PHYSICAL REVIEW D VOLUME 14, NUMBER 10 15 NOVEMBER 1976

Class of the Brans-Dicke Maxwell fields

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(Received 1 March 1976)

A class of exact solutions to the source-free Brans-Dicke Maxwell field equations is obtained which reduces to the Majumdar-Papapetrou class of solutions when the Brans-Dicke scalar ϕ = constant. It has been observed that these solutions are important from the viewpoint of verifying Penrose's suggestion which states that the black holes in the Brans-Dicke theory are identical to those of Einstein's theory.

I. INTRODUCTION

In the study of the electrostatic fields in Einstein's theory with the static metric

$$(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{44}(dt)^2, \tag{1.1}$$

where $g_{44} < 0$ and Greek indices run from 1 to 3, an important relation

$$-g_{44} = 4\pi\psi^2 + A\psi + B, \tag{1.2}$$

which was obtained by Majumdar, states that this is the only functional form dependence of g_{44} on the electrostatic potential ψ. With the particular form of this relation as

$$-g_{44} = 4\pi(\psi \pm \sqrt{2})^2, \tag{1.3}$$

it is observed by Majumdar[1] and Papapetrou[2] independently that the source-free Einstein-Maxwell field equations reduce to a Laplace equation. This suggests that a class of solutions of the source-free Einstein-Maxwell field equations, corresponding to the solutions of Laplace equation, exists which represents the external fields of static sources where charge and mass, in the units $G = c = 1$, are equal. This class, known as the Majumdar-Papapetrou (MP) class of solutions, has received in recent years thorough and critical attention by many authors mainly from the physical point of view.

Anticipating a useful astrophysical role of these solutions, Perjés[3] and Israel and Wilson[4] have independently generalized them to the stationary case. Later, Hartle and Hawking[5] extensively studied the physical character of these solutions and found that the solutions of the MP class corresponding to monopoles can be "analytically extended and interpreted as a system of charged black holes in equilibrium under their gravitational and electrostatic forces." They have observed that these may be the only stationary, asymptotically flat, black-hole solutions obtained by Majumdar-Papapetrou-Israel-Wilson techniques.

The black-hole solutions of this class are quite distinct from the already known black-hole solutions, viz. Schwarzschild, Reissner-Nordström, uncharged Kerr, and charged Kerr solutions. Prior to the findings of Hartle and Hawking,[5] it had been conjectured[6] that the charged Kerr solution to the Einstein-Maxwell field equations—or any one of the three special cases of charged Kerr solutions (Schwarzschild, Reissner-Nordström, and uncharged Kerr solutions.)—is the exterior field of the black hole necessarily produced due to relativistic gravitational collapse in three dimensions. Penrose[7] suggested that this conjecture is also true in the Brans-Dicke (BD) theory of gravity.[8]

With the finding of the MP class of black-hole solutions, the above-mentioned conjecture should be generalized; and in view of Penrose's suggestion the uniqueness of Schwarzschild, Reissner-Nordström, uncharged Kerr, charged Kerr, and the MP class of black-hole solutions should be verified in the BD theory. Thorne and Dykla,[6] in this line, have verified only the uniqueness of Schwarzschild and Kerr black-hole solutions in the BD theory, along with certain other significant observations in support of Penrose's suggestion. They found Schwarzschild's solution to be unique because none of the solutions given by Brans[9] with $\phi \neq$ constant was found to possess an event horizon. But still the question of proving the uniqueness of black holes represented by the Reissner-Nordström and Majumdar-Papapetrou classes of solutions remains. One has, therefore, to depend on the corresponding exact solutions of the source-free BD Maxwell fields. The analog of the Reissner-Nordström solution to the source-free BD Maxwell field has already been obtained by Buchdahl.[10] We feel that this solution will be useful in proving the uniqueness of the Reissner-Nordström black-hole solution in the BD theory.

The analog of the Majumdar-Papapetrou class of solutions to the BD theory, however, is not yet known. With an objective to prove the uniqueness of the black-hole solution of the Majumdar-Papapetrou class, in this paper we have found first the corresponding solutions of the source-free BD Maxwell fields. The solutions, thus obtained, reduce to the MP class of solutions when $\phi=$ constant. For $\phi \neq$ constant we have two different classes of solutions depending on the restrictions on the BD coupling parameter ω. The physical study of these solutions and the problem regarding the uniqueness of the black holes of the MP class will be reported elsewhere.

