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# Approximately relativistic Hamiltonians for interacting particles

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In the relativistic canonical formalism of Bakamjian and Thomas describing direct particle interactions the generators are defined in terms of the total momentum, the center-of-mass position, and a complete set of additional intrinsic canonical variables. In the interaction region of phase space the transformation linking these variables to individual particle coordinates and momenta is not determined by basic principles. In this paper canonical transformations to single-particle variables valid to order $c^{-2}$ and the corresponding approximate Hamiltonians are constructed for a two-particle system; approximate many-body Hamiltonians are then constructed from the two-body ones, maintaining the Lie algebra of the Poincaré group to the same order. If, and only if, the nonrelativistic limit of the potential is velocity independent (except for a possible spin-orbit interaction) it is possible to require, to order $c^{-2}$, transformation properties of the position operators corresponding to the classical world-line conditions. This requirement implies restrictions on admissible canonical transformations to single-particle variables. The cluster separability condition is then automatically satisfied. In the classical limit the class of approximately relativistic Hamiltonians for spinless particles is identical with that obtained by Woodcock and Havas from expansion of an exact Poincaré-invariant Fokker-type variational principle automatically satisfying the world-line conditions. Conversely, direct quantization of their classical Hamiltonians is shown to lead to the approximate quantum-mechanical ones resulting from the Bakamjian-Thomas theory. The relation of these results to various approximately relativistic Hamiltonians built up by several authors starting from the nonrelativistic theory is discussed, as well as their implications for phenomenological nucleon-nucleon potentials.

## I. INTRODUCTION

"The nature of interactions between nuclear particles is understood at present rather poorly." This opening sentence of the classic paper on approximately relativistic equations,[1] written in 1937 by Breit to explain the motivation for his study of wave equations Lorentz invariant only to order $v^2/c^2$, is still appropriate, even though an enormous amount of detailed knowledge of properties of the interactions has been accumulated. In spite of its central importance for nuclear physics the problem seems less fundamental now since the notion of point nucleons without internal structure is, at best, an approximation, the validity of which must be tested. The presumption "that a completely relativistic theory must involve a field"[1] underlies the conventional description of the two-nucleon system in terms of Bethe-Salpeter amplitudes.[2] This presumption is challenged implicitly by much work on the relativistic theory of direct particle interactions[3,4] (which, at least to order $v^2/c^2$, includes the particle interactions following from field theory). Three general schemes allowing large classes of interactions have been developed. The first one, due to Bakamjian and Thomas,[5] makes use of the special properties of the center-of-mass frame of reference to develop a canonical formalism, classical as well as quantum mechanical. The second one, due to Havas,[6] takes a classical Poincaré-invariant "Fokker-type" variational principle as its starting point; the general form, to order $v^2/c^2$, of the approximately relativistic Lagrangian corresponding to this principle has been found recently by Woodcock and Havas[7] for all interactions which for $c \to \infty$ reduce to a static Newtonian potential. The third scheme is based on the relativistic generalization of Newton's second law with two-body forces.[8] These three approaches correspond to the nonrelativistic formalisms of Hamilton, Lagrange, and Newton, respectively, but while the nonrelativistic approaches are equivalent for the usual case of two-body interactions derivable from a potential, in general no such equivalence exists in the relativistic cases discussed above.

All three general methods suffer from various fundamental difficulties that stand in the way of many applications in nuclear and particle physics. No satisfactory methods are known to quantize either the Fokker-type theory of Ref. 6 or the Newton-type theory of Ref. 8. The BT theory, on the other hand, poses difficulties of interpretation. Initially, there was no indication that the corresponding classical canonical formalism would be compatible with the existence of invariant world

lines; indeed, Bakamjian and Thomas abandoned this requirement at the outset, long before formal proofs were provided that the "world-line conditions" implied vanishing interaction.[9] However, for strongly interacting nuclear and subnuclear particles, positions are in fact not observable in the interaction region. Therefore, it seems reasonable to relax the world-line conditions in that region while maintaining the weaker requirement of cluster separability.[10-13]

Nevertheless, the relation of particle position operators to observations remains an important problem in the BT theory. Bound-state wave functions are useful only if they can be used to calculate form factors. Since the particle interaction modifies the representation of the Poincaré group, covariant current densities must acquire interaction terms. The proper determination of these terms is an unsolved problem that inhibits the application of the theory to such problems as quark models. In the context of the classical BT theory Pauri and Prosperi[14] require particle positions (defined as functions of the canonical variables) that satisfy the world-line conditions. These positions cannot satisfy canonical equations of motion.

The essence of the BT theory is the construction of a unitary representation of the Poincaré group as a direct integral of irreducible representations. The generators are defined as functions of the total momentum $\vec{P}$, the center-of-mass position $\vec{X}$, and a complete set of additional intrinsic canonical variables. The canonical transformation linking these variables to individual particle coordinates and momenta is well defined for free particles, but there are no principles that prescribe what this transformation must be in the interaction region of phase space. The BT construction is particularly simple for two particles. For many particles the cluster separability requirement[10-13] presents a serious complication.

Equivalently it is possible in principle to write the generators as functions of individual particle coordinates and momenta and then to construct interaction terms such that the Lie algebra is maintained. It is then easy to verify cluster separability. This construction has been attempted only in a formal expansion in inverse powers of $c$,[10] in particular for electromagnetic interactions.[15-17] The mathematical convergence and domain problems associated with such expansions have not been investigated. Approximately covariant Hamiltonians are obtained from a realization of the Lie algebra of the Galilei group by adding correction terms in various orders of $c^{-1}$ to obtain realizations of the Lie algebra of the Poincaré group to the desired order. Recently Stachel and Havas[18] obtained the most general classical solution that reduces to a Galilei-invariant theory with a static Newtonian potential for $c \to \infty$. The world-line condition was automatically satisfied to order $c^{-2}$, and no difficulty was encountered in satisfying the cluster separability condition to this order (as indeed none was encountered in all earlier work on approximately relativistic Lagrangians or Hamiltonians[19]). The corresponding quantum-mechanical problem was treated by Foldy and Krajcik,[12] who did not require a static potential in the nonrelativistic limit and made cluster separability their main consideration.

The approximate equations obtained by Stachel and Havas[18] reduce to those of WH[7] if it is required that they be compatible with the expansion of an exact theory. The equations of WH, by definition, give a covariant description of world lines[6] as do the equations of Ref. 8 obtained by a relativistic generalization of Newton's second law. The results of Foldy and Krajcik[12] (for particles without spin) may appear, at first sight, to be less general, but after constructing a particular solution they indicate how the most general form can be obtained.

The BT theory has been useful in generating relativistic corrections to order $c^{-2}$ to the nonrelativistic treatment of the deuteron[20] and of nuclear matter[21] and the interaction charge density in the deuteron required by Lorentz covariance to that order has been determined.

In Sec. II we review the quantum-mechanical BT formulation for two particles and discuss sufficient conditions for the validity of the approximations. In Sec. III we construct a special canonical transformation to single-particle variables and the corresponding approximate Hamiltonians. The world-line conditions and consequences are discussed in Sec. IV; it is shown that transformation properties corresponding to the approximate world-line conditions considered in Ref. 18 can be required of the quantum-mechanical position operators provided the nonrelativistic limit of the potential is velocity-independent, except for a possible spin-orbit interaction of the form $\vec{S} \cdot \vec{l} f(r)$. This requirement restricts admissible canonical transformations relating intrinsic coordinates to single-particle coordinates. The cluster separability condition is then automatically satisfied. In the classical limit the class of approximately relativistic Hamiltonians obtained for two particles in Sec. IV and for $N$ particles in Sec. V is the same as that obtained by WH, as is shown in Sec. VI. Conversely, the approximate quantum-mechanical equations (for spinless particles) can be obtained by direct quantization of the equations of WH.[22] Section VII contains a discussion of our results and their implications for phenomenological nucleon-nucleon potentials.

