leave with the infrared photon escort

$$\lim_{\omega\to 0} a^{\mu\,\mathrm{out}}(k) \sim \frac{-1}{(2\pi)^{3/2}}\left(\sum_f \frac{e_f p_f}{p_f\cdot k} - \sum_i \frac{e_i p_i}{p_i\cdot k}\right). \tag{1.16}$$

It is transverse because of charge conservation, $\sum e_f - \sum e_i = 0$. As observed earlier,[14] the retarded representation has, for reasons which we shall review shortly, superselection sectors labeled by the momenta p_i of the incoming charged particles. Thus the initial state must be described by a density matrix $\rho(p_i)\delta(p_i - p_i')$ which is diagonal in the momenta of the incoming charged particles, but a wave function $\phi(p_i)$ with definite phase relations between different momentum components is without meaning. This is adequate for a scattering theory based on cross sections instead of scattering amplitudes. In fact it appears to be the most convenient formulation for practical calculations to typical accelerator experiments and it will be obtained from the present approach as a limiting case in Appendix C. However, the retarded representation does not allow an ordinary quantum-mechanical description of the incoming charged particles, as was pointed out to me by Swieca,[15] for the possibility of a localized or wave-packet state of a charged particle depends on the interference between different momentum components and this is excluded by the superselection rule just mentioned.

Thus attempts to maintain the Gupta-Bleuler condition on the asymptotic state space run into characteristic difficulties whose impact is reinforced by the negative result of rigorous quantum field theory that there exist no localized charged states satisfying this condition.[10] For these reasons we have felt compelled to abandon it. Instead we accept without reservation the infrared-coherence condition (1.12) and define physical states by the more general positivity requirement (1.9).

It is helpful to understand the origin of the superselection rule in the retarded representation. One way of obtaining it is simply to note that because of the low-frequency limit (1.16), the representation space for outgoing photons depends on the momenta of the incoming charged particles. States in different spaces cannot be superposed. Another derivation is obtained by an extension of the ingenious argument of Strocchi and Wightman,[2] who prove the charge superselection rule as follows. The integral form of the Gauss law,

$$Q = \lim_{R\to\infty} \int \vec{E}\cdot d\vec{S},$$

where the closed surface expands to include all space, shows that the total charge equals the integral of the flux of the electric field at spatial infinity. Because all observables are local and $\vec{E}$ is a local field, it follows that Q commutes with all observables, and hence defines superselection sectors. However, the same argument also shows that *the flux per unit solid angle at infinity in any spatial direction also defines a superselection sector.* In the retarded representation the vector potential at large spatial distances approaches the Liénard-Wiechert potential of the incoming charged particles

$$\lim_{|\vec{x}|\to\infty} A^\mu(x) \sim \sum_i \frac{e_i}{4\pi}\frac{p_i^\mu}{[(p_i\cdot x)^2 - p_i^2 x^2]^{1/2}}.$$

This gives for the flux per unit solid angle at spatial infinity in the direction $\hat{x}$

$$\lim_{r\to\infty} r^2 F^{0i}(t,r,\hat{x}) = \sum_a \frac{e_a}{4\pi}\frac{-\hat{x}^i E_a m_a^2}{[m_a^2 + (\vec{p}_a\cdot\hat{x})^2]^{3/2}},$$

$$\lim_{r\to\infty} r^2 F^{ij}(t,r,\hat{x}) = \sum_a \frac{e_a}{4\pi}\frac{(\hat{x}^i p_a^j - p_a^i \hat{x}^j) m_a^2}{[m_a^2 + (\vec{p}_a\cdot\hat{x})^2]^{3/2}}.$$

These quantities define superselection sectors for every $\hat{x}$ and thus for every set of values of the p_a. Consequently in the retarded representation, where incoming particles are dressed at large distances with their retarded Liénard-Wiechert potentials, different momentum components of the incoming charged particles cannot be superposed. Although this is no obstacle to a theory of cross sections, it does not allow a localized or wave-packet description of the incoming charged particles.

The retarded representation corresponds to the traditional method of calculating cross sections, namely by introducing a small photon mass λ, keeping a fixed, finite number of photons in the initial state, and summing over final states in which the number of photons emitted grows without limit. The resulting cross section is independent of λ because of the famous cancellation between real and virtual infrared divergences. However—and this is the important point—*the cancellation will not occur if the initial or final state is described by a wave function $\phi(p)$ for the charged particles.* It only occurs for cross sections, or, in other words, if the initial and final states are diagonal in the momenta of the charged particles. Here we see the superselection rule of the retarded representation at work, outlawing the superposition of momentum components which lies at the very heart of quantum mechanics. In the following sections an alternative approach is described which does allow wave functions for charged particles.

In Sec. II observables are identified by the principle of gauge invariance of the second kind, and it is proved that a substitute Gupta-Bleuler condition provides state spaces that satisfy the positivity requirement. Section III is devoted to the physical interpretation of these spaces and to reconciling the equation of motion on them with Maxwell's equations. In Sec. IV are some concluding reflections. In Appendix A the infrared-coherent space is described with some care and a basis is found in which the Poincaré transformations have the simple free-particle form. This basis is used in Appendix B to effect the spectral resolution of the infrared-coherent space, and the charged field ψ is normalized on-shell by matching the spectral functions of the infrared-coherent space to the spectral function of the electron propagator. In Appendix C it is shown how the present formulation leads to the standard cross-section formula of quantum electrodynamics.

II. GAUGE INVARIANCE AND PHYSICAL STATES

It is a curious fact that the dynamical content of quantum electrodynamics is formulated in terms of unobservable fields[16] A_μ and ψ. However, the principle of gauge invariance allows the identification of observables. Let $\lambda(x)$ be any real c-number solution to the wave equation $\partial^2\lambda(x)=0$. The Lagrangian (1.1) is invariant under the restricted gauge transformations of the second kind characterized by the gauge function λ,

$$A_\mu \to A_\mu + \partial_\mu \lambda \,, \tag{2.1a}$$

$$\psi \to \psi \exp(ie\lambda)\,, \tag{2.1b}$$

$$F_{\mu\nu} \to F_{\mu\nu}\,, \tag{2.1c}$$

$$B \to B. \tag{2.1d}$$

Our basic idea is to take as observables the local operators which are invariant under the gauge transformation (2.1). In order to state this in a convenient form we introduce the generator $G(\lambda)$ of the gauge transformation (2.1) and adopt the following definition:

Definition. The elements $\mathcal{O}$ of the algebra observables $\mathcal{A}$ are all local operators $\mathcal{O}$ on the indefinite-metric space which commute with $G(\lambda)$,

$$[G(\lambda), \mathcal{O}] = 0, \tag{2.2}$$

for all λ with $\partial^2\lambda(x)=0$, where $\lambda(x)$ is a real c number. The generator may be expressed in terms of $B(x)$,

$$G(\lambda) = \int \lambda(x)\overleftrightarrow{\partial}_0 B(x) d^3x\,, \tag{2.3}$$

as may be verified[17] from the commutation relations which hold for the renormalized fields[18,19]

$$[B(x), A_\mu(y)] = i\partial_\mu \Delta(x-y), \tag{2.4a}$$

$$[B(x), \psi(y)] = e\Delta(x-y)\psi(y), \tag{2.4b}$$

$$[B(x), F_{\mu\nu}(y)] = 0, \tag{2.4c}$$

$$[B(x), B(y)] = 0, \tag{2.4d}$$

where $\Delta(x) = (2\pi)^{-1}\epsilon(x^0)\delta(x^2)$ satisfies $\partial^2\Delta(x)=0$, $\Delta(0,x)=0$, and $\dot{\Delta}(0,\vec{x}) = \delta(\vec{x})$. The corresponding relations for the unrenormalized quantities are easily obtained from $\partial^2 B = 0$ and the canonical equal-time commutation relations. It is easy to show that $[G(\lambda), \mathcal{O}]=0$ for all λ with $\partial^2\lambda=0$ is equivalent to the condition that $\mathcal{O}$ commute with $B(x)$. We thus obtain as a necessary and sufficient condition for $\mathcal{O}$ to be an element of the algebra of observables

$$[B(x), \mathcal{O}] = 0, \tag{2.5}$$

which is the form we shall use in the following. This criterion coincides with the definition of strict gauge invariance.[2,20] The justification of our definition[21] is its simplicity and its inclusion of the observable fields *par excellence* $F_{\mu\nu}$ and J_μ. Let it be noted that it also includes B itself. However, the energy-momentum operator P_μ satisfies $[B(x), P_\mu] = i\partial_\mu B(x)$, so P_μ is not an observable and consequently the stress tensor $\theta_{\mu\nu}(x)$ is not either.

Let us now turn to the problem of finding vectors Φ which are elements of the indefinite-metric space $\mathcal{G}$ that are positive on the observables, namely $\langle \mathcal{O}\Phi, \mathcal{O}\Phi\rangle \geq 0$, $\langle\Phi,\Phi\rangle = 1$. Such vectors are physical states. Note first that in any irreducible representation of the algebra of observables, $B(x)$, which is itself an element of the algebra, will, by Schur's lemma, be represented by a c-number function, say $b(x)$, for the elements of the algebra of observables are precisely those operators that commute with $B(x)$. A Hilbert space representation of the algebra may be obtained from the state Φ by means of the Gelfand-Naimark-Segal (GNS) construction:[22] The Hilbert space consists of (the completion in norm of) the subspace of the indefinite-metric space $\mathcal{G}$ whose elements are of the form $\mathcal{O}\Phi$, for which the norm $\langle\mathcal{O}\Phi, \mathcal{O}\Phi\rangle$ is non-negative by hypothesis. Assuming this representation to be irreducible, as it will be if Φ is a pure state, we find that Φ satisfies the condition

$$\langle \mathcal{O}\Phi, B(x)\mathcal{O}'\Phi\rangle = b(x)\langle\mathcal{O}\Phi, \mathcal{O}'\Phi\rangle, \tag{2.6}$$

where $\mathcal{O}$ and $\mathcal{O}'$ are arbitrary elements of the algebra of observables and $b(x)$ is independent of $\mathcal{O}$ and $\mathcal{O}'$ and depends only on Φ. In particular, with $\langle\Phi,\Phi\rangle=1$, we have

$$b(x) = \langle\Phi, B(x)\Phi\rangle. \tag{2.7}$$

Because $B(x)$ is Hermitian and satisfies the wave equation, $b(x)$ is a real c-number solution of the wave equation $\partial^2 b(x) = 0$.

The infrared-coherence condition imposes a restriction on $b(x)$, for we have

$$B^{(-)}(x)=B^{\text{in}(-)}(x)$$
$$=\partial\cdot A^{\text{in}(-)}(x)$$
$$=\frac{-i}{(2\pi)^{3/2}}\int\frac{d^3k}{2\omega}\,k\cdot a^{\text{in}}(k)e^{-ik\cdot x},$$
$$\int d^3x\,\dot{B}^{(-)}(x)=-\tfrac{1}{2}(2\pi)^{3/2}k\cdot a^{\text{in}}(k)\Big|_{\omega=0},$$

and hence by Eq. (1.15)

$$\int\dot{B}^{(-)}(x)d^3x=\tfrac{1}{2}Q\,, \tag{2.8}$$

where Q is the charge operator.[23] Hence we have, with $B^{(+)}(x)=B^{(-)}(x)^\dagger$,

$$\int\dot{b}(x)d^3x=q, \tag{2.9}$$

where q is the charge of the state Φ.

It is clear that a sufficient condition for Eq. (2.6) to hold is that Φ satisfy the condition

$$B^{(-)}(x)\Phi=b^{(-)}(x)\Phi, \tag{2.10}$$

for we have $B(x)=B^{(-)}(x)+B^{(-)}(x)^\dagger$ and for any $\mathcal{O}$ in the algebra of observables

$$B^{(-)}(x)\mathcal{O}\Phi=\mathcal{O}B^{(-)}(x)\Phi=b^{(-)}(x)\mathcal{O}\Phi \tag{2.11}$$

In Appendix A we will prove the following lemma:

Lemma. If $\Phi\in\mathcal{G}$ satisfies the condition $B^{(-)}(x)\Phi=b^{(-)}(x)\Phi$, then Φ has non-negative norm $\langle\Phi,\Phi\rangle\geq0$. Since the condition is linear it in fact defines a linear subspace $\mathcal{G}_{[b]}$ of vectors of non-negative norm. The basic idea of the proof is elementary. The unitary transformation which produces the infrared-coherent space from the Fock space, $a^\mu(k)\to\mathfrak{U}a^\mu(a)\mathfrak{U}^\dagger=a^\mu(k)+s^\mu(k)$, where $s^\mu(k)$ commutes with $a^\mu(k)$, leaves the inner products invariant. Hence the condition $k\cdot a(k)\Psi=0$, which is known to define a subspace of non-negative norm in the Fock space, maps into the condition $k\cdot a(k)\Phi=b(k)\Phi$ in the infrared-coherent space provided only that $s^\mu(k)$ satisfies $k\cdot s(k)=-b(k)$.

