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# Pressure and composition of a weakly degenerate nucleon gas

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The equation of state and species concentrations of a hot neutral gas of neutrons, protons, electrons, positrons, and muons are determined in the temperature range $5\times10^{10}$–$5\times10^{11}$°K and density range $5\times10^{10}$–$10^{14}$ g/cm$^3$. The gas is considered to be in local thermal and chemical equilibrium under the effect of weak and strong interactions. A virial expansion is constructed to include second-order statistical and dynamical terms, the latter through experimental nucleon-nucleon scattering phase shifts. Decreases in pressure and increases in electron number densities are found to be quite modest.

## I. INTRODUCTION

Since the early work on noninteracting models of completely degenerate neutron-star matter,[1,2] considerable work has gone into the effect of nuclear interactions on the equation of state and composition of that matter.[3,4] While the most interesting physical situations do arise when sufficient cooling allows the matter to pass into a state of extreme degeneracy, there are situations in which it is of interest to know something of the nature of neutron-star matter in a condition of partial degeneracy.[5,6] Such partially degenerate neutron-star matter has been given some attention in the noninteracting nucleon approximation.[7] For example, the important question of the relationship between the deposition of neutrino energy and supernova explosions involves a knowledge of the constituent-particle number densities (electron in particular) in a state of low nucleon degeneracy. Although this problem of the role of neutrinos in producing a supernova explosion has gone through a varied history, the related problem of the effect of nuclear interactions on the equation of state of a weakly degenerate nucleon gas in the presence of weak interactions has not been considered, although finite-temperature many-body theory has been applied to a purely neutron gas.[8] In the present note we construct a virial expansion of the grand partition function, and explicitly calculate the second virial coefficients of the nucleon gas. These virial coefficients contain both quantum statistical corrections and dynamical corrections. The dynamical corrections can be related to experimental information on nucleon-nucleon scattering and are thus model independent. The effects upon the equation of state and particle number densities can be expressed, at sufficiently low densities and high temperatures, in terms of these virial coefficients. The interaction turns out to give corrections at most of about 20% in the range of densities and temperatures considered in this work.

## II. THE VIRIAL EXPANSION

The grand canonical partition function for a system containing $s$ species is given by

$$Z_g = \sum_{N_1,\ldots,N_s} \exp\left(\sum_i N_i \mu_i\right) Z(N_1,\ldots,N_s)\,, \quad (1)$$

where $\beta = 1/kT$, $\mu_i$ is the chemical potential of the $i$th species, $N_i$ is the number of particles of the $j$th species present, and $Z(N_1,\ldots,N_s)$ is the canonical partition function

$$Z(N_1,\ldots,N_s) = \mathrm{Tr}\, e^{-\beta H(N_1,\ldots,N_s)}\,. \quad (2)$$

In the above $H(N_1,\ldots,N_s)$ is the Hamiltonian for a system containing $N_i$ particles of the $i$th species. The Hamiltonian can be written as

$$H = H_0 + H_C + H_N\,, \quad (3)$$

where $H_0$ is the kinetic energy of the various species, $H_C$ is the Coulomb interaction energy, and $H_N$ is the nuclear interaction energy. We next decompose $Z_g$ as

$$Z_g = {\sum_{i\le j}}' Z_{ij}\, e^{\beta(\mu_i+\mu_j)} + \delta Z_g\,, \quad (4)$$

where the prime on the sum in the first term indicates a restriction to those species that undergo nuclear interaction. Equation (4) then defines $\delta Z_g$. The quantity $Z_{ij}$ is the appropriate two-particle partition function which will be given below in detail. We next subtract from $Z_{ij}$ the two-particle partition function for a pure Coulomb interaction, $Z_{ij}^C$, and add it into $\delta Z_g$. If at the same time we *neglect* all nuclear interactions in $\delta Z_g$ we then have

$$Z_g = Z_g^C + {\sum_{i\le j}}' (Z_{ij} - Z_{ij}^C)\, e^{\beta(\mu_i+\mu_j)}\,, \quad (5)$$

where $Z_g^C$ is the grand canonical partition function for a system with only Coulomb interactions. We have estimated, on the basis of the random phase approximation, the effects of the Coulomb interaction on the equation of state and species concentrations and found these effects totally negligible for the conditions of temperature and density of interest in the present work. Thus we finally write

$$Z_g = Z_g^0 + \sum_{i\le j} (Z_{ij} - Z_{ij}^C)\, e^{\beta(\mu_i+\mu_j)}\,, \tag{6}$$

where $Z_g^0$ is the grand partition function for a system of noninteracting particles.