## II. TWO-PARTICLE SYSTEMS

We first review the BT quantum theory for two particles with masses $m_1$ and $m_2$. The generators of the Poincaré group for a multiparticle system have the same form as for a single particle if they are written as functions of the total momentum $\vec{P}$, the canonically conjugate center-of-mass position $\vec{X}$, the total spin $\vec{j}$, and the mass operator $h$. That is,

$$\vec{J}=\vec{X}\times\vec{P}+\vec{j} \tag{1}$$

for the angular momentum,

$$H=(\vec{P}^2c^2+h^2c^4)^{1/2} \tag{2}$$

for the Hamiltonian, and

$$\vec{K}=\frac{1}{2c^2}\{\vec{X},H\}-\frac{\vec{j}\times\vec{P}}{H+hc^2} \tag{3}$$

for the generator of the Lorentz transformations. Here and in the following the braces $\{A,B\}$ denote the anticommutator of two operators $A$ and $B$. It follows from the commutation rules

$$[X_k,P_j]=i\delta_{kj}, \tag{4}$$

$$[\vec{j},P_k]=[\vec{j},X_k]=0, \tag{5}$$

and

$$[\vec{j},h]=[\vec{X},h]=[\vec{P},h]=0, \tag{6}$$

that the ten generators of the Poincaré group $\vec{P},H$, $\vec{J},\vec{K}$ satisfy the commutation relations

$$[P_i,P_j]=0, \tag{7}$$

$$[P_i,H]=0, \tag{8}$$

$$[J_i,H]=0, \tag{9}$$

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \tag{10}$$

$$[J_i,P_j]=i\epsilon_{ijk}P_k, \tag{11}$$

$$[J_i,K_j]=i\epsilon_{ijk}K_k, \tag{12}$$

$$[P_i,K_j]=-ic^{-2}\delta_{ij}H, \tag{13}$$

$$[K_i,K_j]=-ic^{-2}\epsilon_{ijk}J_k, \tag{14}$$

$$[K_j,H]=iP_j, \tag{15}$$

where all indices can take the values 1, 2, or 3, summation over repeated indices is understood, and the units are such that $\hbar=1$.

The mass operator $h$ and the total spin $\vec{j}$ are functions of intrinsic canonical variables that commute with $\vec{X}$ and $\vec{P}$. They are the intrinsic relative momentum $\vec{k}$ and position $\vec{x}$, and two spins $\vec{s}_1$ and $\vec{s}_2$. We have then

$$\vec{j}=\vec{x}\times\vec{k}+\vec{s}_1+\vec{s}_2 \tag{16}$$

and

$$h=h_0+c^{-2}v, \tag{17}$$

where $v$ is an interaction operator and

$$h_0=(c^{-2}\vec{k}^2+m_1^2)^{1/2}+(c^{-2}\vec{k}^2+m_2^2)^{1/2}. \tag{18}$$

To choose the total momentum as one of the canonical variables is not the only way in which the generators may be expressed as functions of center-of-mass variables and intrinsic variables. If we define canonically conjugate variables $\vec{Q}$ and $\vec{X}_Q$ by[23,24]

$$\vec{Q}=\zeta^\dagger\vec{P}\zeta \tag{19}$$

and

$$\vec{X}_Q=\zeta^\dagger\vec{X}\zeta, \tag{20}$$

where the unitary operator $\zeta$ is

$$\zeta=\exp[-\tfrac{1}{2}i\{\vec{X},\vec{P}\}\ln(h/M)], \tag{21}$$

it follows that

$$\vec{Q}=\vec{P}M/h. \tag{22}$$

Here

$$M=m_1+m_2, \tag{23}$$

and in the anticommutator of any two vectors the scalar product is understood. The Poincaré generators are now linear functions of the mass operator $h$. We have

$$\vec{P}=\vec{Q}h/M, \tag{24}$$

$$H=E_0h/M, \tag{25}$$

$$\vec{J}=\vec{X}_Q\times\vec{Q}+\vec{j}, \tag{26}$$

and

$$\vec{K}=\tfrac{1}{2}\{\vec{X}_Q,E_0\}-\frac{\vec{j}\times\vec{Q}}{E_0+Mc^2}, \tag{27}$$

where

$$E_0=(\vec{Q}^2c^2+M^2c^4)^{1/2}. \tag{28}$$

In this representation the Lorentz generator $\vec{K}$ does not depend on the interaction operator $v$. Note that while $h$ commutes with $\zeta$ the separate terms $h_0$ and $v$ do not, and the intrinsic momentum

$$\vec{q}=\zeta^\dagger\vec{k}\zeta \tag{29}$$

differs from $\vec{k}$ except when the interaction $v$ vanishes.

The approximations discussed in the following are based on the assumption that the particle velocities are small compared to $c$. Specifically we approximate

$$E_0\cong Mc^2+\frac{\vec{Q}^2}{2M}-\frac{\vec{Q}^4}{8M^3c^2}. \tag{30}$$

It is easy to verify by Taylor expansion that for

any state $\psi$

$$\left\|\left(E_0 - Mc^2 - \frac{\vec{Q}^2}{2M} + \frac{\vec{Q}^4}{8M^3c^2}\right)\psi\right\| \leq \frac{1}{16}\left\|\frac{\vec{Q}^6}{M^5c^4}\psi\right\|, \tag{31}$$

and therefore the approximation (30) is justified if the right-hand side of Eq. (31) is sufficiently small. We also assume that

$$\left\|\frac{h_0 - M}{M}\psi\right\| \ll \|\psi\|. \tag{32}$$

We have therefore the following approximation for $\zeta$:

$$\zeta \cong 1 - \tfrac{1}{2}i\{\vec{X}, \vec{P}\}\frac{h - M}{M}. \tag{33}$$

For $h_0 - M$ we will use the approximation

$$h_0 - M \cong \frac{\vec{q}^2}{2mc^2} - \frac{1}{8}\frac{\mu_1^3 + \mu_2^3}{m^3}\left(\frac{\vec{q}}{c}\right)^4, \tag{34}$$

where $m$ is the reduced mass

$$m = \frac{m_1 m_2}{M}, \tag{35}$$

and

$$\mu_K = \frac{m_K}{M}, \tag{36}$$

for $K = 1, 2$. Since

$$\left\|\left[h_0 - M - \frac{\vec{q}^2}{2mc^2} + \frac{1}{8}\frac{\mu_1^3 + \mu_2^3}{m^3}\left(\frac{\vec{q}}{c}\right)^4\right]\psi\right\| \leq \frac{1}{16}m(\mu_1^5 + \mu_2^5)\left\|\left(\frac{\vec{q}}{mc}\right)^6\psi\right\|, \tag{37}$$

the approximation is justified for states $\psi$ for which

$$\left\|\left(\frac{\vec{q}}{mc}\right)^6\psi\right\| \ll \|\psi\|. \tag{38}$$

These conditions for the states legitimize the formal expansion of the generators in powers of $c^{-2}$ to first order. Convergence of the infinite series is not necessary. The approximate generators are in our representation

$$\vec{K} = M\vec{X}_Q + \frac{\{\vec{X}_Q, Q^2\}}{4Mc^2} - \frac{\vec{j}\times\vec{Q}}{2Mc^2} \tag{39}$$

and

$$H = hc^2\left[1 + \tfrac{1}{2}\left(\frac{\vec{Q}}{Mc}\right)^2\right] - \frac{\vec{Q}^4}{8M^3c^2}. \tag{40}$$

The crucial step for many applications and for the generalization to $N$-particle systems is the transition to single-particle momenta $\vec{p}_1$ and $\vec{p}_2$ and the conjugate position operators $\vec{r}_1$ and $\vec{r}_2$. For noninteracting particles these single-particle variables are well defined. For interacting particles ambiguities arise which will be explicitly exhibited in Sec. IV.