Equation (2.11) tells us that the subspace of non-negative norm $\mathcal{G}_{[b]}$ is invariant under application of any element $\mathcal{O}$ of the algebra of observables. This gives the principal result of this article in the form of the following theorem:

Theorem. If $\Phi\in\mathcal{G}$ satisfies the condition $B^{(-)}(x)\Phi=b^{(-)}(x)\Phi$ then Φ is non-negative on the observables $\langle\mathcal{O}\Phi,\mathcal{O}\Phi\rangle\geq0$ and so, provided only that it have unit norm $\langle\Phi,\Phi\rangle=1$, it defines a physical state.

We have the following remarks:

(1) From what has been said before this condition has a solution only if $b^{(-)}(x)$ is the negative-frequency part of a solution of the wave equation with $\int\dot{b}^{(-)}(x)d^3x=q/2$, where q is an eigenvalue of the charge operator Q.

(2) The linear subspace $\mathcal{G}_{[b]}$ may be completed in the norm[24] to yield a physical Hilbert space $\mathcal{H}_{[b]}$ and the observables $\mathcal{O}$ are promoted to operators on $\mathcal{H}_{[b]}$.

The superposition $\Phi=c_1\Phi_1+c_2\Phi_2$ of two state vectors Φ_1 and Φ_2 with different coherence functions $b_1\neq b_2$ does not define another state vector. To see why this is so, let Φ_i, $i=1,2$ be the normalized completely coherent states

$$\Phi_i=U_i\Omega,$$

where Ω is the vacuum state and

$$U_i=\exp\left\{-\int\frac{d^3k}{2\omega}\,n(k)\cdot[a^\dagger(k)b_i(k)-a(k)b_i^*(n)]\right\}.$$

Here $n^\mu(k)=\tau^\mu/\tau\cdot k$, so $k\cdot n(k)=1$, τ^μ is a unit future timelike vector, and $b_i(k)$ vanishes at the origin so $n^\mu(k)b_i(k)$ is a regular photon wave function. These states satisfy $\langle\Phi_i,\Phi_i\rangle=1$ and also $a^\mu(k)\Phi_i=n^\mu(k)b_i(k)\Phi_i$, so $k\cdot a(k)\Phi_i=b_i(k)\Phi_i$ as required for Φ_i to be a physical state. In order that Φ_1 and Φ_2 be superposable we must have $|\langle\Phi_1,\Phi_2\rangle|\leq1$; however, we find

$$|\langle\Phi_1,\Phi_2\rangle|^2=\exp\left[\int\frac{d^3k}{2\omega}\frac{1}{(\tau\cdot k)^2}|b_1(k)-b_2(k)|^2\right]>1$$

for $b_1\neq b_2$. Only the mixture, defined as usual by $x_1\langle\Phi_1,\mathcal{O}\Phi_1\rangle+x_2\langle\Phi_2,\mathcal{O}\Phi_2\rangle$, with $x_1+x_2=1, 0\leq x_i\leq1$, is a state. If the corresponding linear subspaces $\mathcal{G}_{[b_1]}$ and $\mathcal{G}_{[b_2]}$ are separately completed to different or, in other words, orthogonal Hilbert spaces $\mathcal{H}_{[b_1]}$ and $\mathcal{H}_{[b_2]}$, then, as elements of these spaces, Φ_1 and Φ_2 are thereby declared to be orthogonal, although as elements of the indefinite-metric space they are not orthogonal; the problem is rather that their inner product exceeds 1 in absolute value.

Because observables act within a given Hilbert space $\mathcal{H}_{[b]}$ characterized by a given coherence function $b(x)$, we obtain a vast class of superselection rules, where each superselection sector is characterized by a $b(x)$ subject only to the condition $\partial^2b(x)=0$, $\int\dot{b}(x)d^3x=q$. This is a consequence of the fact that all observables are invariant under the restricted gauge transformations of the second kind.

It is possible for two different state vectors Φ_1, $\Phi_2\in\mathcal{G}$ to yield the same expectation values for all observables $\mathcal{O}$, $\langle\Phi_1,\mathcal{O}\Phi_1\rangle=\langle\Phi_2,\mathcal{O}\Phi_2\rangle$. In such a case we say they are equivalent, $\Phi_1\sim\Phi_2$. Equivalent state vectors must be viewed as representing the same physical state which is best identified with the equivalence class of state vectors. Viewed as vectors in the indefinite-metric space $\mathcal{G}$, equivalent state vectors may look rather different.[25] However, it is sufficient to find one representative from each equivalence class. The question as to whether the substitute Gupta-Bleuler condition $B^{(-)}(x)\Phi = b^{(-)}(x)\Phi$ yields a representative of every physical state in $\mathcal{G}$ is left open. However, it is true that the functions $b(x)$ do provide a complete parametrization of the possible eigenvalues of the generators $G(\lambda)$, Eq. (2.3), of the restricted gauge transformations of the second kind, Eq. (2.1). And for every such function $b(x)$, the substitute Gupta-Bleuler condition does provide a coherent subspace of physical states $\mathcal{G}_{[b]}$ which may be completed in norm to a physical Hilbert space $\mathcal{H}_{[b]}$ and on which the generators $G(\lambda)$ have the eigenvalues

$$G(\lambda)\mathcal{H}_{[b]} = \int \lambda(x)\overleftrightarrow{\partial}_0 b(x) d^3x\, \mathcal{H}_{[b]}. \tag{2.12}$$

III. PHYSICAL INTERPRETATION

In the description of physical states just obtained, the generators of the restricted gauge transformations of the second kind $G(\lambda)$ play the same role as has traditionally been accorded to the charge Q, which is the generator of gauge transformations of the first kind. Physical states fall into different superselection sectors $\mathcal{H}_{[b]}$ characterized by different eigenvalues of $G(\lambda) = \int \lambda(x)\overleftrightarrow{\partial}_0 b(x) d^3x$. The corresponding superselection rules subsume the charge superselection rule, for if $\lambda =$ constant, then $G(\lambda)$ generates a gauge transformation of the first kind with eigenvalue $q\lambda$ where $q = \int \dot{b}(x) d^3x$ is the charge eigenvalue. Thus it is not unnatural that in the charged sectors, $q \neq 0$, we must have $b(x) \neq 0$, although the Gupta-Bleuler condition calls for $b(x) = 0$. What does require interpretation is the unexpected form of Maxwell's equations in $\mathcal{H}_b$,

$$\partial_\mu F^{\mu\nu} = J^\nu - \partial^\nu b, \tag{3.1}$$

and the fact that $\mathcal{H}_b$ is not invariant under Poincaré transformations.

Because $b(x)$ satisfies $\partial^2 b = 0$, the quantity $-\partial^\nu b$ may be interpreted as a conserved classical external current

$$j_\nu^c(x) \equiv -\partial_\nu b(x), \tag{3.2}$$

which is added to the quantum-mechanical current J_ν as a source for the Maxwell field

$$\partial_\mu F^{\mu\nu} = J^\nu + j^{\nu c}. \tag{3.3}$$

The total charge carried by the classical current $\int j_0^c d^3x = -\int \dot{b} d^3x = -q$ is the negative of the eigenvalue q of the quantum-mechanical charge $Q = \int J_0 d^3x$ in the superselection sector $\mathcal{H}_b$, so the classical current may be thought of as being induced by the quantum current. In the neutral sector we may set $j^c = 0$ and maintain the Gupta-Bleuler condition. However, in the charged sectors this is not possible and within a given superselection sector the equations of motion of local quantum electrodynamics must be interpreted as the dynamics in the presence of an external classical current j^c. The total charge which acts as the source of the Maxwell field, namely the sum of the quantum and induced classical charges, is now zero and, as we shall discuss shortly, the $1/r^2$ part of the Maxwell field vanishes at large spacelike distances.

The presence of an external current implies a violation of Poincaré invariance. Since j^c is necessarily different from zero in the charged sectors, the presence of electric charge causes a spontaneous breakdown of Poincaré invariance. This may be expressed formally as

$$U(a,\Lambda)\mathcal{H}_{[b]} = \mathcal{H}_{[b']}, \tag{3.4}$$

where

$$b'(x) = b_{a,\Lambda}(x) = b(\Lambda^{-1}(x-a)), \tag{3.5}$$

which states that the Poincaré transformation $U(a,\Lambda)$ does not act within the physical space $\mathcal{H}_{[b]}$, but maps it onto a different orthogonal space.[26] This relation follows from

$$\begin{aligned} &B^{(-)}(x)\Phi = b^{(-)}(x)\Phi, \\ &B^{(-)}(x)U(a,\Lambda)\Phi = U(a,\Lambda)B^{(-)}(\Lambda^{-1}(x-a))\Phi \\ &\qquad = b^{(-)}(\Lambda^{-1}(x-a))\Phi. \end{aligned} \tag{3.6}$$

Lest it be thought that this does too great a violence to our physical intuition, it should be kept in mind that only local quantities are observables, so the generators of Poincaré transformations are not observable. The function $b(x)$, and hence also $j_\nu^c(x)$, may be chosen to vanish in any given finite region of space-time—for example, in the laboratory while an experiment is performed—consistent with the conditions $\partial^2 b(x) = 0$ and $\int \dot{b}(x) d^3x = q$. It is found that whenever $b(x)$ vanishes, the energy-momentum tensor derived from the Lagrangian (1.1) reduces to the usual Maxwellian energy-momentum tensor. Thus, although it is not possible to maintain traditional electrodynamics globally, it may be done in any given finite region of space-time.

Of course if local quantum electrodynamics is a good guide, then some physical effects should be well accounted for by a finite classical current j^c, or by a mixture of states with different j^c. For example, the scattering of an electron will, to some extent, be influenced by the preceding process which originally produced the electron, say photoionization of an atom. This influence may be adequately accounted for by appropriate classical currents and incident radiation, without introducing the dynamical degrees of freedom of the original atom and the remaining ion. In fact the superselection rule discussed in the Introduction, which prevents the formation of wave-packet states of the electron in the retarded representation (for which $j^c = 0$), suggests that such influences cannot be neglected in experiments involving localization of the electron, such as a time-delay experiment. On the other hand, as shown in Appendix C, if $b(x)$ contains only very low-frequency components, then it has only negligible effects on cross sections where, by definition, intensities as a function of asymptotic momenta are measured.