The pressure can now be calculated by standard means from Eq. (6) as

$$P = \frac{kT}{V}\ln Z_g = P_0 + \delta P\,, \tag{7}$$

where $P_0$ is the noninteracting gas pressure (as a function of temperature and the various chemical potentials) and $\delta P$ is a correction given by

$$\delta P = \frac{kT}{V}\sum_{i\le j} (Z_{ij} - Z_{ij}^C)\, e^{\beta(\mu_i+\mu_j)}. \tag{8}$$

The species number densities can then be determined from the relation

$$n_i = \frac{N_i}{V} = \frac{\partial P}{\partial \mu_i} = n_i^0 + \delta n_i\,, \tag{9}$$

where

$$n_i^0 = \frac{\partial P_0}{\partial \mu_i}$$

and

$$\delta n_i = \frac{1}{V}\sum_{i\le j} e^{\beta(\mu_i+\mu_j)}(Z_{ij} - Z_{ij}^C)\,. \tag{10}$$

We now turn to the computation of $\delta P$ in terms of the two-body nuclear interaction. By separating the motion of the two particles in question into a center-of-mass motion and a relative motion, the two-particle partition function in $\delta P$ can be written

$$\Delta Z_{ij} = Z_{ij} - Z_{ij}^C = \frac{V}{\omega_{ij}} 2^{3/2} \sum_\alpha (e^{-\beta\epsilon_{ij}^\alpha} - e^{-\beta\epsilon_{ij}^{\alpha C}})\,, \tag{11}$$

where

$$\omega_{ij} = (h^2/2\pi m_{ij} kT)^{3/2} \tag{12}$$

and $m_{ij}$ is the sum of the masses of a particle $i$ and a particle $j$. The states labeled by $\alpha$ are internal states and the quantities $\epsilon_{ij}^\alpha$ and $\epsilon_{ij}^{\alpha C}$ are the allowed relative energies for the nuclear plus Coulomb interaction and pure Coulomb interaction, respectively. Equation (11) may be further written as

$$\Delta Z_{ij} = \frac{V}{\omega_{ij}} 2^{3/2}\left[\sum_\alpha g_\alpha e^{\beta B_\alpha} + \int_0^\infty d\epsilon\,(d_{ij}(\epsilon) - d_{ij}^C(\epsilon))\, e^{-\beta\epsilon}\right], \tag{13}$$

where $\alpha$ is now a bound state of the $ij$ nuclear system, $g_\alpha$ is its statistical weight, $B_\alpha$ is its binding energy, and $d_{ij}(\epsilon)$ is the density of continuum states in the presence of nuclear and Coulomb interactions, while $d_{ij}^C(\epsilon)$ is the density of continuum states in the presence of only the Coulomb interaction. Finally, from the work of Beth and Uhlenbeck,[9] the *difference* in the density of states, $d_{ij}(\epsilon) - d_{ij}^C(\epsilon)$, may be written in terms of the scattering phase shifts of the species in question with both nuclear and Coulomb interactions present and those with just Coulomb interaction present. The result is then

$$d_{ij}(\epsilon) - d_{ij}^C(\epsilon) = {\sum_{SLJ}}' \frac{2J+1}{\pi}\frac{d}{d\epsilon}[\delta_{LJS}(\epsilon) - \delta_{LJS}^C(\epsilon)]\,, \tag{14}$$

where we have made a partial-wave decomposition in terms of the orbital, total, and spin angular momentum quantum numbers $L, J, S$. The prime on the summation indicates that only those states allowed by the Pauli exclusion principle are to be included. We now turn to a discussion of the determination of the sum over states from experiments, and the subsequent evaluation of the pressure shift and concentration shifts.