## III. SINGLE-PARTICLE VARIABLES

For noninteracting particles the relation between $\vec{Q}, \vec{q}$ and $\vec{p}_1, \vec{p}_2$ is determined by the requirement that Poincaré generators be the sums of the individual particle generators; in particular $\vec{P} = \vec{p}_1 + \vec{p}_2$. We have[25]

$$\vec{Q} = (\vec{p}_1 + \vec{p}_2)\frac{M}{h_0}, \tag{41}$$

$$\vec{q} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) + \frac{\vec{Q}}{2M}\left[\frac{\vec{Q}\cdot(\vec{p}_1 - \vec{p}_2)}{Mc^2 + E_0} - \frac{\omega_1 - \omega_2}{c^2}\right], \tag{42}$$

where

$$\omega_K = (\vec{p}_K{}^2 + m_K{}^2c^2)^{1/2}, \tag{43}$$

for $K = 1, 2$, and

$$h_0 = c^{-1}(H_0{}^2 - \vec{P}^2)^{1/2} = c^{-1}[(\omega_1 + \omega_2)^2 - (\vec{p}_1 + \vec{p}_2)^2]^{1/2} \tag{44}$$

is the two-particle mass operator defined in Eq. (5) expressed as a function of $\vec{p}_1$ and $\vec{p}_2$. The spin operator $\vec{s}_K$ of particle $K$ is related to the canonical spin $\vec{\sigma}_K$ by[26]

$$\vec{s}_K = \mathcal{R}(\vec{p}_K, \vec{Q})\vec{\sigma}_K, \tag{45}$$

where $\mathcal{R}$ is the Wigner rotation corresponding to the rotationless Lorentz transformation $L(Q)$ that transforms the four-vector $Q = (c^{-2}E_0, \vec{Q})$ to rest, i.e.,[27]

$$L(Q)Q = (M, 0, 0, 0), \tag{46}$$

$$\mathcal{R}(\vec{p}_K, \vec{Q}) = L(L(Q)p_K)L(Q)L^{-1}(p_K), \tag{47}$$

where

$$p_K = (c^{-1}\omega_K, \vec{p}_K). \tag{48}$$

Equations (41), (42), and (45) determine a canonical transformation

$$\vec{Q}, \vec{X}_Q, \vec{q}, \vec{x}_q, \vec{s}_1, \vec{s}_2 \to \vec{p}_1, \vec{r}_1, \vec{p}_2, \vec{r}_2, \vec{\sigma}_1, \vec{\sigma}_2, \tag{49}$$

and clearly there exists a further canonical transformation to variables

$$\tfrac{1}{2}(\vec{p}_1 - \vec{p}_2),\ \vec{r}_1 - \vec{r}_2,\ \vec{p}_1 + \vec{p}_2,\ \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2),\ \vec{\sigma}_1,\ \vec{\sigma}_2.$$

There exists therefore a unitary transformation $U$ such that

$$\vec{\sigma}_K = U\vec{s}_K U^\dagger, \tag{50}$$

for $K = 1$ and 2,

$$\tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) = U\vec{q}U^\dagger, \tag{51}$$

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 = U\vec{x}_q U^\dagger, \tag{52}$$

$$\vec{P} \equiv \vec{p}_1 + \vec{p}_2 = U\vec{Q}U^\dagger, \tag{53}$$

$$\tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) = U\vec{X}_Q U^\dagger. \tag{54}$$

The transformation $U$ gives us all the variables $Q$, $\vec{X}_Q, \vec{q}, \vec{x}_q, \vec{s}_1, \vec{s}_2$ and hence the generators explicitly as functions of the single-particle variables. It is clear from Eqs. (41), (42), and (45) that for $\vec{P}=0$ the transformation $U$ reduces to the identity. The transformation for noninteracting particles was designed to yield

$$H = \sum_K H_K, \tag{55}$$

$$H_K = (\vec{p}_K{}^2c^2 + m_K{}^2c^4)^{1/2}, \tag{56}$$

$$\vec{K} = \sum_K \vec{K}_K, \tag{57}$$

$$\vec{K}_K = \frac{1}{2c^2}\{\vec{r}_K, H_K\} - \frac{\vec{\sigma}_K \times \vec{p}_K}{H_K + m_K c^2}. \tag{58}$$

We now reinstate the interaction by replacing $h_0$ by $h$ in Eq. (41), and retaining the other relations (42) and (45) unchanged. There are no principles that prevent us from introducing other interaction-dependent terms into the transformation $U$. We will discuss this ambiguity at length later.

To order $c^{-2}$ it follows from Eqs. (22) and (42) that

$$\vec{Q} = \vec{P} - \vec{P}(h - M)/M, \tag{59}$$

and

$$\vec{q} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) + \tfrac{1}{2}(\mu_2 - \mu_1)\vec{Q} - \tfrac{1}{2}\vec{Q}\left[\frac{\vec{Q}\cdot\vec{p}}{(Mc)^2} + \frac{\mu_2 - \mu_1}{2mMc^2}\vec{p}^2\right], \tag{60}$$

where $\vec{p}$ is by definition

$$\vec{p} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) + \tfrac{1}{2}(\mu_2 - \mu_1)(\vec{p}_1 + \vec{p}_2) = \mu_2\vec{p}_1 - \mu_1\vec{p}_2. \tag{61}$$

Let the operators $\Phi_0$ and $\Phi_1$ be defined by

$$\Phi_0 = -\tfrac{1}{2}\vec{P}\cdot\vec{r}\left[\frac{\vec{P}\cdot\vec{p}}{(Mc)^2} + \frac{\mu_2 - \mu_1}{2mMc^2}\vec{p}^2\right] - \sum_K \frac{\vec{\sigma}_K\cdot(\vec{P}\times\vec{p}_K)}{2m_K Mc^2} \tag{62}$$

and

$$\Phi_1 = -\tfrac{1}{8}\left\{\{\vec{P}, (\vec{r}_1 + \vec{r}_2)\}, \frac{h - M}{M}\right\}. \tag{63}$$

It is then easy to verify that the operator

$$U = e^{i\vec{P}\cdot\vec{r}(\mu_2 - \mu_1)/2}[1 + i(\Phi_0 + \Phi_1)] \tag{64}$$

has all the required properties.

If we express the generators $\vec{K}$ and $H$ given by Eqs. (22) and (25) as functions of the individual particle variables defined by Eqs. (50)–(54), then it follows that

$$\vec{K} = \vec{K}_1 + \vec{K}_2 + \vec{K}' \tag{65}$$

and

$$H = H_1 + H_2 + H', \tag{66}$$

where a prime denotes the interaction part of any operator. To order $c^{-2}$ we have from Eqs. (54) and (64)

$$\vec{X}_Q = \vec{R} + i[\vec{R}, (\Phi_0 + \Phi_1)], \tag{67}$$

where

$$\vec{R} = \mu_1\vec{r}_1 + \mu_2\vec{r}_2. \tag{68}$$

The transformation $(\vec{p}_1, \vec{r}_1, \vec{p}_2, \vec{r}_2) \to (\vec{P}, \vec{R}, \vec{p}, \vec{r})$ is canonical.

It follows from Eqs. (39), (67), and (64) that

$$\vec{K}' = iM[\vec{R}, \Phi_1'] = \frac{1}{4c^2}\{(\vec{r}_1 + \vec{r}_2), V^0\}, \tag{69}$$

where

$$V^0 = \lim_{c\to\infty} v \tag{70}$$

is the nonrelativistic limit of $v$, and thus

$$\Phi_1' = -\vec{P}\cdot\vec{K}'. \tag{71}$$

From Eqs. (40) it follows that to order $c^{-2}$

$$H' = v(\vec{r}, \vec{p}) + i[V^0, \Phi_0] + i\left[\frac{\vec{p}^2}{2m}, \Phi_1'\right] + i[V^0, \Phi_1] - \frac{V^0\vec{P}^2}{2M^2c^2}. \tag{72}$$

In Eq. (40), the mass operator $h$ was a function of $\vec{q}$ and $\vec{x}_q$, while in Eq. (72) it has been expressed as a function of $\vec{r}$ and $\vec{p}$ defined by Eqs. (51), (52), (61), and (64). Equation (59) was used to obtain the last term.