Let us look at the space-time properties of the radiation field implied by the infrared-coherence condition (1.12). As discussed in Ref. 12, the second term on the right-hand side of the Yang-Feldman equation,

$$A_\mu(x) = A_\mu^{\text{in}}(x) + \int \Delta^{\text{ret}}(x-y) J_\mu(y) d^4y\,, \qquad (3.7)$$

reduces, at early times, to the Liénard-Wiechert potential of the incoming particles

$$\lim_{t\to-\infty} A_\mu(x) = A_\mu^{\text{in}}(x) + \frac{1}{4\pi}\int \frac{d^3p}{2E}\frac{\rho^{\text{in}}(p)p_\mu}{[(p\cdot x)^2 - p^2x^2]^{1/2}}\,, \qquad (3.8)$$

where

$$\rho^{\text{in}}(p) = -e\sum_s [b_s^\dagger(p)b_s(p) - d_s^\dagger(p)d_s(p)]$$

is the charge density operator in momentum space and $b_s(p)$ and $d_s(p)$ are annihilation operators for incoming electrons and positrons, respectively. Let us now evaluate, with suppression of the "in" label on $a_\mu(k)$,

$$A_\mu^{\text{in}}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{2\omega}[a_\mu(k)e^{-ik\cdot x} + a_\mu^\dagger(k)e^{ik\cdot x}] \qquad (3.9)$$

at large distances, or in other words,

$$A_\mu^{\text{in}}(\lambda x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d\hat{k}}{2}\int_0^\infty d\omega[\omega a_\mu(k)e^{-i\lambda\omega\hat{k}\cdot x} + \omega a_\mu^\dagger(k)e^{i\lambda\omega\hat{k}\cdot x}]$$

with $\hat{k}\cdot x \equiv t - \hat{k}\cdot\vec{x}$, for large values of $\lambda > 0$. Let it be understood that matrix elements of this operator equation are taken, so $\omega a_\mu(k)$ and $\omega a_\mu^\dagger(k)$ are smooth functions of ω. One has, by partial integration

$$A_\mu^{\text{in}}(\lambda x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d\hat{k}}{2}\left[\frac{\omega a_\mu(\omega,\hat{k})}{\epsilon + i\lambda\hat{k}\cdot x} + \frac{\omega a_\mu^\dagger(\omega,\hat{k})}{\epsilon - i\lambda\hat{k}\cdot x}\right]\Bigg|_{\omega=0} + O\left(\frac{1}{\lambda^2}\right).$$

The infrared-coherence condition (1.12) states

$$\omega a_\mu(\omega,\hat{k}) = \frac{-1}{(2\pi)^{3/2}}\int \frac{d^3p}{2E}\frac{\rho^{\text{in}}(p)p_\mu}{p\cdot\hat{k}}\,,$$

which gives

$$A_\mu^{\text{in}}(\lambda x) = \frac{-\theta(-x^2)}{4\pi\lambda}\int \frac{d^3p}{2E}\frac{\rho^{\text{in}}(p)p_\mu}{[(p\cdot x)^2 - p^2x^2]^{1/2}} + O\left(\frac{1}{\lambda^2}\right). \qquad (3.10)$$

This looks like the Liénard-Wiechert potential; however, it is restricted to spacelike regions and has the opposite sign. Substitution into Eq. (3.8) gives for $x^0 < 0$

$$A_\mu(\lambda x) = \frac{\theta(x^2)}{4\pi\lambda}\int \frac{d^3p}{2E}\frac{\rho^{\text{in}}(p)p_\mu}{[(p\cdot x)^2 - p^2x^2]^{1/2}} + O\left(\frac{1}{\lambda^2}\right).$$

Thus the effect of the incident infrared-coherent radiation at early times is to cancel the Liénard-Wiechert potential of the incoming charged particles at large spacelike distances. There is an analogous formula for late times, and we may write, for large $\lambda > 0$, as a weak asymptotic limit

$$A_\mu(\lambda x) = \frac{\theta(x^2)}{4\pi\lambda}\int \frac{d^3p}{2E}\frac{[\theta(-x^0)\rho^{\text{in}}(p) + \theta(x^0)\rho^{\text{out}}(p)]p_\mu}{[(p\cdot x)^2 - p^2x^2]^{1/2}} + O\left(\frac{1}{\lambda^2}\right). \qquad (3.11)$$

We observe that at fixed t and large r, the $1/r$ term of the vector potential $A_\mu(t,x)$ has been canceled, and hence also the $1/r^2$ in the Maxwell field $F_{\mu\nu}(t,\vec{x})$. Although this at first appears to contradict some of our earliest experience with electromagnetic theory, reflection suggests that this experience can be adequately accounted for by the Liénard-Wiechert potentials inside the future and past cones at asymptotic distances.

The weak limit (3.11) may easily be verified in lowest-order perturbation theory. To order e we have

$$A_\mu(x) = A_\mu^{\text{in}}(x) + [\Delta^{\text{ret}}(-e):\bar{\psi}^{\text{in}}\gamma_\mu\psi^{\text{in}}:](x)\,, \qquad (3.12\text{a})$$

$$\psi(x) = \psi^{\text{in}}(x) + [S^{\text{ret}}(-e)A^{\text{in}}\psi^{\text{in}}](x)\,. \qquad (3.12\text{b})$$

As an example, take the expectation value of $A_\mu(x)$ on the charged vector $\Phi = \int d^4x\,\bar{\psi}(x)\phi(x)\Omega$, where Ω is the vacuum state and $\phi(x)$ is a smearing spinor function. One finds, for large $\lambda > 0$, to order e,

$$\langle\Phi, A_\mu(\lambda x)\Phi\rangle = \frac{-e\theta(x^2)}{4\pi\lambda}\int d^4p\,\frac{p^\mu}{[(p\cdot x)^2 - p^2x^2]^{1/2}} \times \bar{\phi}(p)(\not{p} + m)\phi(p)\delta(p^2 - m^2)\theta(p^0) + O(1/\lambda^2),$$

where $\phi(p) = (2\pi)^{-3/2}\int e^{ip\cdot x}\phi(x)d^4x$. Here the Liénard-Wiechert potential produced by the source contribution to A_μ, Eq. (3.12a), is canceled at large spacelike distances by the dependence of the vector Φ on A^{in} through Eq. (3.12b).

As a final remark, we observe that Eqs. (3.12) allow us to calculate the commutator of A and ψ to lowest order in e. Setting $A_\mu(x) = \int A_\mu(k) e^{-ik\cdot x} d^4k$, $\psi(x) = \int \psi(p) e^{-ip\cdot x} d^4p$, we find, for k_μ arbitrarily close to 0 and p^2 arbitrarily close to m^2,

$$[A_\mu(k), \psi(p)] = -ep_\mu \left[\frac{\epsilon(p^0)\delta(p\cdot k)}{k^2} + \frac{\epsilon(k^0)\delta(k^2)}{p\cdot k}\right]\psi(p). \tag{3.13}$$

(It is likely that this commutation relation is exact.) The first term represents the Liénard-Wiechert potential and the second is the coherent infrared radiation. The same cancellation occurs at large spacelike separation, for we find, for large $\lambda > 0$ and p^2 arbitrarily close to m^2,

$$[A_\mu(\lambda x), \psi(p)] = \frac{e}{4\pi\lambda} \frac{\theta(x^2)p}{[(p\cdot x)^2 - p^2x^2]^{1/2}} \psi(p) + O\left(\frac{1}{\lambda^2}\right). \tag{3.14}$$

IV. CONCLUDING REMARKS

We have outlined a physical interpretation of quantum electrodynamics formulated in terms of the unobservable fields A_μ and ψ which are operators on an indefinite-metric space $\mathcal{G}$. It is based on the identification of observables as all local quantities which are invariant under the restricted gauge transformations of the second kind,

$$A_\mu \to A_\mu + \partial_\mu \lambda, \quad \psi \to \psi e^{ie\lambda}, \tag{4.1}$$

with $\partial^2\lambda = 0$, $\lambda = c$ number, or, in other words, all local quantities which commute with the generators

$$G(\lambda) = \int \lambda(x) \overleftrightarrow{\partial}_0 B(x) d^3x, \tag{4.2}$$

where $B(x) = \partial\cdot A(x)$ in the Feynman gauge. A physical state is defined to be any vector in $\mathcal{G}$ which is positive on the observables.

Although it is attractive to contemplate a formulation in terms of observables acting on a positive metric space, such a formulation will not, in the opinion of this author, avoid the issues raised here. In particular,

$$\lim_{r\to\infty} r^2 F_{\mu\nu}(t, r, \hat{x}), \tag{4.3}$$

the flux per unit solid angle of the electromagnetic field at spatial infinity on the direction $\hat{x}$ necessarily commutes with all observables and hence defines superselection sectors. (Note that the flux at infinity of a charge Q at rest is not Lorentz-covariant.) In the local formulation presented here this flux is automatically zero. On the other hand, the generators (4.2) of the restricted gauge transformations of the second kind commute with all observables and hence, by Schur's lemma, they define superselection sectors labeled by the c-number function $b(x)$, which is the value of $B(x)$ in a particular superselection sector. The coherence function $b(x)$ is restricted by

$$\partial^2 b(x) = 0 \tag{4.4a}$$

because $\partial^2 B(x) = 0$, and by

$$\int \dot{b}(x) d^3x = q, \tag{4.4b}$$

where q is an eigenvalue of the charge operator, in order that (4.2) generate a gauge transformation of the first kind when $\lambda(x) = \text{constant}$.

Those who find the indefinite metric a high price to pay for manifest covariance and locality should bear in mind that the indefinite-metric space provides a concise encoding system for positive metric representations of the algebra of observables, for as we have shown, for every function $b(x)$ consistent with Eqs. (4.4), the indefinite-metric space $\mathcal{G}$ contains a subspace of non-negative norm $\mathcal{G}_{[b]}$, defined by the condition

$$B^{(-)}(x)\Phi = b^{(-)}(x)\Phi, \tag{4.5}$$

where $B^{(-)}$ and $b^{(-)}$ are negative-frequency parts, and furthermore every such subspace $\mathcal{G}_{[b]}$ is invariant under the observables so that it may be completed in norm to a physical Hilbert space $\mathcal{H}_{[b]}$. The existence of numerous positive subspaces $\mathcal{G}_{[b]}$ in $\mathcal{G}$ corresponds to strong positivity conditions on the Wightman functions, as exemplified by Eq. (B36).

Finally we comment on the experimental predictions of the present formulation of quantum electrodynamics. Two types of experiments must be distinguished. The first is the standard cross-section experiment in which intensities as a function of the incident and outgoing charged particle momenta are measured, and influences such as the mode of preparation of the incident particles are minimized. For such cross sections the predictions of the present time-symmetric formalism coincide with the traditional or retarded formalism (in which the number of incident photons is finite and a sum over the final photon number is effected), as shown in Appendix C. However, there is a second type of experiment in which the phase relations of the wave function of the incident charged particles are important. One example is a time-delay measurement. Another occurs when the initial state is prepared by refocusing a pair of particles which emerge from a common space-time event such as a decay or scattering process. The traditional formalism is inadequate for the second type of experiment because the famous cancellation of real and virtual infrared divergences will not occur. [If the initial state is represented by a density matrix $\rho(p_i, p_i')$ which is not diagonal in p_i and p_i', the real infrared divergence will depend on $p_i \cdot p_i'$ whereas the virtual infrared divergence is independent of this variable.] In the present formulation this transition probability is finite. It depends in an essential way on the coherence function $b(k)$. The support of $b(k)$ may be thought of as representing the distance of the charged particles from their mass shell which corresponds to the time elapsed since their previous interaction. Although the distant Coulombic tail does not cause appreciable recoil, and thus does not affect cross sections, it does produce significant delays which depend on the incident momenta and the time elapsed since previous interaction.

ACKNOWLEDGMENT

This work was undertaken because of the difficulties with the retarded representation which Professor Swieca pointed out to me in the course of several pleasurable and illuminating discussions. I wish to express my gratitude to Professor Swieca and to Richard Brandt, Dieter Maison, and Filippo Neri for stimulating conversations.

APPENDIX A: INFRARED-COHERENT SPACE IN THE MOMENTUM BASIS

The representation space on which the infrared-coherence condition (1.12) holds can be constructed by shifting the annihilation operator

$$a^\mu(k) = a_f^\mu(k) + \frac{1}{(2\pi)^{3/2}} \int d^4p \, \frac{\rho(p)p^\mu}{p\cdot k} f(k), \qquad \text{(A1)}$$

where $a_f^\mu(k)$ is a Fock representation and $f(k) = f(\omega, \hat{k})$ is a cutoff function with $f(0,\hat{k}) = 1$. However, this is not convenient because it makes the Poincaré transformations appear complicated, with an intricate dependence on $f(k)$. We will instead construct the momentum basis from scratch.[27]

Let $\mathcal{G}_\gamma^{(1)}$ be the one-photon representation space whose elements are the one-photon wave functions $\phi^\mu(k) = \phi^\mu(\omega, \hat{k})$ with inner product

$$\langle \phi_1, \phi_2 \rangle = \int \frac{d^3k}{2\omega}\, \phi_1^{\mu *}(k)(-g_{\mu\nu})\phi_2^\nu(k), \qquad \text{(A2a)}$$

$$\langle \phi_1, \phi_2 \rangle = \int d\hat{k} \int_0 \frac{d\omega}{2\omega} [\omega\phi_1^{\mu *}(k)](-g_{\mu\nu})[\omega\phi_2^\nu(k)]. \qquad \text{(A2b)}$$

We wish to extend this scheme to include wave functions ϕ which behave like $ep/p\cdot k$ or, more generally, like $\phi_R(\hat{k})/\omega$ at the origin, and for which the inner product (A2b) is divergent at the origin.