II. FIELD EQUATIONS AND THE SOLUTIONS

The source-free BD Maxwell field equations are

$$R_{ij}=-\frac{8\pi}{\phi}E_{ij}-\frac{\omega}{\phi^2}\phi_{,i}\phi_{,j}-\frac{1}{\phi}\phi_{;ij}, \qquad (2.1)$$

$$(3+2\omega)\phi_{;k}^{;k}=0, \qquad (2.2)$$

$$F^{ij}_{;j}=0, \qquad (2.3)$$

and

$$F_{ij,k}+F_{jk,i}+F_{ki,j}=0, \qquad (2.4)$$

with

$$E_{ij}=g^{ab}F_{ai}F_{bj}-\tfrac{1}{4}g_{ij}F_{mn}F^{mn}, \qquad (2.5)$$

where latin indices run from 1 to 4 and the subscripted comma and semicolon denote partial derivative and covariant derivative, respectively. With a choice of the skew-symmetric electromagnetic tensor F_{ij} as

$$F_{ij}=\psi_{j,i}-\psi_{i,j},$$

(2.4) is identically satisfied. Here ψ_i is the electromagnetic four-potential. For the electrostatic case, $\psi_4=\psi$ survives and the field equations (2.1)–(2.3), for the metric (1.1), become[11]

$$R_{\alpha\beta}\equiv\overline{R}_{\alpha\beta}+\frac{1}{V}V_{;\alpha\beta}=-\frac{8\pi}{\phi}\left[\frac{1}{V^2}(\tfrac{1}{2}g_{\alpha\beta}g^{\sigma\nu}\psi_{,\tau}\psi_{,\sigma}-\psi_{,\alpha}\psi_{,\beta})\right]-\frac{\omega}{\phi^2}\phi_{,\alpha}\phi_{,\beta}-\frac{1}{\phi}\phi_{;\alpha\beta}, \qquad (2.6)$$

$$R_{44}\equiv-Vg^{\sigma\nu}V_{;\sigma\nu}=-\frac{4\pi}{\phi}g^{\sigma\nu}\psi_{,\sigma}\psi_{,\nu}+\frac{V}{\phi}g^{\sigma\nu}V_{,\sigma}\phi_{,\nu}, \qquad (2.7)$$

$$g^{\sigma\nu}\phi_{;\sigma\nu}+\frac{1}{V}g^{\sigma\nu}V_{,\sigma}\phi_{,\nu}=0 \text{ for all values of } \omega, \qquad (2.8)$$

and

$$Vg^{\sigma\nu}\psi_{;\sigma\nu}-g^{\sigma\nu}V_{,\sigma}\psi_{,\nu}=0, \qquad (2.9)$$

where $\overline{R}_{\alpha\beta}$ is the Ricci tensor corresponding to the spatial metric $(d\overline{s})^2=g_{\alpha\beta}dx^\alpha dx^\beta$, $g_{44}=-V^2$, and ϕ is independent t.

To solve these field equations, we assume a functional relationship between V, ϕ, and ψ, as

$$V\equiv V(\phi,\psi), \qquad (2.10)$$

where ϕ and ψ are independent of each other. This mutual independence is in the spirit of the Brans-Dicke assumption that the Lagrangian density of matter is not functionally dependent on ϕ. In view of the assumption (2.10) and the field equations (2.8) and (2.9), (2.7) reduces to

$$\left(V_{\phi\phi}-\frac{V_\phi{}^2}{V}+\frac{V_\phi}{\phi}\right)g^{\sigma\nu}\phi_{,\sigma}\phi_{,\nu}+\left(V_{\psi\psi}+\frac{V_\psi{}^2}{V}-\frac{4\pi}{\phi V}\right)g^{\sigma\nu}\psi_{,\sigma}\psi_{,\nu}+\left(2V_{\phi\psi}+\frac{V_\psi}{\phi}\right)g^{\sigma\nu}\phi_{,\sigma}\psi_{,\nu}=0, \qquad (2.11)$$

where $V_\phi=\partial V/\partial\phi$, $V_\psi=\partial V/\partial\psi$, $V_{\phi\phi}=\partial^2V/\partial\phi^2$, $V_{\psi\psi}=\partial^2V/\partial\psi^2$, and $V_{\phi\psi}=\partial^2V/\partial\phi\partial\psi$.

This equation is identically satisfied when the coefficients of $g^{\sigma\nu}\phi_{,\sigma}\phi_{,\nu}$, $g^{\sigma\nu}\psi_{,\sigma}\psi_{,\nu}$, and $g^{\sigma\nu}\phi_{,\sigma}\psi_{,\nu}$ are simultaneously zero. This suggests the functional form of (2.10) to be

$$V^2=\phi^{-1}(4\pi\psi^2+A\psi+B), \qquad (2.12)$$

where A and B are arbitrary constants of integration. Comparing (2.12) with (1.2), we find that for $\phi=\text{const}=1$, they agree with each other. Thus, in view of (2.12), (2.7) is identically satisfied. We now express (2.12) as

$$V^2=\phi^{-1}4\pi(\psi\pm\sqrt{2})^2, \qquad (2.13)$$

by a proper choice of the arbitrary constants A and B. In view of (2.13), the field equations (2.6), (2.8), and (2.9) reduce, respectively, to