The operator $\Phi_1$ was defined by Eq. (63) to meet the requirement $P = \vec{p}_1 + \vec{p}_2$, and hence

$$\vec{P}' = i[\vec{P}, \Phi_1'] + \frac{V^0\vec{P}}{Mc^2} = 0. \tag{73}$$

We now assume that the nonrelativistic limit of the potential is velocity-independent, i.e., that

$$[\vec{r}, V^0] = 0, \quad V^0 = V^0(\vec{r}). \tag{74}$$

The significance of this restriction will be discussed in the next section. The assumption implies that the momentum-space kernel $(\vec{p}'|v|\vec{p})$ depends only on the difference $\vec{p}' - \vec{p}$ in the nonrelativistic limit, i.e.,

$$\lim_{c\to\infty}(\vec{p}'|v|\vec{p}) = \tilde{V}^0(\vec{p}' - \vec{p}), \tag{75}$$

where $\tilde{V}^0(\vec{p}' - \vec{p})$ is the Fourier transform of $V^0(\vec{r})$. We assume further that the relativistic corrections to $V^0$ are obtained by expansion of $(\vec{p}'|v|\vec{p})$ in powers of $(\vec{p}' + \vec{p})/(2mc)$. We can then write

$$(\vec{p}'|v|\vec{p}) = (w'w)^{1/2}\tilde{V}(\vec{p}'-\vec{p}) - \frac{1}{2}\left(\frac{\vec{p}'+\vec{p}}{2mc}\right)^2\tilde{A}(\vec{p}'-\vec{p}) - \frac{1}{2}\left[\frac{(\vec{p}'+\vec{p})\cdot\vec{r}}{2mc}\right]^2\tilde{B}(\vec{p}'-\vec{p}), \quad (76)$$

where $\vec{r}$ denotes $-i\nabla_{\vec{p}'-\vec{p}}$ and

$$w = \frac{m_1 m_2 c^2}{[(\vec{p}^2+m_1^2c^2)(\vec{p}^2+m_2^2c^2)]^{1/2}}. \quad (77)$$

The last two terms in Eq. (76), which involve two arbitrary functions $\tilde{A}$ and $\tilde{B}$, represent the most general spin-independent expression that may arise in the expansion of $v$. There is no term proportional to $(\vec{p}'+\vec{p})\cdot\vec{r}/(2mc)$ in Eq. (76) since we do not wish to pursue the possibility that $v$ might not be time-reversal invariant. The function $\tilde{V}(\vec{p}'-\vec{p})$ need not be equal to the limit $\tilde{V}^0(\vec{p}'-\vec{p})$. There is no advantage in a formal expansion of $\tilde{V}(\vec{p}'-\vec{p})$, or its Fourier transform $V(\vec{r})$, in powers of $c^{-1}$.

For any function $\tilde{f}(\vec{p}'-\vec{p})$ we define

$$f(r) = \frac{1}{(2\pi)^{3/2}}\int d^3(\vec{p}'-\vec{p})\tilde{f}(\vec{p}'-\vec{p})e^{i(\vec{p}'-\vec{p})\cdot\vec{r}}. \quad (78)$$

Momentum-space kernels of the form

$$(p_j'+p_j)\tilde{f}(\vec{p}'-\vec{p}) \quad (79)$$

and

$$(p_i'+p_i)(p_j'+p_j)\tilde{f}(\vec{p}'-\vec{p}) \quad (80)$$

thus represent operators $\{p_j, f(\vec{r})\}$ and $\{p_i,\{p_j,f(\vec{r})\}\}$, respectively. In order to simplify the notation it will be convenient in the following to have it understood that complete symmetrization of operator products is implied. This means that we will write $p_j f(\vec{r})$ for $\frac{1}{2}\{p_j,f(\vec{r})\}$ and either $p_i p_j g(\vec{r})f(\vec{r})$ or $p_i g(\vec{r})p_j f(\vec{r})$ for $\frac{1}{4}\{p_i,\{p_j,g(\vec{r})f(\vec{r})\}\}$. This unambiguous convention is precisely the Weyl prescription[28] for the quantization of the classical Hamiltonian.

The form of the first term of Eq. (76) was chosen for later convenience; expanding $(w'w)^{1/2}$ in powers of $c^{-2}$ we obtain

$$v = V(r) + \frac{\vec{p}^2V}{mMc^2} + \left(1-\frac{2m}{M}\right)\frac{\nabla^2V}{2m^2c^2} - \frac{1}{2}\left(\frac{\vec{p}}{mc}\right)^2[V(\vec{r})+A(\vec{r})] - \frac{1}{2}\left(\frac{\vec{p}\cdot\vec{r}}{mc}\right)^2B(\vec{r}). \quad (81)$$

Without loss of generality we can change the definition of $V(r)$ to include the term proportional to $\nabla^2V$. Since this term is of order $c^{-2}$, this change in definition does not affect the terms that are already of order $c^{-2}$ and the net effect is to delete the $\nabla^2V$ term from Eq. (81). For the same reason we may replace $V^0$ by $V$ in Eqs. (69), (71), (72), and (73) without affecting their validity.

From Eqs. (62) and (63) it follows that

$$i[V,\Phi_0] = \frac{\vec{P}\cdot\vec{r}\,\vec{P}\cdot\nabla V}{2M^2c^2} + \vec{P}\cdot\vec{r}\,\vec{p}\cdot\nabla V\frac{\mu_2-\mu_1}{2mMc^2} - \frac{i[(\mu_2\vec{\sigma}_1-\mu_1\vec{\sigma}_2)\cdot\vec{P}\times\vec{p},V]}{2mMc^2} \quad (82)$$

and

$$i\left[\frac{\vec{p}^2}{2m},\Phi_1'\right] + i[V,\Phi_1] = \frac{\vec{P}\cdot\vec{p}V(\vec{r})(\mu_2-\mu_1)}{2mMc^2}. \quad (83)$$

In the absence of tensor forces we have

$$\nabla V = \frac{\vec{r}}{r}\frac{dV}{dr}, \quad r\equiv|\vec{r}|. \quad (84)$$

From Eqs. (72), (81), (82), and (83) it follows that

$$H' = V(r) - \frac{i[(\mu_2\vec{\sigma}_1-\mu_1\vec{\sigma}_2)\cdot(\vec{P}\times\vec{p}),V]}{2mMc^2} - \frac{1}{2c^2}\left\{\frac{\vec{p}_1\cdot\vec{p}_2}{m_1m_2}V - \frac{\vec{p}_1\cdot\vec{r}\,\vec{p}_2\cdot\vec{r}}{m_1m_2}\frac{1}{r}\frac{dV}{dr} + \left(\frac{\vec{p}}{m}\right)^2\left[V\left(1-\frac{m}{M}\right)+A\right] + \left(\frac{\vec{p}\cdot\vec{r}}{m}\right)^2\left(B-\frac{m}{M}\frac{1}{r}\frac{dV}{dr}\right)\right\}. \quad (85)$$

The relatively simple expression (85) has been obtained by assuming $V$ to be a central velocity-independent potential. The expressions (69) and (70) are independent of these assumptions. A more general form would have resulted from the addition of an arbitrary interaction term $\Phi_2$ to $\Phi_0+\Phi_1$ in Eq. (64). The interaction terms in $\vec{K}$ and $H$ would then be

$$\vec{K}_{\rm int} = \vec{K}' + \vec{K}'' \quad (86)$$

and

$$H_{\rm int} = H' + H'', \quad (87)$$

where, in analogy to Eqs. (69) and (72), we get

$$\vec{K}'' = iM[\vec{R},\Phi_2] \quad (88)$$

and

$$H'' = i[(\vec{p}^2/2m+V),\Phi_2]. \quad (89)$$

In order to preserve the conditions that $\vec{P}$ and $\vec{J}$ have no interaction term, i.e., that $\vec{P}'=0$ and $\vec{J}'=0$, we must require that $\Phi_2$ be invariant under rotations and translations. Since we want $H''$ to be time-reversal invariant, $\Phi_2$ must be odd under time reversal. In the next section we examine

possible additional restrictions on $\Phi_2$ imposed by the assumption that $\vec{r}_1$ and $\vec{r}_2$ are observable particle positions.