Before we can effect the extension, we must specify a bit more precisely the original space $\mathcal{G}_\gamma^{(1)}$ which we wish to extend. Because $(-g_{\mu\nu})$ is indefinite, the inner product (A2) is indefinite and does not serve to define a Hilbert space. To deal with this problem, it has been traditional,[1,2] as a matter of mathematical convenience, to introduce a Hilbert space topology, the so-called "large Hilbert space" by means of the noncovariant and nonphysical Euclidean inner product

$$(\phi_1, \phi_2) \equiv \int \frac{d^3k}{2\omega} \sum_{\mu=0}^{3} \phi_1^{\mu *}(k)\phi_2^\mu(k).$$

In fact this topology is not suitable for extension to wave functions with a singularity at the origin such as $1/\omega$, for the Hilbert space topology ignores behavior at a single point (it identifies wave functions which are equal almost everywhere) and so it is not natural as a starting point for extension to a class of functions whose behavior is specified at a single point, namely the origin. We choose instead a Schwartz topology[28] and stipulate that the wave functions $\phi^\mu(\omega, \hat{k}) \in \mathcal{G}_\gamma^{(1)}$ be continuous infinitely differentiable functions on the product of the unit sphere $\hat{k}^2 = 1$ and the half-line $\omega \geq 0$, and that are of fast decrease as $\omega \to \infty$.

To accommodate wave functions which behave at the origin like $1/\omega$, we introduce the extended one-photon space $\mathcal{G}_\gamma^{(1)E} \supset \mathcal{G}_\gamma^{(1)}$ which consists of wave functions $\phi^\mu(k)$ such that $\omega\phi^\mu(k)$ is regular:

$$\phi^\mu(k) \in \mathcal{G}_\gamma^{(1)E} \text{ if and only if } \omega\phi^\mu(k) \in \mathcal{G}_\gamma^{(1)}. \qquad \text{(A3)}$$

We call $\phi(k) \in \mathcal{G}_\gamma^{(1)E}$ a "regular" wave function if $\phi(k) \in \mathcal{G}_\gamma^{(1)}$, and we call it singular otherwise. If ϕ_1 and ϕ_2 are both singular, the inner product (A2) is divergent.

We wish to construct an inner product $\langle \phi_1, \phi_2 \rangle$ for $\phi_1, \phi_2 \in \mathcal{G}_\gamma^{(1)E}$ with the following properties:

(1) Hermitian

$$\langle \phi_1, \phi_2 \rangle = \langle \phi_2, \phi_1 \rangle^*, \qquad \text{(A4a)}$$

(2) sesquilinear

$$\langle a\phi_1, b\phi_2 \rangle = a^* b \langle \phi_1, \phi_2 \rangle, \qquad \text{(A4b)}$$

(3) it must coincide with the old inner product (A2) for ϕ_1, ϕ_2 regular,

(4) translational invariance

$$\langle \phi_1^a, \phi_2^a \rangle = \langle \phi_1, \phi_2 \rangle \text{ for } \phi^{a\mu}(k) = e^{ik\cdot a}\phi^\mu(k), \qquad \text{(A4c)}$$

(5) Lorentz invariance

$$\langle \phi_1^\Lambda, \phi_2^\Lambda \rangle = \langle \phi_1, \phi_2 \rangle \text{ for } \phi^{\Lambda\mu}(k) = \Lambda^\mu_\nu \phi^\nu(\Lambda^{-1}k). \qquad \text{(A4d)}$$

We postpone the discussion of positivity and do not require it at present. The obvious regularization of (A2), namely

$$\langle\phi_1,\phi_2\rangle_1 \equiv -\tfrac{1}{2}\int d\hat{k}\int_0^\infty d\omega \ln\omega\,\frac{\partial}{\partial\omega}\times[\omega\phi_1^{\mu *}(k)(-g_{\mu\nu})\omega\phi_2^{\nu *}(k)] \tag{A5}$$

satisfies all requirements except Lorentz invariance.

To discuss Lorentz invariance we note some elementary facts. The transformation law

$$k'^{\mu} = \Lambda^{\mu}_{\nu} k^{\nu} \tag{A6}$$

for lightlike vectors $k^{\mu} = \omega(1, \hat{k})$ implies that light rays $\hat{k}$ transform into each other according to

$$\hat{k}'^{i} = \frac{\Lambda^{i}_{0} + \Lambda^{i}_{j}\hat{k}^{j}}{\Lambda^{0}_{0} + \Lambda^{0}_{j}\hat{k}^{j}} \tag{A7a}$$

and that frequencies transform multiplicatively

$$\omega' = (\Lambda^{0}_{0} + \Lambda^{0}_{j}\hat{k}^{j})\omega, \tag{A7b}$$

so that the tip of the light cone $\omega = 0$ is an invariant point. It is convenient to introduce the four-index quantity

$$\hat{k}^{\mu} \equiv k^{\mu}/\omega = (1, \hat{k}) \tag{A8}$$

and write these transformation laws more compactly,

$$\hat{k}'^{\mu} = \frac{\Lambda^{\mu}_{\nu}\hat{k}^{\nu}}{\Lambda^{0}_{\nu}\hat{k}^{\nu}}, \tag{A9a}$$

$$\omega' = \Lambda^{0}_{\nu}\hat{k}^{\nu}\omega\,. \tag{A9b}$$

Because the measure

$$d^3k/2\omega = \tfrac{1}{2}\,d\hat{k}(d\omega/\omega)\omega^2 = \tfrac{1}{2}\,d\hat{k}'(d\omega'/\omega')\omega'^2$$

is invariant, and $d\omega/\omega = d\omega'/\omega'$ by Eq. (A9b), one has for the Jacobian of the transformation (A7a) or (A9a)

$$d\hat{k}' = \frac{d\hat{k}}{(\Lambda^{0}_{\nu}\hat{k}^{\nu})^2}. \tag{A10}$$

A simple calculation now gives for the inner product (A5)

$$\langle\phi_1^{\Lambda},\phi_2^{\Lambda}\rangle_1 = \langle\phi_1,\phi_2\rangle_1 + \tfrac{1}{2}\int d\hat{k}\ln(\Lambda^{0}_{\mu}\hat{k}^{\mu})\phi^{\mu *}_{1,R}(\hat{k})(-g^{\mu\nu})\phi^{\nu}_{2,R}(\hat{k}) \tag{A11}$$

instead of the Lorentz invariance condition (A4d). Here we have introduced the residue of the wave function at zero frequency

$$\phi^{\mu}_{R}(\hat{k}) \equiv [\omega\phi^{\mu}(\omega,\hat{k})]|_{\omega=0}\,, \tag{A12}$$

which depends on the light ray $\hat{k}$ and transforms according to

$$\phi^{\Lambda\mu}_{R}(\hat{k}') = (\Lambda^{0}_{\lambda}\hat{k}^{\lambda})\Lambda^{\mu}_{\nu}\phi^{\nu}_{R}(\hat{k}), \tag{A13}$$

where $\hat{k}'$ and $\hat{k}$ are related by Eq. (A9a).

We seek to correct the inner product $\langle\,,\rangle_1$ and make it Lorentz-invariant by adding to it a term which vanishes when both functions have vanishing residue. The most obvious candidate,

$$\langle\phi_1,\phi_2\rangle_2 \equiv \tfrac{1}{2}\int d\hat{k}\,\phi^{\mu *}_{1,R}(\hat{k})(-g_{\mu\nu})\phi^{\nu}_{2,R}(\hat{k}), \tag{A14}$$

turns out to be Lorentz-invariant

$$\langle\phi_1^{\Lambda},\phi_2^{\Lambda}\rangle_2 = \langle\phi_1,\phi_2\rangle_2\,, \tag{A15}$$

so it cannot correct the Lorentz noninvariance of $\langle\,,\rangle_1$. Because it also satisfies all the other requirements (A4), any real multiple of it may be added to the inner product. Another possibility which suggests itself is

$$\int d\hat{k}_1 d\hat{k}_2 \phi^{\mu}_{1,R}(\hat{k}_1)\,\frac{(-g_{\mu\nu})}{1-\hat{k}_1\cdot\hat{k}_2}\,\phi_{2,R}(\hat{k}_2).$$

It appears to satisfy all the requirements (A4), but it diverges at $\hat{k}_1 = \hat{k}_2$. Consider instead its regularized form

$$\langle\phi_1,\phi_2\rangle_0 \equiv \frac{1}{8\pi}\int d\hat{k}_1 d\hat{k}_2 \phi^{\mu *}_{1,R}(\hat{k}_1)\,\frac{(-g_{\mu\nu})}{1-\hat{k}_1\cdot\hat{k}_2}\times[\phi^{\nu}_{2,R}(\hat{k}_2) - \phi^{\nu}_{2,R}(\hat{k}_1)]\,, \tag{A16}$$

with normalization chosen for later convenience. Despite appearances it is Hermitian-symmetric because the difference $\langle\phi_1,\phi_2\rangle_0 - \langle\phi_2,\phi_1\rangle_0^*$ has an integrand which is odd under the interchange $\hat{k}_1 \leftrightarrow \hat{k}_2$. It also satisfies the other requirements except Lorentz invariance. Instead, from Eqs. (A10) and (A13) one has

$$\langle\phi_1^{\Lambda},\phi_2^{\Lambda}\rangle_0 = \langle\phi_1,\phi_2\rangle_0 - \tfrac{1}{2}\int d\hat{k}\ln(\Lambda^{0}_{\lambda}\hat{k}^{\lambda})\phi^{\mu *}_{1R}(\hat{k})(-g_{\mu\nu})\phi^{\nu}_{2R}(\hat{k}), \tag{A17}$$

where we have used

$$\int d\hat{k}'\,\frac{1}{1-\hat{k}\cdot\hat{k}'}\left(1-\frac{\Lambda^{0}_{\mu}\hat{k}^{\mu}}{\Lambda^{0}_{\mu}\hat{k}'}\right) = -4\pi\ln(\Lambda^{0}_{\mu}\hat{k}^{\mu})\,. \tag{A18}$$

Upon comparison with Eq. (A11), one observes that the sum $\langle\,,\rangle_0 + \langle\,,\rangle_1$ is Lorentz-invariant. Hence we take as the inner product, defined on the extended one-photon space $\mathcal{G}_{\gamma}^{(1),E}$, and satisfying all the requirements (A4),

$$\langle\phi_1,\phi_2\rangle \equiv \langle\phi_1,\phi_2\rangle_0 + \langle\phi_1,\phi_2\rangle_1 + r\langle\phi_1,\phi_2\rangle_2\,. \tag{A19}$$

Here r is any real number. In dealing with the many-photon situation it is convenient to write this inner product symbolically in order to indicate which variable is eliminated in the contraction

$$\int dk\,\phi_1^{\mu *}(k)(-g_{\mu\nu})\phi_2^{\nu}(k) \equiv \langle \phi_1, \phi_2 \rangle . \tag{A20}$$

In a final notational simplification we suppress $(-g_{\mu\nu})$ and the vector indices and write simply

$$\int dk\,\phi_1^*(k)\phi_2(k) \equiv \langle \phi_1, \phi_2 \rangle . \tag{A21}$$

The extended many-photon space $\mathcal{G}_\gamma^E$ is obtained from the one-photon space by the Fock construction.[29] Let $a^\dagger(\phi)$ and $a(\phi)$ be creation and annihilation operators with nonvanishing commutation relations

$$[a(\phi_1), a^\dagger(\phi_2)] = \langle \phi_1, \phi_2 \rangle \tag{A22}$$

for $\phi_1, \phi_2 \in \mathcal{G}_\gamma^{(1)E}$. The extended Fock space $\mathcal{G}_\gamma^E$ is generated by applying powers of $a^\dagger(\phi)$ to the vacuum state Ω, with $a(\phi)\Omega = 0$, $\langle \Omega, \Omega \rangle = 1$. The resulting elements Φ of $\mathcal{G}_\gamma^E$ are sequences of n-photon wave functions

$$\Phi = \{\Phi_n\}, \quad n = 0, 1, 2 \ldots \tag{A23a}$$

$$\Phi_n = \Phi_n(k_1 \cdots k_n)^{\mu_1 \cdots \mu_n} , \tag{A23b}$$

which are symmetric under interchange of particle variables and which are elements of $\mathcal{G}_\gamma^E$ in each variable. A suitable restriction on the behavior of the wave functions at large n is required in order to ensure the convergence of infinite sums. The inner product $\langle \Phi, \Psi \rangle$ for $\Phi, \Psi \in \mathcal{G}_\gamma^E$ is

$$\langle \Phi, \Psi \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int dk_1 \cdots dk_n \Phi_n^*(k_1 \cdots k_n) \times \Psi_n(k_1 \cdots k_n), \tag{A24}$$

where here and in the following the integration with "measure" dk_i represents the contraction (A21). The creation and annihilation operators act according to

$$[a^\dagger(\phi)\Phi]_n(k_1 \cdots k_n)^{\mu_1 \cdots \mu_n} = \sum_{i=1}^{n} \phi^{\mu_i}(k_i)\Phi_{n-1}(k_1 \cdots \hat{k}_i \cdots k_n)^{\mu_1 \cdots \hat{\mu}_i \cdots \mu_n} , \tag{A25}$$

where the caret over a variable means it does not appear, and

$$[a(\phi)\Phi]_n(k_1 \cdots k_n)^{\mu_1 \cdots \mu_n} = \int dk\,\phi^{\mu *}(k)(-g_{\mu\nu})\Phi_{n+1}(k, k_1 \cdots k_n)^{\nu\mu_1 \cdots \mu_n} . \tag{A26}$$