## III. AN INTERACTING NUCLEON GAS

When the two species labels $i$ and $j$ represent protons and neutrons, hereafter labeled $p$ and $n$, we are dealing with an interacting nucleon gas. We assume that the neutron and proton masses are equal. The rapid convergence of the virial expansion in Sec. II depends on the quantities $\beta\mu_n$ and $\beta\mu_p$ being large and negative, i.e., on the nucleons being essentially nondegenerate (and also nonrelativistic). We define a degeneracy factor

$$D = (n_n^0 + n_p^0)\,\omega\,,$$

where $\omega$ is defined in Eq. (12) for nucleons. If $D \ll 1$, then the conditions of nondegeneracy are met. The scattering data of Mac Gregor, Arndt, and Wright[10] may now be organized for inclusion in Eq. (14). The experimental phase shifts are nuclear bar phase shifts,[11] i.e., phase shifts using the bar parameterization with the direct Coulomb effect removed,

$$\delta = \bar{\delta}_N = \bar{\delta} - \delta_C\,. \tag{15}$$

The bar and subscript $N$ will be suppressed in future discussion. The Mac Gregor *et al.* data are for neutron-proton scattering in the spin singlet-odd parity and triplet-even cases and proton-proton for the singlet-even and triplet-odd with the exception that $^1S_0$ neutron-proton is also included. The following groupings can thus be made, with a degree of approximation noted below:

$$\Delta_{0+}^{pp}(\epsilon)=[\delta({}^1S_0^{pp};\epsilon)+5\delta({}^1D_2^{pp};\epsilon)+9\delta({}^1G_4^{pp};\epsilon)]/\pi\,, \tag{16a}$$

$$\Delta_{0+}^{nn}(\epsilon)=\Delta_{0+}^{np}(\epsilon)=[\delta({}^1S_0^{np};\epsilon)+5\delta({}^1D_2^{pp};\epsilon)+9\delta({}^1G_4^{pp};\epsilon)]/\pi\,, \tag{16b}$$

$$\Delta_{0-}^{np}(\epsilon)=[3\delta({}^1P_1^{np};\epsilon)+7\delta({}^1F_3^{np};\epsilon)+11\delta({}^1H_5^{np};\epsilon)]/\pi\,, \tag{16c}$$

$$\Delta_{1+}^{np}(\epsilon)=[3\delta({}^3S_1^{np};\epsilon)+3\delta({}^3D_1^{np};\epsilon)+7\delta({}^3D_3^{np};\epsilon)+7\delta({}^3G_3^{np};\epsilon)+9\delta({}^3G_4^{np};\epsilon)+11\delta({}^3G_5^{np};\epsilon)]/\pi\,, \tag{16d}$$

and

$$\Delta_{1-}^{np}(\epsilon)=\Delta_{1-}^{nn}(\epsilon)=\Delta_{1-}^{pp}(\epsilon)=[\delta({}^3P_0^{pp};\epsilon)+3\delta({}^3P_1^{pp};\epsilon)+5\delta({}^3P_2^{pp};\epsilon)+5\delta({}^3F_2^{pp};\epsilon)+7\delta({}^3F_3^{pp};\epsilon)+9\delta({}^3F_4^{pp};\epsilon)+9\delta({}^3H_4^{pp};\epsilon)+11\delta({}^3H_5^{pp};\epsilon)+13\delta({}^3H_6^{pp};\epsilon)]/\pi\,. \tag{16e}$$

Substitution of $n$-$p$ data for $n$-$n$ and $p$-$p$ data for $n$-$n$ and $n$-$p$ is not serious because of the charge independence of nuclear forces and because of the inclusion of the important ${}^1S_0^{np}$ phase shift in Eq. (16b). The experimental data cover the relative energy range 0.5–230 MeV. The $S$-wave phase shifts $\delta({}^3S_1^{np})$, $\delta({}^1S_0^{np})$, and $\delta({}^1S_0^{pp})$ are taken from the effective range[12,13] theory below 0.5 MeV, while the others may be interpolated linearly in that range. The temperature range of interest is $5\times10^{10}$–$5\times10^{11}$ °K (4.3–43 MeV) so the scattering-data energy spectrum is more than adequate to ensure accuracy in calculation of quantities such as Eq. (12).