## IV. THE WORLD-LINE CONDITIONS

The world-line condition for particles without spin can be stated as the requirement that the commutation relations of the individual particle positions $\vec{r}_K$ with the Lorentz generator $\vec{K}$ are the same with and without interaction,[29] namely

$$[r_{Ki}, K_j] = \frac{1}{2c^2}\{r_{Kj}, [r_{Ki}, H]\}. \tag{90}$$

The obvious generalization to noninteracting particles with spin is because of Eq. (58):

$$[r_{Ki}, K_j] = \frac{1}{2c^2}\{r_{Kj}, [r_{Ki}, H]\} + \frac{[r_{Ki}, H](\vec{\sigma}_K \times \vec{p}_K)_j}{(H_K + m_K c^2)^2} - i\frac{\epsilon_{ijk}\sigma_{Kk}}{H_K + m_K c^2} \cong \frac{1}{2c^2}\{r_{Kj}, [r_{Ki}, H]\} - i\frac{\epsilon_{ijk}\sigma_{Kk}}{2m_K c^2}. \tag{91}$$

Even for interacting particles the last term in the approximate form is independent of the interaction. It is well known that the classical condition cannot be satisfied exactly for interacting particles.[9] However, it can be satisfied approximately to order $c^{-2}$. The requirement that Eq. (91) be satisfied to order $c^{-2}$ implies restrictions for both $V$ and $U$. From Eq. (69) the interaction term $\vec{K}' + \vec{K}''$ of $\vec{K}$ and the nonrelativistic limit of the potential must satisfy the relation

$$[r_{Ki}, (\vec{K}' + \vec{K}'')_j] = \frac{1}{2c^2}[r_{Ki}, \{r_{Kj}, V\}]. \tag{92}$$

Since by definition (68)

$$\vec{r}_1 = \vec{R} + \mu_2 \vec{r} \tag{93}$$

and

$$\vec{r}_2 = \vec{R} - \mu_1 \vec{r}, \tag{94}$$

it follows from Eq. (92) and

$$[\vec{R}, V] = 0 \tag{95}$$

that

$$[R_i, K_j''] = \frac{\mu_1\mu_2}{2c^2}[r_i, \{r_j, V\}] \tag{96}$$

and

$$(\mu_1 - \mu_2)[R_i, K_j''] = -\mu_1\mu_2[r_i, (\vec{K}' + \vec{K}'' - c^{-2}\vec{R}V)_j]. \tag{97}$$

If $\vec{K}''$ commutes with $\vec{R}$ it follows from Eqs. (96) and (97) that the potential $V$ is velocity-independent, i.e., $[\vec{r}, V] = 0$, and that

$$[r_i, K_j''] = 0. \tag{98}$$

In any case it follows from Eqs. (96) and (97) and the Jacobi identity that

$$[r_k, [r_i\{r_j, V\}]] = 0. \tag{99}$$

Hence $V$ can depend at most linearly on $\vec{p}$. As a consequence of the assumptions adopted in Sec. III, there are no terms proportional to $\vec{p}\cdot\vec{r}$. Thus, the only nontrivial scalar quantity that is linear in $\vec{p}$ is a spin-orbit coupling of the form $\vec{s}\cdot\vec{l}f(r)$, where $\vec{l} = \vec{r}\times\vec{p}$. If such a nonrelativistic spin-orbit coupling is present, then Eq. (96) requires that

$$[R_i, K_j''] = iM[R_i, [R_j, \Phi_2]] \neq 0. \tag{100}$$

If we consider either spinless particles or averages over all possible spin orientations, then the world-line conditions imply the conditions (74), (98), and

$$[R_i, K_j''] = 0. \tag{101}$$

We must find the most general form of $\Phi_2$ which is translation and rotation invariant, odd under time reversal, and consistent with Eqs. (97) and (101). Terms in $\Phi_2$ containing higher powers of the momenta than the first would lead to terms in $H''$ of third or higher powers in the momenta. If we assume that the interaction terms of $H$ contain the momenta only through the velocities $\vec{P}/M$ and $\vec{p}/m$, such terms will be of order $c^{-3}$ or higher. Thus the most general form of $\Phi_2$ satisfying all our requirements is

$$\Phi_2 = -\frac{\vec{P}\cdot\vec{r}}{2Mc^2}W(r) - \frac{1}{2}\left\{\frac{\vec{p}\cdot\vec{r}}{2mc^2}, Z(r)\right\}, \tag{102}$$

where the factors are chosen for later convenience, and $W(r)$ and $Z(r)$ are arbitrary functions. Then it follows from Eqs. (88) and (89) that

$$\vec{K}'' = \tfrac{1}{2}\vec{r}W(r) \tag{103}$$

and

$$H'' = \frac{\vec{r}\cdot\nabla V}{mc^2}Z(r) - \frac{\vec{P}\cdot\vec{p}}{2mMc^2}W - \frac{\vec{P}\cdot\vec{r}\,\vec{p}\cdot\vec{r}}{2mMc^2}\frac{1}{r}\frac{dW}{dr} - \frac{1}{2}\left(\frac{\vec{p}}{mc}\right)^2 Z - \frac{1}{2}\left(\frac{\vec{p}\cdot\vec{r}}{mc}\right)^2\frac{1}{r}\frac{dZ}{dr}. \tag{104}$$

The first term in Eq. (104) is a function of $r$ which can be included in the definition of $V(r)$ as discussed for Eq. (81). The last two terms are, respectively, of the same form as the terms proportional to $A$ and $B$ in Eq. (85). As long as $A$ and $B$ are arbitrary functions, the further arbitrary function $Z$ does not lead to more general results. Thus we have for spinless particles (or after taking averages over spin)

$$H'+H''=V(r)-\frac{1}{2c^2}\left\{\frac{1}{m_1m_2}\left[\vec{p}_1\cdot\vec{p}_2V(r)-\vec{p}_1\cdot\vec{r}\,\vec{p}_2\cdot\vec{r}\,\frac{1}{r}\frac{dV}{dr}\right]\right.$$
$$+\left(\frac{\vec{p}_1}{m_1}-\frac{\vec{p}_2}{m_2}\right)^2(V+X)+\left[\left(\frac{\vec{p}_1}{m_1}-\frac{\vec{p}_2}{m_2}\right)\cdot\vec{r}\right]^2Y+\frac{\vec{p}_1}{m_1}\cdot\left(\frac{\vec{p}_1}{m_1}-\frac{\vec{p}_2}{m_2}\right)W$$
$$\left.+\frac{\vec{p}_2\cdot\vec{r}}{m_2}\left(\frac{\vec{p}_1\cdot\vec{r}}{m_1}-\frac{\vec{p}_2\cdot\vec{r}}{m_2}\right)\frac{1}{r}\frac{dW}{dr}\right\}, \tag{105}$$

where by definition

$$X=A-\frac{m}{M}V+Z-\frac{m_2}{M}W,$$
$$Y=B-\frac{m}{M}\frac{1}{r}\frac{dV}{dr}+\frac{1}{r}\frac{dZ}{dr}+\frac{m_1}{M}\frac{1}{r}\frac{dW}{dr}. \tag{106}$$

## V. *N*-PARTICLE SYSTEMS

As noted in the Introduction, the cluster separability requirement is not readily satisfied in the BT theory. Therefore, instead of proceeding with an expansion of the exact BT theory, we shall construct an approximate many-body Hamiltonian from the approximate two-body Hamiltonians obtained in Secs. II and III, maintaining the Lie algebra to the order required.

For any cluster of two particles $I$ and $K$ we adopt the notation $V \to V_{IK}$ and similarly for all other two-body operators. From Eqs. (86), (69), (87), (72), and (89) if follows that the two-body interaction terms in the generators $\vec{K}$ and $H$ are

$$\vec{K}_{IK}=\tfrac{1}{4}\{(\vec{r}_I+\vec{r}_K),V_{IK}\}+\vec{K}''_{IK} \tag{107}$$

and

$$H_{IK}=v_{IK}(\vec{r}_{IK},\vec{p}_{IK})+i[V_{IK},\Phi_{0IK}]$$
$$+i\left[\left(\frac{p_{IK}^2}{2m_{IK}}+V_{IK}\right),(\Phi_{1IK}+\Phi_{2IK})\right]-\frac{\vec{P}_{IK}^2V_{IK}}{2M_{IK}^2c^2}. \tag{108}$$