Note that because $\Phi_n(k_1 \cdots k_n)$ are smooth functions, rather than Lebesgue measurable functions, it is possible to define an annihilation operator $a^\mu(k)$ depending on a single momentum variable

$$[a^\mu(k)\Phi]_n(k_1 \cdots k_n)^{\mu_1 \cdots \mu_n} \equiv \Phi_{n+1}(k, k_1 \cdots k_n)^{\mu, \mu_1 \cdots \mu_n} , \tag{A27}$$

although its Hermitian conjugate $a^\mu(k)^\dagger$ does not exist as an operator on $\mathcal{G}_\gamma^E$, but gives instead a distribution. It is also convenient to introduce the residue of the annihilation operator at zero frequency

$$a_R^\mu(\hat{k}) \equiv [\omega a^\mu(k)]\big|_{\omega=0} , \tag{A28}$$

which depends on a direction $\hat{k}$ and picks out the residues of the wave functions

$$[a_R^\mu(\hat{k})\Phi]_n(k_1 \cdots k_n)^{\mu_1 \cdots \mu_n} = \omega\Phi_{n+1}(k, k_1 \cdots k_n)^{\mu, \mu_1 \cdots \mu_n}\big|_{\omega=0} . \tag{A29}$$

The Fock space over regular wave functions $\mathcal{G}_\gamma$ is the linear subspace of $\mathcal{G}_\gamma^E$ defined by the condition of vanishing residues of all wave functions

$$a_R^\mu(\hat{k})\Phi = 0. \tag{A30}$$

Call $\mathcal{G}_\gamma^+$, the subspace of $\mathcal{G}_\gamma$ consisting of transverse wave functions, $\Phi \in \mathcal{G}^+$ if and only if

$$a_R^\mu(\hat{k})\Phi = 0, \quad k \cdot a(k)\Phi = 0 . \tag{A31}$$

Of course the inner product is non-negative on $\mathcal{G}_\gamma^+$, and it may be completed in norm to give the familiar Hilbert space of free photons $\mathcal{H}_\gamma^{\text{free}}$. However, $\mathcal{G}_\gamma^E$ also contains other subspaces of non-negative norm, namely the image $U\mathcal{G}_\gamma^+$ of $\mathcal{G}_\gamma^+$ under any unitary operator U, for example, $U = \exp[a^\dagger(\phi) - a(\phi)]$, $\phi \in \mathcal{G}_\gamma^{(1)E}$. This possibility will be used to construct the asymptotic space for quantum electrodynamics once the charged particles are introduced.

Let $\mathcal{G}_m$ be the usual Fock space of free massive particles (which, however, it is natural at this stage to also provide with the Schwartz topology) and let $b_a^\dagger(p)$ and $b_a(p)$ be the usual creation and annihilation operators for particles of type a. The extended space $\mathcal{G}^E$ is defined by the product of $\mathcal{G}_m$ with the extended photon space $\mathcal{G}_\gamma^E$,

$$\mathcal{G}^E \equiv \mathcal{G}_m \times \mathcal{G}_\gamma^E . \tag{A32}$$

It contains as a subspace $\mathcal{G}^{\text{free}}$ the space of free particles and free photons

$$\mathcal{G}^{\text{free}} \equiv \mathcal{G}_m \times \mathcal{G}_\gamma . \tag{A33}$$

Obviously it may also be characterized as the subspace of all vectors $\Phi \in \mathcal{G}^E$ which satisfy

$$a_R^\mu(\hat{k})\Phi = 0 . \tag{A34}$$

The asymptotic vector space of quantum electrodynamics $\mathcal{G}$ may be similarly defined. It is the linear subspace of $\mathcal{G}^E$ which satisfies the infrared-coherence condition (1.12) $\Phi \in \mathcal{G}$ if and only if

$$a^\mu_R(\hat{k})\Phi = \frac{-1}{(2\pi)^{3/2}} \int d^4p\,\rho(p)\,\frac{p^\mu}{p^0 - \vec{p}\cdot\hat{k}}\,\Phi\,, \quad \text{(A35)}$$

where ρ is the charge density operator in momentum space

$$\rho(p) = \sum_a \theta(p^0)\delta(p^2 - m_a{}^2) b_a^\dagger(p) b_a(p)\,. \quad \text{(A36)}$$

In terms of wave functions

$$\Phi = \{\Phi_{s,n}\}, \quad s, n = 0, 1, 2 \ldots \quad \text{(A37a)}$$

$$\Phi_{s,n} = \Phi_{sn}(p_1 \cdots p_s, k_1 \cdots k_n), \quad \text{(A37b)}$$

this condition reads

$$\omega \Phi_{s,n+1}(p_1 \cdots p_s, k, k_1 \cdots k_n)^{\mu,\mu_1\cdots\mu_n}\big|_{\omega=0} = \frac{-1}{(2\pi)^{3/2}} \sum_{i=1}^{s} \frac{e_i p_i^\mu}{E_i - \vec{p}_i\cdot\hat{k}}\,\Phi_{s,n}(p_1\cdots p_s, k_1 \cdots k_n)^{\mu_1\cdots\mu_n}\,. \quad \text{(A38)}$$

All operators and wave functions should of course bear "in" and "out" labels which are suppressed.

We are now ready to prove the lemma used in Sec. II: If $\Phi \in \mathcal{G}$ satisfies $B^{(-)}(x)\Phi = b^{(-)}(x)\Phi$ then Φ has non-negative norm. We use $B(x) = B^{\text{in}}(x) = \partial\cdot A^{\text{in}}(x)$ so the hypothesis of the lemma reads

$$\partial \cdot A^{\text{in}}(x)\Phi = b^{(-)}(x)\Phi\,. \quad \text{(A39)}$$

With

$$A^{\text{in}}_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega}\,[a^{\text{in}}_\mu(k)e^{-ik\cdot x} + a^{\text{in}\,\dagger}_\mu(k)e^{ik\cdot x}] \quad \text{(A40)}$$

and

$$b(x) = \frac{i}{(2\pi)^{3/2}} \int \frac{d^3k}{2\omega}\,[b(k)e^{-ik\cdot x} - b^*(k)e^{ik\cdot x}] \quad \text{(A41)}$$

and suppression of the "in" label, the hypothesis becomes

$$k\cdot a(k)\Phi = -b(k)\Phi\,. \quad \text{(A42)}$$

On setting $\omega = 0$ we find, from the infrared-coherence condition (A35), with $b(k) = b(\omega, \hat{k})$

$$\frac{1}{(2\pi)^{3/2}}\,Q\Phi = b(0,\hat{k})\Phi\,, \quad \text{(A43)}$$

where we have used

$$\begin{aligned} k\cdot a(k)\big|_{\omega=0} &= [a^0_R(\hat{k}) - \hat{k}\cdot\vec{a}_R(\hat{k})] \\ &= -(2\pi)^{-3/2}\int d^4p\,\rho(p) \\ &= -(2\pi)^{-3/2}\,Q\,. \end{aligned} \quad \text{(A44)}$$

This equation has solutions only for Φ which are eigenstates of the charge operator Q. We assume this is so, with corresponding eigenvalue q, so we have

$$b(0,\hat{k}) = (2\pi)^{-3/2} q\,. \quad \text{(A45)}$$

It is natural to set

$$b(k) = \frac{q}{(2\pi)^{3/2}}\,f(k) \text{ with } f(0) = 1\,, \quad \text{(A46)}$$

so the hypothesis of the lemma reads

$$k\cdot a(k)\Phi = \frac{-q}{(2\pi)^{3/2}}\,f(k)\Phi\,. \quad \text{(A47)}$$

[The proof is easily generalized to the case $q = 0$, $b(k) \neq 0$, but this is of no particular interest.]

Let $\phi^\mu_p(k)$ be the singular photon wave function depending on the four-momentum vector p

$$\phi^\mu_p(k) \equiv \frac{-1}{(2\pi)^{3/2}}\,\frac{p^\mu}{p\cdot k}\,f(k)\,, \quad \text{(A48)}$$

and let $U(f)$ be the unitary operator on $\mathcal{G}^E$

$$U(f) \equiv \exp \int d^4p\,\rho(p)[a^\dagger(\phi_p) - a(\phi_p)] \quad \text{(A49)}$$

satisfying

$$[a^\mu(k), U(f)] = U(f)\,\frac{(-1)f(k)}{(2\pi)^{3/2}} \int d^4p\,\rho(p)\,\frac{p^\mu}{p\cdot k}\,. \quad \text{(A50)}$$

Observe that $U(f)$ is the unitary transformation which effects the shift (A1). The cutoff function $f(k)$ has emerged unexpectedly, to within normalization, as the eigenvalue of $k\cdot a(k)$ or, stated loosely, as the eigenvalue of the generators of restricted gauge transformations of the second kind. From Eq. (A50) we have

$$[k\cdot a(k), U(f)] = U(f)\,\frac{(-1)}{(2\pi)^{3/2}}\,f(k)Q\,, \quad \text{(A51)}$$

$$[a^\mu_R(\hat{k}), U(f)] = U(f)\,\frac{(-1)}{(2\pi)^{3/2}} \int d^4p\,\rho(p)\,\frac{p^\mu}{p^0 - p\cdot\hat{k}}\,. \quad \text{(A52)}$$

The last relations shows that $U^\dagger(f)$ maps the infrared-coherent space $\mathcal{G}$ onto the free-particle space $\mathcal{G}^{\text{free}}$, for if $\Phi \in \mathcal{G}$ satisfies Eq. (A35) then $U^\dagger(f)\Phi$ satisfies (A34), so $U^\dagger(f)\Phi \in \mathcal{G}^{\text{free}}$. Similarly if Φ satisfies Eq. (A47), which is the hypothesis of the lemma, then by the last relation but one, $k\cdot a(k)U^\dagger(f)\Phi = 0$. So $U^\dagger(f)\Phi$ is a transverse free-particle vector. Its norm is non-negative

$$0 \leq \langle U^\dagger(f)\Phi, U^\dagger(f)\Phi\rangle = \langle\Phi,\Phi\rangle$$

Q.E.D.

We conclude this appendix by computing the explicit form of the inner product in $\mathcal{G}$ in terms of the wave functions satisfying Eq. (A38). Let $\phi_2^\mu(k)\in\mathcal{G}_\gamma^{(1)E}$ be a one-photon wave function with residue $\phi_{2R}^\mu(\hat{k})=cp^\mu/(E-\vec{p}\cdot\hat{k})$. Then Eq. (A18), with Λ_μ^0 replaced by p_μ/m, allows us to rewrite the inner product (A16) in the simpler form

$$\langle\phi_1,\phi_2\rangle_0=\tfrac{1}{2}\int d\hat{k}\,\phi_{1R}^{\mu}(\hat{k})(-g_{\mu\nu})\frac{cp^\nu}{E-\vec{p}\cdot\hat{k}}\times\ln\left(\frac{E-\vec{p}\cdot\hat{k}}{m}\right). \tag{A53}$$

Consequently we may reexpress the desired covariant inner product, Eq. (A19), as

$$\langle\phi_1,\phi_2\rangle=\tfrac{1}{2}\int d\hat{k}\,\phi_{1R}^{\mu}(\hat{k})(-g_{\mu\nu})\frac{cp^\nu}{E-\vec{p}\cdot\hat{k}}\ln\frac{(E-\vec{p}\cdot\hat{k})\Delta}{ml}+\langle\phi_1,\phi_2\rangle_\Delta\,, \tag{A54a}$$

where $l=e^{-\gamma}$. We have separated the contribution of the residues, which is written explicitly, from the continuum contribution

$$\langle\phi_1,\phi_2\rangle_\Delta=-\tfrac{1}{2}\int d\hat{k}\int_0^\infty d\omega\ln\frac{\omega}{\Delta}\frac{\partial}{\partial\omega}[\omega\phi_1^{\mu}{}^*(k)(-g_{\mu\nu})\omega\phi_2^\nu(k)]\equiv\int(dk)_\Delta\phi_1^*(k)\phi_2(k)\,. \tag{A54b}$$

Here we have introduced the "phase space"

$$\int(dk)_\Delta\cdots\equiv-\tfrac{1}{2}\int d\hat{k}\int_0^\infty d\omega\ln\left(\frac{\omega}{\Delta}\right)\frac{\partial}{\partial\omega}\,\omega^2\cdots\,, \tag{A55}$$

which differs from the usual photon phase space

$$\int d^3k/2\omega\cdots=\int d\hat{k}\int_0^\infty d\omega/2\omega\,\omega^2\cdots$$

by a partial integration. The covariant inner product (A54) is independent of the arbitrary parameter Δ which cancels out between the residue and continuum contributions. However, it is convenient to retain Δ as a reminder that the separation between the residue and continuum contributions is arbitrary, and it may be assigned any convenient value.