We can now write

$$\Delta Z_{np}=\frac{V}{\omega}2^{3/2}\left[(3e^{\beta B_d}-3)+\beta\int_0^{230}d\epsilon\,(\Delta_{0+}^{np}(\epsilon)+\Delta_{0-}^{np}(\epsilon)+\Delta_{1+}^{np}(\epsilon)+\Delta_{1-}^{np}(\epsilon))\,e^{-\beta\epsilon}\right], \tag{17a}$$

$$\Delta Z_{pp}=\frac{V}{\omega}2^{3/2}\beta\int_0^{230}d\epsilon\,(\Delta_{0+}^{pp}(\epsilon)+\Delta_{1-}^{pp}(\epsilon))\,e^{-\beta\epsilon}\,, \tag{17b}$$

and

$$\Delta Z_{nn}=\frac{V}{\omega}2^{3/2}\beta\int_0^{230}d\epsilon\,(\Delta_{0+}^{nn}(\epsilon)+\Delta_{1-}^{nn}(\epsilon))\,e^{-\beta\epsilon}\,. \tag{17c}$$

In Eq. (17a), $B_d$ is the deuteron binding energy of 2.225 MeV. Equations (17a) through (17c) can be evaluated by numerical quadrature so that the nucleonic pressure and number densities can be found as a function of temperature and chemical potential, and the corrections due to nuclear interactions can be determined.

The nucleon gas, as defined here, includes all other particles in statistical equilibrium through the weak interaction with each other and with the dominant neutrons and less populous protons. However, for the temperatures and densities of interest, the only particles with significant number densities are $e^+$, $e^-$, $\mu^+$, and $\mu^-$. The leptonic number densities and partial pressures of these constituents are taken to be those of an ideal relativistic Fermi-Dirac gas with spin $\frac{1}{2}$ throughout:

$$P_i=\frac{2}{\beta}\int\frac{d^3p}{(2\pi\hbar)^3}\ln(1+e^{\beta(\mu_i-E_i)}) \tag{18}$$

and

$$n_i=2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{1+e^{\beta(E_i-\mu_i)}}\,, \tag{19}$$

where $i=e^+$, $e^-$, $\mu^+$, and $\mu^-$ and $E_i=(c^2p^2+m_i^2c^4)^{1/2}$. The total pressure of the system, then, is the sum of the nucleonic pressure and the leptonic partial pressures. Chemical, or statistical, equilibrium is imposed by minimization of the Gibbs free energy with the result

$$\mu_n=\mu_p+\mu_{e-}\,, \tag{20a}$$

$$\mu_{e+}=-\mu_{e-}\,, \tag{20b}$$

$$\mu_{\mu-}=\mu_{e-}\,, \tag{20c}$$

and

$$\mu_{\mu+}=-\mu_{\mu-}\,. \tag{20d}$$

Total specification of the six-component gas requires two more constraints, which are the specification of the total charge and mass density

$$\sum_i Z_i n_i=Q=0 \tag{21}$$

and

$$\sum_i m_i n_i=\rho\,, \tag{22}$$

where $\rho$ is the total mass density and the system is taken as neutral. The pressure and number densities are then calculated, for a given temperature, subject to the six constraints. The

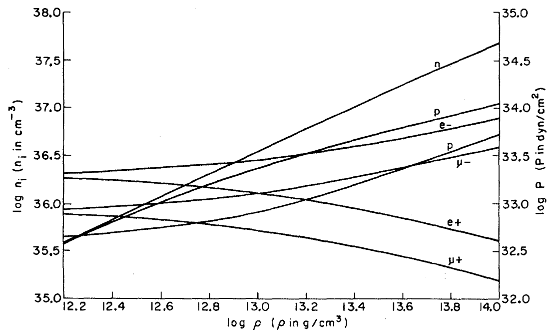


FIG. 1. Composition and pressure of nucleon gas at $T = 5\times10^{11}$ °K without strong interactions. Logarithms are to base 10.