If we tentatively assume that the Poincaré generators are of the form

$$\vec{P}=\sum_K\vec{p}_K, \tag{109}$$
$$\vec{J}=\sum_K\vec{J}_K, \tag{110}$$
$$H=\sum_K H_K+\sum_{I<K}\sum H_{IK}, \tag{111}$$
$$\vec{K}=\sum_K\vec{K}_K+\sum_{I<K}\sum\vec{K}_{IK}, \tag{112}$$

then the cluster separability requirement is manifestly satisfied, as are the commutation relations (7)–(12). However, we have to verify the remaining relations (13)–(15). We have

$$[P_i,K_j]=\sum_K[P_{Ki},K_{Kj}]$$
$$+\tfrac{1}{2}\sum_{I<K}\sum[(\vec{p}_I+\vec{p}_K)_i,K_{IKj}]$$
$$=-i\delta_{ij}\left(\sum_K H_K+\sum_{I<K}\sum H_{IK}\right), \tag{113}$$

and thus Eq. (13) is satisfied. Furthermore,

$$[K_i,K_j]=\sum_K[K_{Ki},K_{Kj}]$$
$$+\tfrac{1}{2}\sum_{I<K}\sum\{[(\vec{K}_I+\vec{K}_K)_i,K_{IKj}]$$
$$+[K_{IKi},(\vec{K}_I+\vec{K}_K)_j]\}, \tag{114}$$

and since

$$\vec{K}_{IK}=iM_{IK}[\vec{R}_{IK},(\Phi_{1IK}+\Phi_{2IK})], \tag{115}$$

it follows from the Jacobi identity that the second term in Eq. (114) vanishes. Equation (14) is therefore verified. On the other hand,

$$[H,\vec{K}]=\sum_K[H_K,\vec{K}_K]+\sum_{I<K}\sum\{[(H_I+H_K),\vec{K}_{IK}+[H_{IK},(\vec{K}_I+\vec{K}_K)]\}$$
$$+\sum_{\substack{I<K,\,I<L\\K\neq L}}\sum\sum[H_{IK},\vec{K}_{IL}]+\sum_{I<L,\,K<L}\sum\sum[H_{IL},\vec{K}_{KL}]+\sum_{I<K<L}\sum\sum\{[H_{KL},\vec{K}_{IK}]+[H_{IK},\vec{K}_{KL}]\}. \tag{116}$$

Since the single sum equals $-i\vec{P}$ and the double sums vanish, Eq. (15) is satisfied if and only if all triple sums in Eq. (116) add up to zero. This is the case if the $V$'s and $W$'s are only functions of the particle positions. Otherwise they must be canceled by adding a suitable three-body interaction of order $c^{-2}$ to the Hamiltonian $H$.[12]

It should also be noted that independently of these considerations any three- (or more) body terms in $H$ which only depend on the various particle sep-

arations and are of order $c^{-2}$ are compatible with the commutation relations (7)–(15) to that order, and thus could be introduced in Eq. (111) as possible relativistic corrections. However, it would be more appropriate to consider such terms only if many-body terms are already included in the lowest order. We shall therefore not pursue this possibility here.

Summarizing our results, for the case in which the nonrelativistic interactions are velocity-independent and spin-independent (or averaged over spins), the many-body Hamiltonian to order $c^{-2}$ is given by

$$H=\sum_K\left(m_Kc^2+\frac{\vec{p}_K^{\,2}}{2m_K}-\frac{\vec{p}_K^{\,4}}{8m_K^{\,3}c^2}\right)+\sum_{I<K}\sum V_{IK}$$
$$-\frac{1}{2c^2}\sum_{I<K}\sum\left\{\frac{\vec{p}_I\cdot\vec{p}_K}{m_Im_K}-\frac{\vec{p}_I\cdot\vec{r}_{IK}\,\vec{p}_K\cdot\vec{r}_{IK}}{m_Im_Kr_{IK}}\frac{dV_{IK}}{dr_{IK}}+\left(\frac{\vec{p}_I}{m_I}-\frac{\vec{p}_K}{m_K}\right)^2(V_{IK}+X_{IK})+\left[\left(\frac{\vec{p}_I}{m_I}-\frac{\vec{p}_K}{m_K}\right)\cdot\vec{r}_{IK}\right]^2Y_{IK}\right.$$
$$\left.+\left(\frac{\vec{p}_I^{\,2}}{m_I^{\,2}}-\frac{\vec{p}_I\cdot\vec{p}_K}{m_Im_K}\right)W_{IK}-\left[\left(\frac{\vec{p}_K\cdot\vec{r}_{IK}}{m_K}\right)^2-\frac{\vec{p}_I\cdot\vec{r}_{IK}\,\vec{p}_K\cdot\vec{r}_{IK}}{m_Im_K}\right]\frac{1}{r_{IK}}\frac{dW_{IK}}{dr_{IK}}\right\} \tag{117}$$

[see Eqs. (66), (85), (87), and (105)]. The corresponding Lorentz generator $\vec{K}$ is according to Eqs. (65), (69), and (104)

$$\vec{K}=\sum_I\left(m_I+\frac{\vec{P}_I^{\,2}}{2m_Ic^2}\right)\vec{r}_I$$
$$+\frac{1}{2c^2}\sum_{I<K}\sum[(\vec{r}_I+\vec{r}_K)V_{IK}+(\vec{r}_I-\vec{r}_K)W_{IK}]\,. \tag{118}$$

We have treated all particles as distinguishable. Note that in a theory of direct particle interactions there is no spin-statistics theorem, and thus for identical particles the symmetry (or antisymmetry) of the wave function under interchange of a pair is an independent assumption.[11]

## VI. APPROXIMATELY RELATIVISTIC HAMILTONIANS RELATED TO POINCARÉ-INVARIANT VARIATIONAL PRINCIPLES

The Hamiltonian (117), constructed to provide an approximate realization of the Lie algebra (7)–(15) of the Poincaré group, was based on a two-body Hamiltonian obtained by an expansion of the exact two-body BT Hamiltonian (3).

Results that are equivalent to order $c^{-2}$ follow also from a different exact theory proposed by Havas,[6] which, in contrast to the BT theory, is formulated in terms of physically significant single-particle variables from the outset. It starts from a Poincaré-invariant "Fokker-type" variational principle

$$\delta I=\delta(I_1+I_2)=0\,, \tag{119}$$

where $I_1$ is defined by[30]

$$I_1=-\sum_{I<K}\sum\int_{-\infty}^{\infty}\int d\tau_I\,d\tau_K\Lambda_{IK}(s_{IK}^\mu,v_I^\mu,v_K^\mu)\,, \tag{120}$$
$$s_{IK}^\mu\equiv z_I^\mu(\tau_I)-z_K^\mu(\tau_K)\,,$$

where $z_I^\mu$ is the world line of the $I$th particle parameterized by the proper time $\tau_I$ in the Minkowski space of metric

$$\eta_{\mu\nu}=0,\quad \mu\neq\nu,$$
$$\eta_{00}=1,\quad \eta_{11}=\eta_{22}=\eta_{33}=-c^{-2}, \tag{121}$$
$$\eta_{\mu\rho}\eta^{\rho\nu}=\delta_\mu^\nu\,,$$

and $v_I^\mu$ is the four-velocity defined by

$$v_I^\mu\equiv\frac{dz_I^\mu}{d\tau_I}\,. \tag{122}$$

Since from Eqs. (121) and (122) we have

$$v_{I\mu}v_I^\mu=1,\quad v_I^\mu\,\delta v_{I\mu}=0\,, \tag{123}$$

the variations of the components of each world line are not independent; the term $\delta I_2$ is introduced to maintain the conditions (123). It is defined by

$$\delta I_2=-\sum_K\int_{-\infty}^{\infty}d\tau_K M_K(\tau_K)c^2v_K^\mu\delta v_{K\mu}\,, \tag{124}$$

where the $M_K(\tau_K)c^2$ are Lagrange multipliers.