Having found a simple explicit form for the inner product when the wave function has residue $cp^\mu/(E-\vec{p}\cdot\hat{k})$, we may apply it to the wave functions $\Phi\in\mathcal{G}$ which satisfy the infrared-coherence condition (A38). It will prove convenient to effect all the p integrations after the k integrations. We formalize this by introducing, for every set of vectors $p_1\cdots p_s$ the subspace

$$\mathcal{G}_{\gamma p_1\cdots p_s}$$

of the extended photon space,

$$\mathcal{G}_{\gamma p_1\cdots p_s}\subset\mathcal{G}_\gamma^E$$

consisting of vectors $\Phi_{p_1\cdots p_s}\in\mathcal{G}_{\gamma p_1\cdots p_s}$ which satisfy

$$a_R^\mu(\hat{k})\Phi_{p_1\cdots p_s}=\frac{-1}{(2\pi)^{3/2}}\sum_{i=1}^{s}\frac{e_ip_i^\mu}{E_i-\vec{p}_i\cdot\hat{k}}\Phi_{p_1\cdots p_s}\,. \tag{A56}$$

For every set of vectors $p_1\cdots p_s$ we write

$$\Phi_{p_1\cdots p_s}(k_1\cdots k_n)\equiv\Phi(p_1\cdots p_s,k_1\cdots k_n) \tag{A57}$$

and define $\Phi_{p_1\cdots p_s}\in\mathcal{G}_{\gamma p_1\cdots p_s}$ by its wave functions

$$\Phi_{p_1\cdots p_s}=[\Phi_{p_1\cdots p_s}(k_1\cdots k_n)],\quad n=0,1,2\ldots. \tag{A58}$$

This allows us to express the inner product in $\mathcal{G}$ as

$$\langle\Phi_1,\Phi_2\rangle=\sum_{s=0}^\infty\frac{1}{s!}\int\frac{d^3p_1}{2E_1}\cdots\frac{d^3p_s}{2E_s}\times\langle\Phi_{1,p_1\cdots p_s},\Phi_{2,p_1\cdots p_s}\rangle \tag{A59}$$

(with $s!\to\pi_a s_a!$ if there are several types a of charged particle). All the k integrations are contained in

$$\langle\Phi_{1,p_1\cdots p_s},\Phi_{2,p_1\cdots p_s}\rangle,$$

which is the inner product in $\mathcal{G}_{\gamma,p_1\cdots p_s}$. A short calculation using Eq. (A54) gives the explicit form

$$\langle\Phi_{1p_1\cdots p_s},\Phi_{2p_1\cdots p_s}\rangle=\exp[-K(\Delta)]\langle\Phi_{1p_1\cdots p_s},\Phi_{2p_1\cdots p_s}\rangle_\Delta\,, \tag{A60}$$

where

$$K(\Delta)=\frac{-1}{(2\pi)^3}\,\tfrac{1}{2}\int d\hat{k}\sum_i\frac{e_ip_i^\mu}{E_i-\vec{p}_i\cdot\hat{k}}(-g_{\mu\nu})\sum_i\frac{e_ip_i^\nu}{E_i-\vec{p}_i\cdot\hat{k}}\ln\frac{(E_i-\vec{p}_i\cdot\hat{k})\Delta}{m_il}\,, \tag{A61a}$$

$$\langle\Phi_{1,p_1\cdots p_s},\Phi_{2,p_1\cdots p_s}\rangle_\Delta=\sum_{n=0}^\infty\frac{1}{n!}\int(dk_1)_\Delta\cdots(dk_n)_\Delta\Phi_{1,p_1\cdots p_s}^*(k_1\cdots k_n)\Phi_{2,p_1\cdots p_s}(k_1\cdots k_n)\,. \tag{A61b}$$

The latter contraction makes use of the "phase space" $dk_\Delta \cdots = -\frac{1}{2} d\hat{k}\, d\omega \ln(\omega/\Delta)(\partial/\partial\omega)\omega^2 \cdots$ in each photon variable. Because of the infrared-coherence condition on the many-photon wave functions, the residue contribution now appears multiplicatively in the exponential $\exp[-K(\Delta)]$ instead of as a subtraction. In particular the arbitrary constant $l = e^{-r}$, which results from the arbitrary constant r in the one-photon inner product, is now seen to produce a harmless multiplicative factor l^{-B} to the inner product, which is absorbed in the normalization of the wave function. Here the scalar B is given by

$$B = \frac{-1}{(2\pi)^3} \frac{1}{2} \int d\hat{k} \left(\sum_i \frac{-1}{E_i - \vec{p}_i \cdot \hat{k}} \right)^2$$

$$= \frac{-1}{(2\pi)^2} \sum_{i,j} \frac{e_i e_j \psi_{ij}}{\tanh\psi_{ij}} , \qquad \text{(A62)}$$

where $\psi_{ij} \geq 0$ is the hyperbolic angle between p_i and p_j, $p_i \cdot p_j = m_i m_j \cosh\psi_{ij}$.

APPENDIX B: SPECTRAL DECOMPOSITION OF THE COHERENT SPACE AND NORMALIZATION OF THE CHARGED FIELD

The spectral decomposition of the inner product

$$\langle \Phi_1, \Phi_2 \rangle = \int d^4Q\, \rho(Q), \qquad \text{(B1)}$$

where Φ_1 and Φ_2 are elements of the infrared-coherent space $\mathcal{G}$, is expressed in terms of the spectral function $\rho(Q)$ given by[30]

$$\rho(Q) = \frac{1}{(2\pi)^4} \int d^4x\, e^{-iQ\cdot x} \langle \Phi_1, U(x)\Phi_2 \rangle , \qquad \text{(B2)}$$

where $U(x)$ is the operator of translation by x^μ. Making use of the explicit form of the inner product (A59), we have

$$\rho(Q) = \sum_{s=0}^{\infty} \frac{1}{s!} \int \frac{d^3p_1}{2E_1} \cdots \frac{d^3p_s}{2E_s}\, \rho_{p_1\cdots p_s}(Q - \sum p_i) , \qquad \text{(B3)}$$

where $\rho_{p_1\cdots p_s}(L)$ is the spectral function in $\mathcal{G}_{\gamma p_1 \cdots p_s}$,

$$\rho_{p_1\cdots p_s}(L) = \frac{1}{(2\pi)^4} \int d^4x\, e^{-iL\cdot x} \langle \Phi_{1p_1\cdots p_s}, U_\gamma(x)\Phi_{2p_1\cdots p_s} \rangle . \qquad \text{(B4)}$$

Here $\Phi_{ip_1 \cdots p_s}$ is an element of $\mathcal{G}_{\gamma p_1 \cdots p_s} \subset \mathcal{G}_\gamma^E$ and $U_\gamma(x)$ is the translation operator on $\mathcal{G}_\gamma$. By Eq. (A60) we have

$$\rho_{p_1\cdots p_s}(L) = \frac{1}{(2\pi)^4} e^{-K(\Delta)} \int d^4x\, e^{-iL\cdot x} \sum_{n=0}^{\infty} \frac{1}{n!} \int (dk_1)_\Delta \cdots (dk_n)_\Delta e^{i\Sigma k_i \cdot x} \Phi^*_{1p_1\cdots p_s}(k_1 \cdots k_n) \Phi_{2p_1\cdots p_s}(k_1 \cdots k_n). \qquad \text{(B5)}$$

We are interested in the spectral function when the total momentum Q is close to the sum of the momenta of the charged particles, Σp_i, so that the momentum in the radiation field $L = Q - \Sigma p_i$ is small. For this purpose we replace the momentum L by λL and we will let λ get small. On setting $y = \lambda x$ we find

$$\rho_{p_1\cdots p_s}(\lambda L) = e^{-K(\Delta)} \frac{1}{(2\pi)^4} \frac{1}{\lambda^4} \int d^4y\, e^{iL\cdot y} \sum_{n=0}^{\infty} \frac{1}{n!} \int (dk_1)_\Delta \cdots (dk_n)_\Delta e^{i(\Sigma k_i \cdot y/\lambda)} \Phi^*_{1p_1\cdots p_s}(k_1 \cdots k_n) \times \Phi_{2p_1\cdots p_s}(k_1 \cdots k_n). \qquad \text{(B6)}$$

For small values of λ we may use the Riemann-Lebesgue lemma and replace the wave functions by their zero-frequency limit since they are smooth functions [after multiplication by the ω^2 which appears in $(dk)_\Delta$, Eq. (A55)]

$$\Phi_{ip_1\cdots p_s}(k_1 \cdots k_n)^{\mu_1 \cdots \mu_n} \to \Phi_i(p_1 \cdots p_s) \prod_{i=1}^{n} \left[\frac{-1}{(2\pi)^{3/2}} \sum_a \frac{e_a p_a^{\mu_i}}{p_a \cdot k_i} \right], \qquad \text{(B7)}$$

where $\Phi_i(p_1 \cdots p_s)$ is the no-photon wave function. This gives

$$\rho_{p_1\cdots p_s}(\lambda L) = \Phi_1^*(p_1 \cdots p_s)\Phi_2(p_1 \cdots p_s) F(\lambda L), \qquad \text{(B8)}$$

$$F(\lambda L) = e^{-K(\Delta)} \frac{1}{(2\pi)^4} \frac{1}{\lambda^4} \int d^4y\, e^{-iL\cdot y} \exp\left\{ \int (dk)_\Delta e^{ik\cdot y/\lambda} \left[-\frac{1}{(2\pi)^3} \left(\sum_a \frac{e_a p_a}{p_a \cdot k} \right)^2 \right] \right\}. \qquad \text{(B9)}$$

The spectral function for four-momentum in the radiation field $\rho_{p_1 \cdots p_s}(L)$ has, for small values of L, factorized into the product of the no-photon wave functions and a universal spectral function $F(L)$ depending on a set of particle charges and momenta

$$F(L) = F(L, e_i, p_i) . \qquad \text{(B10)}$$

If the charges were zero we would have

$$F(L)=\delta^4(L)\,. \tag{B11}$$

Its deviation from a δ function is the result of the singular coupling of charged particles to the infrared radiation field. From the definition of the "phase space" $(dk)_\Delta$, Eq. (A54b), we find

$$F(\lambda L)=e^{-K(\Delta)}\frac{1}{(2\pi)^4}\frac{1}{\lambda^4}\int d^4y\, e^{-iL\cdot y}e^J\,,$$
$$J=\frac{1}{(2\pi)^3}\tfrac{1}{2}\int d\hat{k}\sum_{i,j}\frac{e_ie_jp_i\cdot p_j}{p_i\cdot\hat{k}p_j\cdot\hat{k}}\int_0^\infty d\omega\ln\frac{\omega}{\Delta}\frac{\partial}{\partial\omega}e^{-(\epsilon-i\hat{k}\cdot y)\omega/\lambda}\,, \tag{B12}$$

where, for a four-vector a, $\hat{k}\cdot a\equiv a^0-\hat{k}\cdot\vec{a}$,

$$J=\frac{1}{(2\pi)^3}\tfrac{1}{2}\int d\hat{k}\sum_{i,j}\frac{e_ie_jp_i\cdot p_j}{p_i\cdot\hat{k}p_j\cdot\hat{k}}\left[\ln\frac{\Delta(\epsilon-i\hat{k}\cdot y)}{\lambda}+C\right],$$

where $C=-\int_0^\infty du\ln u\, e^{-u}$ is Euler's constant. Use of Eqs. (A61) and (A62) for $K(\Delta)$ and B gives

$$F(L)=(le^C)^{-B}\int\frac{d^4y}{(2\pi)^4}e^{-iL\cdot y}\exp\left(\frac{-1}{(2\pi)^3}\tfrac{1}{2}\int d\hat{k}\left(\frac{e_ip_i}{p_i\cdot\hat{k}}\right)\left\{\sum_j\frac{e_jp_j}{p_j\cdot\hat{k}}\left[\ln\frac{p_j\cdot\hat{k}}{m_j(\epsilon-i\hat{k}\cdot y)}\right]\right\}\right), \tag{B13a}$$

$$F(\lambda L)=\lambda^{-4+B}F(L)\,. \tag{B13b}$$

The arbitrary constant Δ has canceled out as it should. This equation gives a complete description of the distribution of four-momentum in the infrared radiation field in the presence of charged particles of four-momenta $p_1\cdots p_s$. Observe that $F(L)$ is the Fourier transform of a function (distribution) which is analytic in the future tube, so it vanishes if L lies outside the future cone, as it must. It is homogeneous of degree $-4+B$ so the infrared renormalization has caused a change in the infrared asymptotic dimension.