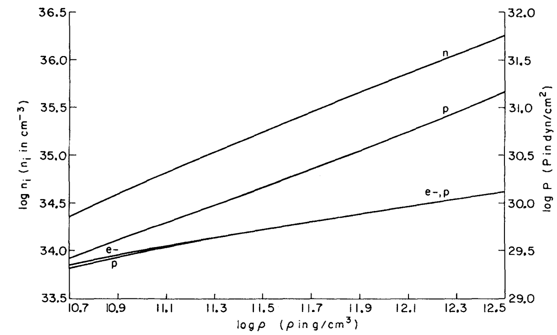


FIG. 3. Composition and pressure of nucleon gas at $T = 5\times10^{10}$ °K without strong interactions. Logarithms are to base 10.

potentials are varied and an iterative procedure involving the Newton-Raphson method is used and brings about quick convergence to arbitrary accuracy on the nonlinear conditions in Eqs. (21) and (22).

## IV. RESULTS

For the three temperatures $T = 5\times10^{10}$, $10^{11}$, and $5\times10^{11}$ °K, mass densities $\rho$ are chosen so that the nucleonic degeneracy factor $D$ is less than 1. The pressure and constitution of the gas can be calculated both with and without the nuclear interaction contributions. For the noninteracting case, the total pressure and number densities are shown versus mass density in Figs. 1–3; these represent, graphically, the equation of state, as the chemical potentials are just parameters and no longer appear. The results for the interacting case cannot be effectively shown on those plots, owing to their logarithmic nature, and so the fractional decreases in pressure and the increase in electron number density are shown in Figs. 4 and 5, respectively. In the pressure graph, the fraction $P/P^0$ is shown as a function of total mass density, while the electron population graph shows the quantity $n_{e^-}/n_{e^-}^0$.

In all graphs, for each temperature shown, the nucleonic degeneracy factor $D$ ranges from about 0.01 to 1 over the mass density range shown; the electrons are fairly degenerate at the lower density and highly degenerate at the upper density limit.

Figure 1 is for $T = 5\times10^{11}$ °K ($kT = 43$ MeV) and shows that all six species being considered are present in at least some abundance. As will be the case for all three temperatures, the nuclear interactions make no difference whatsoever at the lower densities and only modest changes in the composition and pressure at the higher densities. We will therefore focus our attention on the high-density end, insofar as the nuclear interactions are concerned. At $\log\rho = 14.0$ (all units are cgs), the nuclear interactions have caused a 15% decrease in total pressure, measured from the noninteracting case. A pressure decrease is expected since the phase-shift sum is positive over the entire energy range, indicating an overall attraction among the nucleons. At this density, the electron population has increased by about 3.5% while the neutron number density is down by less than 1%. The pressure will follow the number density of the

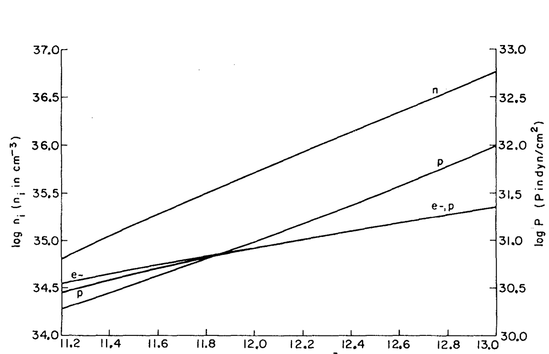


FIG. 2. Composition and pressure of nucleon gas at $T = 10^{11}$ °K without strong interactions. Logarithms are to base 10.