Since the $z_I^\mu$ are four-vectors under the Poincaré group, the world-line condition is automatically satisfied, but no exact canonical formulation of the form (7)–(15) is possible. If the interaction is such that it possesses a Newtonian limit in the sense that for $c\to\infty$ Eq. (119) is equivalent to

$$\delta I=0,$$
$$I=\int_{-\infty}^{\infty}dt\,L[\vec{r}_K(t),\vec{v}_K(t)],\quad K=1,\ldots,N \tag{125}$$

with

$$L=T-V,\quad T\equiv\tfrac{1}{2}\sum_K m_K\vec{v}_K^{\,2},\quad \vec{v}_K=\frac{d\vec{r}_K}{dt}\,,$$
$$V\equiv\sum_{I<K}\sum V_{IK}(r_{IK}),\quad r_{IK}\equiv|\vec{r}_I(t)-\vec{r}_K(t)|\,, \tag{126}$$

then, as shown in WH,[7] the variational principle (119) is equivalent to a variational principle of the

form (125) to order $c^{-2}$, but with a Lagrangian differing from that of (126) [see Eq. (WH75)]. This approximately relativistic Lagrangian is equivalent to an approximately relativistic Hamiltonian [Eq. (WH105)] of the form

$$H = H_2 + V - I_{\mathrm{PN}}\,, \tag{127}$$

where

$$H_2 = \sum_K \left( m_K c^2 + \frac{\vec{p}_K^{\,2}}{2m_K} - \frac{\vec{p}_K^{\,4}}{8m_K^{\,3}c^2} \right), \tag{128}$$

$V$ is given by (126), and the "post-Newtonian" interaction $I_{\mathrm{PN}}$ is given by

$$I_{\mathrm{PN}} = \frac{1}{2c^2} \sum_{I<K} \sum \left\{ \frac{\vec{p}_I \cdot \vec{p}_K}{m_I m_K} V_{IK} - \frac{\vec{p}_I \cdot \vec{r}_{IK}\, \vec{p}_K \cdot \vec{r}_{IK}}{m_I m_K r_{IK}} \frac{dV_{IK}}{dr_{IK}} \right.$$
$$+ \left( \frac{\vec{p}_I}{m_I} - \frac{\vec{p}_K}{m_K} \right)^2 (V_{IK} + X_{IK}) + \left[ \left( \frac{\vec{p}_I}{m_I} - \frac{\vec{p}_K}{m_K} \right) \cdot \vec{r}_{IK} \right]^2 Y_{IK}$$
$$\left. + \left( \frac{\vec{p}_I^{\,2}}{m_I^{\,2}} - \frac{\vec{p}_I \cdot \vec{p}_K}{m_I m_K} \right) W_{IK} - \left[ \left( \frac{\vec{p}_K \cdot \vec{r}_{IK}}{m_K} \right)^2 - \frac{\vec{p}_I \cdot \vec{r}_{IK}\, \vec{p}_K \cdot \vec{r}_K}{m_I m_K} \right] \frac{1}{r_{IK}} \frac{dW_{IK}}{dr_{IK}} \right\}, \quad \vec{p}_I \equiv \frac{\partial L}{\partial \vec{v}_I}\,. \tag{129}$$

Here $V_{IK}$ is the Newtonian potential, which can be defined from the relativistic function $\Lambda_{IK}$ by a procedure described in WH. Similarly, $X_{IK}$, $Y_{IK}$, and $W_{IK}$ are functions which are defined by $\Lambda_{IK}$ and in general are independent of $V_{IK}$. The particular form of their dependence on $\Lambda_{IK}$ is not needed here; indeed, the problem for any application of this formalism would be to determine a possible $\Lambda_{IK}$ from the four experimentally found functions rather than to calculate these functions from a known $\Lambda_{IK}$.

The total momentum and angular momentum are

$$\vec{P} = \sum_K \vec{p}_K \tag{130}$$

and

$$\vec{J} = \sum_K \vec{r}_K \times \vec{p}_K\,. \tag{131}$$

$H$ and the components of $\vec{P}, \vec{J}$ and the center-of-mass quantity

$$\vec{G} \equiv \vec{K} - \vec{P}t = \sum_I \left[ m_I + \frac{\vec{p}_I^{\,2}}{2m_I c^2} + \frac{1}{2c^2} \sum_{K \neq I} (V_{IK} + W_{IK}) \right] \vec{r}_I - Pt\,, \quad W_{KI} \equiv -W_{IK}, \quad I<K \tag{132}$$

are ten constants of the motion, whose Poisson brackets satisfy the Lie algebra of the Poincaré group to order $c^{-2}$.[7,18]

In Eq. (132) a definition of antisymmetry for $W_{IK}$ was introduced purely as a matter of convenience to obtain a $\vec{K}$ or $\vec{G}$ of the same general structure as in the Newtonian case, i.e., with a sum of terms each multiplied by an individual position vector (which is the form previously found for all particular approximately relativistic theories investigated by various authors[19]). Whether such a definition (which is not needed anywhere in the considerations of WH) is introduced or not, Eq. (132) can be written

$$\vec{G} = \sum_I \left( m_I + \frac{\vec{p}_I^{\,2}}{2m_I c^2} \right) \vec{r}_I + \frac{1}{2c^2} \sum_{I<K} \sum \left[ (\vec{r}_I + \vec{r}_K) V_{IK} + (\vec{r}_I - \vec{r}_K) W_{IK} \right] - \vec{P}t\,. \tag{133}$$

Since the components of $\vec{G}$ satisfy the same Lie brackets as those of $\vec{K}$, all the results of the preceding sections can readily be transcribed from $\vec{K}$ to $\vec{G}$, and conversely for the results of WH and of Ref. 18. To obtain the classical limit of our earlier results, we just have to replace all (possibly noncommuting) quantum-mechanical operators by the corresponding (commuting) classical quantities. It is then immediately apparent that our Eqs. (117) and (118) reduce to the classical results of WH [our Eqs. (129) and (133)].

Conversely, if we restrict ourselves to spinless particles, the classical Hamiltonian (127) differs from the nonrelativistic one only by a more complicated dependence on positions and momenta, and thus the corresponding quantum-mechanical Hamiltonian can be obtained as usual by replacing the classical positions and momenta by the corresponding quantum-mechanical operators. The only difficulty in this procedure lies in finding the proper order of these noncommuting operators for the various terms in $I_{\mathrm{PN}}$, given by Eq. (129), which are products of the momenta and of various functions of $r_{IK}$ and the components of $\vec{r}_{IK}$. The question of the correct order is an old one, and various schemes for resolving the ambiguities

have been suggested. One method, based on group-theoretical considerations, is due to Weyl[28]; explicit results for one dimension were obtained by McCoy[31] and for an arbitrary number of dimensions by Daughaday and Nigam.[32] An alternative, generally inequivalent, method is due to Born and Jordan,[33] and was also generalized in Ref. 32. However, Weyl's method appears preferable,[34] and we shall not consider any others here.

It is sufficient to state the quantization rules only for the case of two particles $I$ and $K$. Furthermore, because of the form (129) of $I_{\rm PN}$, we need only state the rules for terms which are bilinear in the components of $\vec{p}_I$ and $\vec{p}_K$, or quadratic in the components of either. Wherever we have a classical expression

$$f^{\alpha,2-\alpha} \equiv \varphi(r_{IK}, \vec{r}_I, \vec{r}_K) p_{Ik}{}^{\alpha} p_{Kl}{}^{2-\alpha}, \quad \alpha = 0,\ 1,\ \text{or}\ 2 \tag{134}$$

the corresponding quantum-mechanical operator $F^{\alpha,2-\alpha}$ [following from Eq. (22) of Ref. 32] is given by

$$F^{0,2} \equiv \tfrac{1}{4}\{\{\varphi, p_{Kl}\}, p_{Kl}\}$$
$$= \tfrac{1}{4}(p_{Kl}{}^2\varphi + 2p_{Kl}\varphi p_{Kl} + \varphi p_{Kl}{}^2), \tag{135}$$
$$F^{1,1} \equiv \tfrac{1}{4}\{\{\varphi, p_{Ik}\}, p_{Kl}\}$$
$$= \tfrac{1}{4}(p_{Ik}p_{Kl}\varphi + p_{Ik}\varphi p_{Kl} + p_{Kl}\varphi p_{Ik} + \varphi p_{Ik}p_{Kl}), \tag{136}$$
$$F^{2,0} \equiv \tfrac{1}{4}\{\{\varphi, p_{Ik}\}, p_{Ik}\}$$
$$= \tfrac{1}{4}(p_{Ik}{}^2\varphi + 2p_{Ik}\varphi p_{Ik} + \varphi p_{Ik}{}^2) \tag{137}$$

(without summation over repeated indices), where the braces have the same meaning as in Eq. (3). These rules correspond precisely to the symmetrization rules introduced in Sec. III, and thus the quantum-mechanical operator corresponding to the classical Hamiltonian (127) becomes formally identical with the operator (117) based on the BT theory.