Let us evaluate the original spectral function $\rho(Q)$ for vectors Φ_1 and Φ_2 containing a single electron. In this case only the term $s=1$ contributes to the sum (B3) and we have

$$\rho(Q)=\int\frac{d^3p}{2E}\Phi_1^*(p)\Phi_2(p)F(Q-p). \tag{B14}$$

Let Q lie close to the mass shell of the electron. Because $F(Q-p)$ has support only in the future cone, $Q^0-p^0\geq 0$, $(Q-p)^2\geq 0$, where $M=\sqrt{Q^2}$ lies close to m, the integration over p only gets contributions when the four-vector p^μ lies close to the four-vector Q^μ. In the space $\mathcal{G}$ we are using, the wave functions $\Phi(p)$ are smooth, so we have

$$\rho(Q)=\Phi_1^*\left(\frac{mQ}{M}\right)\Phi_2\left(\frac{mQ}{M}\right)P(Q), \tag{B15}$$

where $P(Q)$ is the universal function

$$P(Q)=\frac{l_1^\beta}{(2\pi)^4}\int\frac{d^3p}{2E}\int d^4x\, e^{-i(Q-p)\cdot x}\exp\left[\frac{-e^2}{(2\pi)^3}\tfrac{1}{2}\int d\hat{k}\frac{m^2}{(p\cdot\hat{k})^2}\ln\frac{p\cdot\hat{k}}{m(\epsilon-i\hat{k}\cdot x)}\right], \tag{B16}$$

and $l_1=le^C$, $\beta=(e/2\pi)^2=\alpha/\pi$. Lorentz invariance of the last integral allows it to be evaluated in the frame where $Q=(M,0,0,0)$,

$$P(Q)=\frac{l_1^\beta}{(2\pi)^4}\int\frac{d^3p}{2E}\int d^3x\, e^{-ip\cdot x}\int dt\, e^{-i(M-E)t}\exp\left[\frac{-e^2}{(2\pi)^3}\tfrac{1}{2}\int d\hat{k}\frac{m^2}{(E-\vec{p}\cdot\hat{k})^2}\ln\frac{(E-\vec{p}\cdot\hat{k})/m}{\epsilon-i(t-\hat{k}\cdot\vec{x})}\right]. \tag{B17}$$

Let $\delta=M-m$ and make the change of variables $\vec{p}\to|\delta^{1/2}|\vec{p}$, $\vec{x}\to|\delta^{-1/2}|\vec{x}$, $t\to|\delta^{-1}|t$. For small δ we find

$$P(Q)=\frac{l_1^\beta}{(2\pi)^4}\frac{1}{2m}\frac{1}{|\delta|}\int d^3p\,d^3x\,dt\,e^{-i\vec{p}\cdot\vec{x}}\left(\frac{\epsilon-it}{|\delta|}\right)^\beta\exp\left[-i\left(\operatorname{sgn}\delta-\frac{p^2}{2m}\right)t\right]. \tag{B18}$$

Integration over $\vec{x}$ followed by integration over $\vec{p}$ is now trivial and gives, after the change of variable $t\to|\delta|t$, with $\delta=(M-m)$,

$$P(Q)=\frac{l_1^\beta}{2\pi}\frac{1}{2m}\int dt\, e^{-i(M-m)t}(\epsilon-it)^\beta. \tag{B19}$$

Writing this in the form

$$P(Q)=\frac{l_1^\beta}{2\pi}\frac{1}{2m}\frac{\partial}{\partial M}\int_{-\infty}^\infty dt\, e^{-i(M-m)t}(\epsilon-it)^{\beta-1} \tag{B20}$$

allows us to use the representation

$$(\epsilon-it)^{\beta-1}=\frac{1}{\Gamma(1-\beta)}\int_0^\infty d\lambda\, e^{-(\epsilon-it)\lambda}\lambda^{-\beta}\,,$$

valid for $\beta = \alpha/\pi < 1$, which gives

$$P(Q) = \frac{l_1^\beta}{2m} \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial M} \frac{\theta(M-m)}{(M-m)^\beta} . \tag{B21}$$

We may exploit the smallness of $M - m$ to rewrite this as

$$P(Q) = \frac{(2ml_1)^\beta}{2m^2} \frac{1}{\Gamma(1-\beta)} Q^\mu \times \frac{\partial}{\partial Q^\mu} \left(\frac{\theta(Q^2 - m^2)\theta(Q^0)}{(Q^2 - m^2)^\beta} \right), \tag{B22}$$

which is a well-defined distribution in Q for $\beta = \alpha/\pi < 1$. Note that for $\beta = 0$ we recover the free-particle spectral function

$$P(Q) = \delta(Q^2 - m^2)\theta(Q^0).$$

With this result, the goal of calculating the spectral function in the infrared-coherent space $\mathcal{G}$, for four-momentum Q in the neighborhood of the electron mass shell $Q^2 = m^2$, is achieved. If the infrared-coherent space $\mathcal{G}$ is the space which would be obtained by reconstruction from the Wightman functions, this expression must agree with the mass-shell singularity of the two-point function of the electron. The electron propagator

$$G(x) = \langle 0 | T[\psi(x)\overline{\psi}(0)] | 0 \rangle \tag{B23}$$

with Fourier transform $G(Q)$

$$G(x) = \frac{1}{(2\pi)^4} \int e^{-iQ\cdot x} G(Q) d^4Q \tag{B24}$$

has the mass-shell singularity in the Feynman gauge given by

$$G(Q) = z\Gamma(1+\beta) \frac{i(\not{Q}+m)}{Q^2 - m^2 + i\epsilon} \left(\frac{2ml_1}{m^2 - Q^2 - i\epsilon} \right)^\beta, \tag{B25a}$$

$$G(Q) = -z\Gamma(\beta) \frac{i(\not{Q}+m)}{2m^2} Q^\mu \frac{\partial}{\partial Q^\mu} \left(\frac{2ml_1}{m^2 - Q^2 - i\epsilon} \right)^\beta, \tag{B25b}$$

where z is an arbitrary finite constant that defines the normalization of the charged field ψ. This result has not been proved rigorously in perturbation theory, but it is believed to be exact.[8] The propagator is related to the Wightman function $W(Q)$,

$$\langle \psi(x)\overline{\psi}(0) \rangle = \frac{1}{(2\pi)^3} \int d^4Q \, e^{-iQ\cdot x} W(Q) , \tag{B26}$$

by $W(Q) = [G_+(Q) - G_-(Q)]/2\pi$ which gives

$$W(Q) = z \frac{\not{Q}+m}{\Gamma(1-\beta)} \frac{1}{2m^2} Q^\mu \times \frac{\partial}{\partial Q^\mu} \left[\left(\frac{2ml_1}{Q^2 - m^2} \right)^\beta \theta(Q^2 - m^2)\theta(Q^0) \right], \tag{B27}$$

where we have used $\Gamma(\beta)\Gamma(1-\beta) = \pi/\sin\pi\beta$. Comparison with Eq. (B22) gives the relation at the mass shell between this Wightman function and the spectral function $P(Q)$ of the infrared-coherent space, $\mathcal{G} = \mathcal{G}^{\text{in}} = \mathcal{G}^{\text{out}}$,

$$W(Q) = z(\not{Q}+m)P(Q), \quad Q^2 \approx m^2 . \tag{B28}$$

This simple relation allows a physical interpretation of the normalization of the charged field ψ near the mass shell. Let the vectors Φ_i, $i = 1, 2$ be given by

$$\Phi_i = \int d^4x \, \overline{\psi}(x)\phi_i(x)\Omega , \tag{B29}$$

where Ω is the vacuum state and $\phi_i(x)$ is a four-spinor smearing function. They have inner product

$$\langle \Phi_1, \Phi_2 \rangle = \int d^4Q \, \overline{\phi}_1(Q)W(Q)\phi_2(Q) , \tag{B30}$$

where

$$\phi_i(Q) = \frac{1}{(2\pi)^{3/2}} \int \phi_i(x) e^{iQ\cdot x} d^4x . \tag{B31}$$

The contribution to this inner product at the mass shell is, by Eq. (B28),

$$z \int d^4Q \, \overline{\phi}_1(Q)(\not{Q}+m)\phi_2(Q)P(Q), \quad Q^2 \approx m^2 \tag{B32a}$$

or, with $u_\alpha(Q)$, $\alpha = 1, 2$, a pair of Dirac spinors, such that $\Sigma_\alpha u_\alpha(Q)\overline{u}_\alpha(Q) = (\not{Q}+m)$,

$$z \int d^4Q \sum_\alpha \phi_{1\alpha}^*(Q)\phi_{2\alpha}(Q)P(Q), \tag{B32b}$$

where $\phi_{i\alpha}(Q) = \overline{u}_\alpha(Q)\phi_i(Q)$. This is precisely the form of the inner product, Eqs. (B1) and (B15), and we may identify, for $Q^2 = m^2$, the one-particle wave functions $\Phi_{i\alpha}^{\text{in}}(Q)$ and $\Phi_{i\alpha}^{\text{out}}(Q)$, where we have restored the spin label α and the "in" or "out" labels

$$\Phi_{i\alpha}^{\text{in}}(Q) = z^{1/2} \frac{1}{(2\pi)^{3/2}} \int \overline{u}_\alpha(Q)\phi_i(x) e^{iQ\cdot x} d^4x = \Phi_{i\alpha}^{\text{out}}(Q) . \tag{B33}$$

This is precisely the relation that obtains at the mass shell in theories without massless particles.

Equation (B29) provides an exceptionally simple charged state which satisfies the substitute Gupta-Bleuler condition provided $\phi(Q)$ is of the form

$$\phi(Q) = e^{-Q\cdot\tau} e^{iQ\cdot a} h(Q^2)\theta(Q^0)u , \tag{B34}$$

where τ is a future timelike vector, u is a constant four-spinor, and $h(x)$ is a $c-\infty$ function satisfying $h(x) = 1$ for $x \geq m^2$ and $h(x) = 0$ for $x \leq 0$. This may be verified by writing the commutator

$[B(x),\psi(y)]=e\Delta(x-y)\psi(y)$ in momentum space. The coherence function $b(x)$, expressed in terms of its Fourier components on the light cone according to Eq. (A41), is given by

$$b(k)=-e(2\pi)^{-3/2}e^{-k\cdot\tau}e^{ik\cdot a}. \quad \text{(B35)}$$

This yields a strong positivity condition on $W(Q)$, the Wightman function of the electron in the Feynman gauge,

$$\int d^4Q\,\bar{u}\,W(Q)u\,e^{-2Q\cdot\tau}\geq 0. \quad \text{(B36)}$$

APPENDIX C: CROSS-SECTION FORMULA REGAINED

In this section we shall derive a quantum electrodynamical cross-section formula which is independent of the coherence function $f(k)$. It is obtained by letting the support of $f(k)$ approach the origin and summing over final states. In accordance with the superselection rule on charged particle momenta discussed in the Introduction, no such limit exists for quantities, such as delay times, which depend on relative phases between different momentum components of the charged particles.