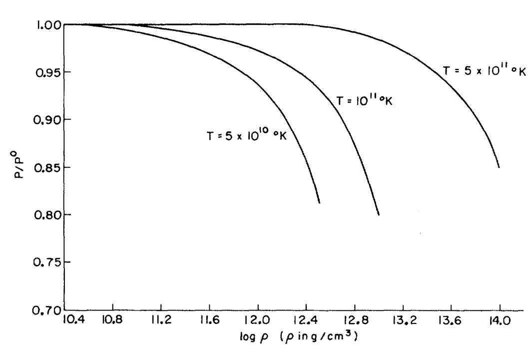


FIG. 4. Effect of strong interactions on pressure in nucleon gas. Logarithm is to base 10.

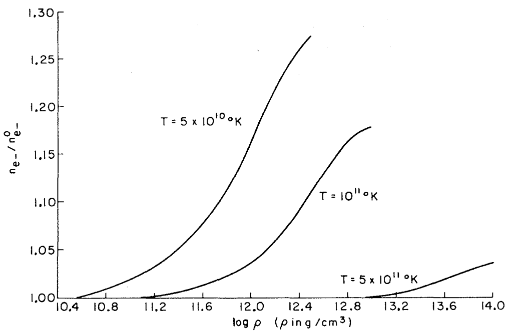


FIG. 5. Effect of strong interactions on electron number density in nucleon gas. Logarithm is to base 10.

dominant particle; at this temperature, the pressure rises slowly at low densities because electrons are dominant and their number is increasing slowly with density, but when the neutrons emerge and climb rapidly in number, the pressure also increases quickly. At the other two temperatures, the neutrons will dominate the other species completely at all densities shown and the pressure and neutron number density will follow nearly parallel tracks. The fractional increase in the protons is, here and at the other two temperatures, almost exactly the same as that for the electrons, owing to charge neutrality and the relative scarcity of the other charged particles.

At $T = 10^{11}$ °K, or $kT = 9.6$ MeV (see Fig. 2), the muons and positrons are much less important over the entire density range, so that the gas is essentially just neutrons, protons, and electrons. At $\log\rho = 13.0$, the pressure is down 20% from the noninteracting case, the electrons are up 18%, and the neutrons are down by less than 1% again. The neutrons are so dominant that their abundance cannot change much for a given overall mass density. The picture is similar at $5\times10^{10}$ °K, or $kT = 4.3$ MeV (Fig. 3), as the muons fade completely and the positrons are several orders lower than the electrons and nucleons, making no appearance on the plot. Again from the pressure and electron number density graphs (Figs. 4 and 5), we see that at the high-density end ($\log\rho = 12.5$) the total pressure is down 18% from the noninteracting case, while the electron population has increased 27% owing to the shift in chemical potentials caused indirectly by the nuclear interactions. Again the fractional change in the neutron population is almost nil and the proton increase matches that of the electron.

Caution must be exercised in reading the pressure and electron number density graphs. Owing to the logarithmic nature of the density scale, it is dangerous to extrapolate any of the curves to higher density. At higher nucleonic degeneracies, third- and higher-order terms in the virial expansion are necessary to prevent a fictitious pressure collapse. When the virial expansion fails to converge rapidly, the techniques of reference must then be employed.

However, a pattern does emerge from the graphs, in their region of validity. At high densities, which is the area of interest, there is complete neutron dominance, unchallenged by any proton or electron growth owing to the presence of the nuclear interactions. There is a pressure decrease of about 15–20% when the nucleon degeneracy factor is unity and this decrease is due almost entirely to the self-attraction of the nucleons, not to any reshuffling of the leptonic populations. Indeed, since the weak interaction which allows statistical equilibrium to be achieved conserves the sum of neutrons and protons, while the number of electrons has increased, the pressure drop will be even greater in a pure neutron gas of the same overall mass density. The electron and proton growth is about 3.5–27%, depending on temperature and density, and is caused by the changing chemical potentials, i.e., the changing constitution of the gas is an indirect effect of the nuclear interactions, transmitted by the mechanism of the weak interaction. In view of the strength of the nuclear interactions and the small mean nucleonic separation at these high densities, the equation of state of the nucleon gas is very modestly affected by the interactions.

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