The approximate classical Hamiltonian (127) as well as the corresponding quantum-mechanical one are not symmetric under interchange of the particle variables $\vec{r}_I(t), \vec{p}_I(t)/m_I$ and $\vec{r}_K(t), \vec{p}_K(t)/m_K$, unless $W_{IK}$ vanishes, just like the approximate Hamiltonian (117). However, this lack of symmetry of (117) arose from the relation of the variables describing the system in the exact Hamiltonian (2) to the single-particle variables rather than from any immediately recognizable property of the exact Hamiltonian, while the lack of symmetry of the approximate Hamiltonian (127) reflects a corresponding lack of symmetry of the exact relativistic action (120) under interchange of the particle variables $z_I^{\mu}(\tau_I), v_I^{\mu}(\tau_I)$ and $z_K^{\mu}(\tau_K)$, $v_K^{\mu}(\tau_K)$. It should be noted that this interchange in the exact interaction involves an interchange of times as well as positions, as required in a manifestly Lorentz-covariant formalism.

## VII. DISCUSSION

In this paper we investigated the problem of finding the most general form of Hamiltonians which are invariant under the Poincaré group to order $c^{-2}$, and which represent approximations to theories which are exactly invariant under this group. We showed that both the classical and the quantum-mechanical approximate Hamiltonians following from two exact theories (the quantum-mechanical canonical theory of Ref. 5, and the classical "Fokker-type" theory of Ref. 6), for spinless particles,[35] are formally identical to order $c^{-2}$. The approximate Hamiltonians are characterized by the appearance of three independent functions of the separation $r$ for each pair of particles in addition to the Newtonian potential energy $V$.

While the Hamiltonian are formally identical, physically some differences remain. The classical theory started from an action integral in which the integrand was a function of the four-coordinates and four-velocities of the particles, which are the correct physical quantities transforming appropriately under the Poincaré group by assumption; the four independent functions $V$, $X$, $Y$, and $W$ followed unambiguously from the exact relativistic interaction $\Lambda$ (and any nonsymmetric contribution to the approximate interaction $I_{\rm PN}$ was a consequence of an absence of symmetry in the exact interaction). The BT theory, on the other hand, started from a Hamiltonian with an interaction term which was written as a function of variables describing the entire system without a clear physical interpretation; the appearance of the independent functions $W$ and $Z$ in the approximate interaction was a consequence of the ambiguity of the unitary transformation relating the original variables to single-particle variables (and any nonsymmetry in the approximate interaction is a reflection of this ambiguity), i.e., the physical content of the exact theory is only determined through this transformation.

In lowest order $c^0$ both theories yield the familiar velocity-independent Hamiltonians of Newtonian or Schrödinger theory. However, this is by no means the only possible nonrelativistic (i.e., Galilei-invariant) Hamiltonian even for spinless particles. The restriction to this particular form was arrived at differently for the two theories. For the theory based on a Poincaré-invariant variational principle, the class of relativistic in-

teractions was restricted in its dependence on the relativistic two-body invariants so as to ensure the existence of the desired Newtonian limit, as discussed in detail in WH. For the BT theory, on the other hand, this limit followed from requiring the validity of the world-line condition, as discussed in Sec. IV. As noted above, this condition is automatically satisfied in the relativistic Lagrangian theory. The reason that this does not restrict the nonrelativistic limit in the same way as in the BT theory is that in the latter the world-line condition operates in conjunction with the canonical formalism (i.e., the requirement of a realization of the Lie algebra of the Poincaré group) from the outset, while in the former this requirement is not imposed automatically.

Recent work by Pauri and Prosperi[14] on the classical BT theory for two particles illuminates the same problems from a different point of view. Because they consider the classical theory they naturally insist on the definition of particle positions $\vec{x}_1$ and $\vec{x}_2$ that satisfy the world-line condition everywhere. It follows that, for interacting particles, these particle positions cannot be obtained by a canonical transformation and do not satisfy canonical equations of motion. Instead they satisfy Newtonian-type equations of motion that, to order $c^{-2}$, can be derived from a Lagrangian. Therefore, to order $c^{-2}$ there exists a Hamiltonian function of $\vec{x}_1, \vec{x}_2$ and the canonically conjugate momenta $\vec{p}_1$ and $\vec{p}_2$. To order $c^{-2}$ their definition of $\vec{x}_1$ and $\vec{x}_2$ involves two arbitrary functions that correspond to our functions $Z$ and $W$. From the expansion of $v$ they obtain four functions compared to our three because they expand $V(r)$ in powers of $c^{-2}$ and we do not.

Various authors[36-38] have given special expressions for interactions in approximately relativistic Hamiltonians within the framework of the canonical formalism, sometimes claiming to have obtained the most general expressions. Their results do not include our nonsymmetric $W$ terms. For the comparison it is important to note that when the nonrelativistic two-body potential $V^{\mathrm{NR}}$ is defined to be a rotationally invariant operator that commutes with $\vec{P}$ and $\vec{R}$,

$$[\vec{P}, V^{\mathrm{NR}}] = [\vec{R}, V^{\mathrm{NR}}] = 0 , \tag{138}$$

then it may include our $X$ and $Y$ terms.

In Secs. II–IV we considered particles with spin, and then dropped further consideration of spin to facilitate comparison with the results of WH. There are two ways in which the results of WH could be made to include spin: first, consideration of a classical variational principle for particles with spin,[35] and second, by introducing spin only through the direct quantization of the WH Hamiltonian for spinless particles (127).[39]

As shown in WH, no terms in $c^{-1}$ can arise in the approximate interaction Hamiltonian (129) even if the exact interaction (120) is not time-reversal invariant. However, such terms are not necessarily absent if the interactions depend on spin or isospin.[35] They could have been included in our approximate interaction Hamiltonian (105) through terms in $v$ that are not time-reversal invariant or through $\Phi_2$.

We now turn to the question: What can be learned from our analysis about nucleon-nucleon potentials? Are the approximate world-line conditions a physically reasonable requirement? Clearly it makes sense only to the extent that we have structureless point particles. The notion of a pion world line in the vicinity of a nucleon seems inappropriate since the pion-nucleon interaction involves absorption of the pion. For nucleons it is possible and reasonable to satisfy the approximate world-line condition, but the requirement is not necessarily compelling. It is violated by most phenomenological potentials, but it is satisfied by the one-boson exchange potentials.

The approximate world-line conditions do not impose any restrictions on the velocity-dependent terms. For simple field interactions the classical theory predicts the unambiguous result $W=0$, $Y=0$, $X=0$ $(-V)$ for scalar (vector) fields in Eq. (105).[7] On the other hand, conventional potential constructions from quantum-field theory are ambiguous. For instance, the folded diagram method yields one-boson potentials for equal masses $(m/M=\frac{1}{4})$ and scalar bosons such that $W=0$ and $Z=0$ in Eq. (104), and such that in Eq. (85)

$$A = -\tfrac{1}{4}\lambda^2 V \tag{139}$$

and

$$B = -\tfrac{1}{4}\lambda^2 \frac{1}{r}\frac{dV}{dr} , \tag{140}$$

where $\lambda^2$ is an arbitrary parameter, with $0 \leq \lambda^2 \leq 1$.[40] The method of Blankenbecler and Sugar[41] yields $\lambda^2 = 0$. Another widely used prescription gives $\lambda^2 = 1$.[42,4] Hamiltonians with different values of the parameter $\lambda^2$ are canonically equivalent; they can be transformed into each other by appropriate choices of $Z \neq 0$. The classical result follows for

$$Z = \tfrac{1}{4}(\lambda^2 + 1)v . \tag{141}$$

It can therefore be obtained only by an additional canonical transformation, $Z \neq 0$. This situation is well known in the context of electromagnetic interactions,[43] but generally ignored in the literature on nucleon-nucleon potentials.[44]

Thus we have the following observations about

nucleon-nucleon potentials. (1) If the approximate world-line condition is imposed the nonrelativistic limit of the potential must be velocity-independent. (2) If the potential is derived from simple covariant fields the velocity-dependent terms of order $c^{-2}$ are unambiguously determined by the classical correspondence as functions of the nonrelativistic potential.

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