We represent the S operator considered as an operator on $\mathcal{G}^E$ in terms of its Wick expansion

$$S=\sum_{m,n,s,t}\frac{1}{m!n!s!t!}\int\prod_{f=1}^{s}[dp_f b^\dagger(p_f)]\prod_{f=1}^{m}[dk_f a^\dagger(k_f)]S_{s,m;t,n}(p_f,k_f;p_i,k_i)\prod_{i=1}^{t}[dp_i b(p_i)]\prod_{i=1}^{n}[dk_i a(k_i)]. \quad \text{(C1)}$$

It has the same form in either the "in" or "out" basis, so we may safely suppress this label. The normalization factor $(s!t!)^{-1}$ should be replaced by $\pi_a(s_a!t_a!)^{-1}$ if there are several types a of charged particle. Here $\int dp$ represents $\int d^3p(2E)^{-1}=\int d^3p[2(m^2+\vec{p}^2)^{1/2}]^{-1}$ and $\int dk$ represents the infrared renormalized inner product defined in Appendix A. Because the scattering operator S represents an observable, it commutes with the generators of restricted gauge transformations

$$[k\cdot a(k),S]=[k\cdot a^\dagger(k),S]=0, \quad \text{(C2)}$$

which means that the S-matrix elements $S_{s,m;t,n}(p_f,k_f;p_i,k_i)$ are, as usual, transverse in the photon polarization indices. Similarly, S preserves the infrared coherence condition $s^\mu(\hat{k})\Phi=0$, where

$$s^\mu(\hat{k})=a_R^\mu(\hat{k})+\frac{1}{(2\pi)^{3/2}}\int d^4p\,\frac{\rho(p)p^\mu}{p^0-\vec{p}\cdot\hat{k}},$$

by commuting with $s^\mu(\hat{k})$,

$$[s^\mu(\hat{k}),S]=[s^{\mu\dagger}(\hat{k}),S]=0. \quad \text{(C3)}$$

This implies that the S-matrix elements satisfy

$$\lim_{\omega_m\to 0}\{\omega_m[S_{s,m;t,n}(p_f,k_1\cdots k_m;p_i,k_i)-c_{fi}(k_m)S_{s,m-1;t,n}(p_f,k_1\cdots k_{m-1};p_i,k_i)]\}=0, \quad \text{(C4a)}$$

$$\lim_{\omega_n\to 0}\{\omega_n[S_{s,m;t,n}(p_f,k_f;p_i,k_1\cdots k_n)-c_{if}(k_n)S_{s,m;t,n-1}(p_f,k_f;p_i,k_1\cdots k_{n-1})]\}=0, \quad \text{(C4b)}$$

where

$$c_{fi}^\mu(k)=-c_{if}^\mu(k)=c_f^\mu(k)-c_i^\mu(k)$$
$$=\frac{-1}{(2\pi)^{3/2}}\left(\sum_f\frac{e_f p_f^\mu}{p_f\cdot k}-\sum_i\frac{e_i p_i^\mu}{p_i\cdot k}\right). \quad \text{(C5)}$$

The S-matrix elements may be obtained from a reduction formula using the distances from the mass shell $\delta_a=p_a{}^2-m_a{}^2$ as infrared regulators,[13] or by use of photon mass or dimensional regularization.[31]

If the S operator is applied to the state $\Psi=\{\Psi_{t,n}(p_1\cdots p_t,k_1\cdots k_n)\}$ there results the state with wave functions

$$(S\Psi)_l(p,k_1\cdots k_l)=\int dp'\sum_{m=0}^{l}\sum_P\sum_{n=0}^{\infty}\frac{1}{n!}\int dk_1'\cdots dk_n' S_{m,n}(p,k_{P_1}\cdots k_{P_m};p',k_1'\cdots k_n')$$
$$\times\Psi_{l-m+n}(p',k_{P_{m+1}}\cdots k_{P_l},\;k_1'\cdots k_n'). \quad \text{(C6)}$$

The charged particle variables have been represented only symbolically because our interest here centers on the photon variables. The sum over permutations P represents the sum over the $l![m!(l-m)!]^{-1}$ ways that l objects may be partitioned into two sets of m and $l-m$ objects each.

Let the initial state Ψ be obtained from a free-particle state Ψ^0 by application of U, Eq. (A49),

$$\Psi = U\Psi^0, \tag{C7a}$$

$$U = \exp\left(\int d^4p\, \rho(p)[a^\dagger(\phi_p) - a(\phi_p)]\right), \tag{C7b}$$

where

$$\phi_p(k) = \frac{-1}{(2\pi)^{3/2}} \frac{p}{p\cdot k} f(k), \tag{C8}$$

and $f(k)$ is the coherence function. Suppose for simplicity of notation that the state $\Psi^0 = \{\Psi^0_{t,n}\}$ contains only a fixed number τ of charged particles but no photons, so its wave functions are given by

$$\Psi^0_{t,n} = \delta_{t,\tau}\delta_{n,0}\psi(p_i), \quad i = 1, \ldots, \tau \tag{C9}$$

with $\int dp_1 \cdots dp_n \psi^*(p_i)\psi(p_i) = 1$. We have

$$U = \exp\left[-\tfrac{1}{2}\int d^4p_1 d^4p_2 \rho(p_1)\rho(p_2) \int dk\, \phi^*_{p_1}(k)\,\phi_{p_2}(k)\right] \times \exp\left[\int d^4p\, \rho(p) a^\dagger(\phi_p)\right] \exp\left[-\int d^4p\, \rho(p) a(\phi_p)\right], \tag{C10}$$

so the state $\Psi = U\Psi_0 = \{\Psi_{t,n}\}$ is a completely coherent state of the radiation field with wave functions given by

$$\Psi_{t,n}(p_i, k_i) = N\epsilon_i(k_1)\cdots\epsilon_i(k_n)\delta_{t\tau}\psi(p_i), \tag{C11}$$

where

$$\epsilon^\mu_i(k) = c^\mu_i(k) f(k) = \frac{-1}{(2\pi)^{3/2}} \sum_i \frac{e_i p^\mu_i}{p_i\cdot k} f(k), \tag{C12}$$

$$N = \exp\left[-\tfrac{1}{2}\int dk\, \epsilon^*_i(k)\epsilon_i(k)\right]. \tag{C13}$$

If this is substituted into Eq. (C6) one obtains

$$(S\Psi)_l(p, k_1\cdots k_l) = \int dp' N \sum_{m=0}^{l} \sum_P \epsilon_i(k_{P_{m+1}})\cdots\epsilon_i(k_{P_l}) \sum_{n=0}^{\infty} \frac{1}{n!}\int dk'_1\cdots dk'_n S_{m,n}(p, k_{P_1}\cdots k_{P_m}; p', k'_1\cdots k'_n) \times \epsilon_i(k'_1)\cdots\epsilon_i(k'_n)\psi(p'). \tag{C14}$$

Suppose that the support of the coherence function $f(k)$, and hence of $\epsilon_i(k) = c_i(k) f(k)$, only includes such low frequencies that S_{mn} may be replaced by its low-frequency limit, Eq. (C4), when contracted with $\epsilon_i(k)$,

$$\int dk'_n S_{m,n}(p, k; p', k'_1\cdots k'_n)\epsilon_i(k'_n) \to S_{m,n-1}(p, k; p', k'_1\cdots k'_{n-1}) \int dk'\, c_{if}(k')\epsilon_i(k'), \tag{C15}$$

where $c_{if}(k)$ is given in Eq. (C5). This yields

$$(S\Psi)_l(p, k_1\cdots k_l) = \int dp' NM \sum_{m=0}^{l} \sum_P \epsilon_i(k_{P_{m+1}})\cdots\epsilon_i(k_{P_l}) S_{m,0}(p, k_{P_1}\cdots k_{P_m}; p')\psi(p'), \tag{C16}$$

$$M = \exp\left[-\int dk\, c_{fi}(k)\epsilon_i(k)\right]. \tag{C17}$$

If this initial ψ has sharp values for the momenta of the incident charged particles, this simplifies to

$$(S\Psi)_l(p_f, k_1\cdots k) = NM \sum_{m=0}^{l} \sum_P \epsilon_i(k_{P_{m+1}})\cdots\epsilon_i(k_{P_l}) S_{m,0}(p_f, k_{P_1}\cdots k_{P_m}; p_i). \tag{C18}$$

Suppose that one wishes to calculate a cross section corresponding to a typical accelerator experiment: *Only initial and final particle intensities as a function of momenta are measured*, but no relative phase between different momentum components. It is sufficient as usual to assume that the initial state has sharp momenta (these could be averaged over) and that the final state detector is characterized by a density (efficiency) matrix ρ which is diagonal in the particle momenta

$$\rho = \{\rho_n(p, k_1\cdots k_n)\}, \quad 0 \le \rho_n \le 1. \tag{C19}$$

The unobservability of zero-energy photons corresponds to the analytic statement

$$\lim_{\omega_n\to 0} \rho_n(p, k_1\cdots k_n) = \rho_{n-1}(p, k_1\cdots k_{n-1}). \tag{C20}$$

The cross section $\sigma(\rho)$ for the initial state with momenta p_i to trigger the final state detector ρ is given by

$$\sigma(\rho)=\int dp_f|NM|^2\sum_{l=0}^{\infty}\frac{1}{l!}\int dk_1\cdots dk_l(S\Psi)_l^*(p_f,k_1\cdots k_l)\rho_l(p_f,k_1\cdots k_l)(S\Psi)_l(p_f,k_1\cdots k_l)\delta^4(\textstyle\sum p_f-\sum p_i+\sum k), \tag{C21}$$

where $S\Psi$ is given above, Eq. (C18), and a δ function of energy-momentum conservation has been factored out of S. We suppose that the support of $f(k)$, and hence also of $\epsilon_i(k)=c_i(k)f(k)$, is sufficiently close to the origin that the substitution

$$\rho_l(p,k_1\cdots k_l)\epsilon_i(k_l)\to\rho_{l-1}(p,k_1\cdots k_{l-1})\epsilon_i(k_l) \tag{C22}$$

is justified, by virtue of the low-frequency limit of ρ_l, Eq. (C20). Then making use also of the substitution (C15) applied to a final photon variable one obtains after a brief combinatoric exercise

$$\sigma(\rho)=\int dp_f\sum_{n=0}^{\infty}\frac{1}{n!}dk_1\cdots dk_nS_{n,0}^*(p_f,k_1\cdots k_n;p_i)\rho_n(p_f,k_1\cdots k_n)S_{n,0}(p_f,k_1\cdots k_n;p_i)\delta^4(\textstyle\sum p_f-\sum p_i+\sum k). \tag{C23}$$

This formula may be given an explicit form using Eqs. (A60) and (A61),

$$\sigma(\rho)=\int dp_fe^{-K_{fi}(\Delta)}\sum_{n=0}^{\infty}\frac{1}{n!}\int(dk_1)_\Delta\cdots(dk_n)_\Delta S_{n,0}^*(p_f,k_1\cdots k_n;p_i)\rho_n(p_f,k_1\cdots k_n)S_{n,0}(p_f,k_1\cdots k_n;p_i), \tag{C24a}$$

where $(dk_\Delta)\cdots=-\frac{1}{2}d\hat{k}d\omega\ln(\omega/\Delta)(\partial/\partial\omega)\cdots$ and

$$K_{fi}(\Delta)=\frac{-1}{(2\pi)^3}\frac{1}{2}\int d\hat{k}\sum_a\frac{e_a\eta_ap_a^\mu}{E_a-\vec{p}\cdot\hat{k}}(-g_{\mu\nu})\sum_a\frac{e_a\eta_ap_a^\nu}{E_a-\vec{p}_a\cdot\hat{k}}\ln\frac{(E_a-\vec{p}_a\cdot\hat{k})\Delta}{m_al}. \tag{C24b}$$

Here a is an index that runs over all initial and final charged particles and η_a is a sign function with $\eta_f=+1$, $\eta_i=-1$. This is the quantum electrodynamical cross-section formula obtained earlier.[13, 31] It is notable, however, that if the initial or the final state is not diagonal in the charged particle momenta, then the cross section does not have a well-defined limit as the support of $f(k)$ shrinks to zero.

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*Work supported in part by the National Science Foundation, under Grant No. PHY74-2177A01.

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[20]Ref. 19, Eq. (6.23), and Ref. 2, Eqs. (2.183)–(2.187).

[21]A less strict criterion for observables adapted specifically to the Gupta-Bleuler method has been considered in Ref. 2. If a less strict criterion for observables were adopted, the class of observables would become larger, so the positivity condition (1.9) would become stricter and the set of states which satisfy it would be a subset of the states found here.

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[25]This notion of equivalence of state vectors is more general than the one which occurs in the usual Gupta-Bleuler method, Ref. 2, Eq. (2.192).

[26]This phenomenon also occurs in the Coulomb gauge.

[27]We develop here the approach suggested in footnote 13 of Ref. 12.

[28]There does not appear to be any obstacle to effecting the constructions of Refs. 1 and 2 in this topology. In particular the transversality condition $k_\mu \phi^\mu(k)$, with solution $\phi^0(\omega, \hat{k}) = \hat{k} \cdot \vec{\phi}(\omega, \hat{k})$, defines a subspace on which the inner product (A2) is non-negative and which may be completed in the norm to give a physical one-photon Hilbert space $\mathcal{H}_\gamma^{(1)}$.

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