$$\overline{R}_{\alpha\beta}=-\tfrac{1}{2}h_{;\alpha\beta}-(\omega+\tfrac{5}{4})h_{,\alpha}h_{,\beta}+\tfrac{1}{2}(h_{,\beta}k_{,\alpha}+k_{,\beta}h_{,\alpha})-k_{;\alpha\beta}+k_{,\alpha}k_{,\beta}-g_{\alpha\beta}g^{\sigma\nu}k_{,\sigma}k_{,\nu}, \qquad (2.14)$$

$$g^{\sigma\nu}h_{;\sigma\nu}+\tfrac{1}{2}g^{\sigma\nu}h_{,\sigma}h_{,\nu}+g^{\sigma\nu}h_{,\sigma}k_{,\nu}=0, \qquad (2.15)$$

and

$$g^{\sigma\nu}k_{;\sigma\nu}+\tfrac{1}{2}g^{\sigma\nu}k_{,\sigma}h_{,\nu}=0, \qquad (2.16)$$

where $e^{2k}=4\pi(\psi\pm\sqrt{2})^2$ and $e^h=\phi$.

By defining a three-space conformal transformation as

$$g'_{\alpha\beta}=e^{h+2k}g_{\alpha\beta}, \qquad (2.17)$$

the equations (2.14)–(2.16) reduce, respectively, to

$$\overline{R}'_{\alpha\beta}=-(\omega+\tfrac{3}{2})h_{,\alpha}h_{,\beta},$$

$$g'^{\sigma\nu}h_{:\sigma\nu}=0,$$

and

$$g'^{\sigma\nu}k_{:\sigma\nu}=g'^{\sigma\nu}k_{,\sigma}k_{,\nu},$$

which, by substituting $p=(2\omega+3)^{1/2}h$ and $k=-\ln(1+W)$, further reduce to

$$\bar{R}'_{\alpha\beta} = -\tfrac{1}{2} p_{,\alpha} p_{,\beta}, \tag{2.18}$$

$$g'^{\sigma\nu} p_{:\sigma\nu} = 0, \tag{2.19}$$

and

$$g'^{\sigma\nu} W_{:\sigma\nu} = 0. \tag{2.20}$$

where the subscripted colon denotes covariant differentiation with respect to $g'_{\alpha\beta}$. It can be seen that for $\phi = \text{constant}$, (2.18)–(2.20) reduce to a single Laplace equation and hence admit the Majumdar-Papapetrou class of solutions. Otherwise they suggest a more general class of solutions. For $2\omega + 3 > 0$, the equations (2.18) and (2.19) correspond to Einstein's vacuum field equations for the metric

$$(ds)^2 = \bar{e}^{p} g'_{\alpha\beta} dx^2 dx^\beta - e^{p}(dt)^2, \tag{2.21}$$

where (2.20) suggests W to be behaving as a test field in the conformal space associated with the Einstein vacuum spacetime.

Thus, we can always obtain a class of electrostatic solutions of the source-free BD Maxwell field equations for the form of the given Einstein vacuum fields with the help of the following theorem:

Given any Einstein vacuum field solution $(\bar{e}^{p} g'_{\alpha\beta}, e^{p})$ the source-free BD Maxwell field equations for electrostatic fields always admit, for $2\omega + 3 > 0$, a solution

$$[e^{-p/(2\omega+3)1/2}\{(1+W)^2 g'_{\alpha\beta}, (1+W)^{-2}\}],$$

with the BD scalar $\phi = e^{p/(2\omega+3)1/2}$ and the electrostatic potential $\psi = (1/2\sqrt{\pi})(1+W)^{-1} \mp \sqrt{2}$, where W satisfies

$$g'^{\alpha\beta} W_{:\alpha\beta} = 0.$$

Another class of solutions arises for $\omega = -\frac{3}{2}$. In this case (2.18) suggests $g'_{\alpha\beta} = \eta_{\alpha\beta}(+1, +1, +1)$, so that (2.19) and (2.20) reduce to two Laplace equations determining the scalar field and the electrostatic field, respectively. The metric (1.1), in this case, reduces to

$$(ds)^2 = \bar{e}^{h}[(1+W)^2(dx^2 + dy^2 + dz^2) - (1+W)^{-2}(dt)^2], \tag{2.22}$$

which is clearly conformal to the Majumdar-Papapetrou metric. Thus given any Majumdar-Papapetrou solution $[(1+W)^2(dx^2+dy^2+dz^3) - (1+W)^{-2}(dt)^2]$ with the electrostatic potential $\psi = (1/2\sqrt{\pi})(1+W)^{-1} \mp \sqrt{2}$, the BD electrostatic field equations admit, for $\omega = -\frac{3}{2}$, a solution $\bar{e}^{h}[(1+W)^2 \times (dx^2+dy^2+dz^2) - (1+W)^{-2}(dt)^2]$ with the electrostatic field $\psi = (1/2\sqrt{\pi})(1+W)^{-1} \pm \sqrt{2}$ and the BD scalar $\phi = e^{h}$, where h is the solution of the Laplace equation.

III. CONCLUSION

Thus, with a well-defined motivation we have generalized the Majumdar-Papapetrou class of solutions to the BD theory to obtain two different classes of solutions depending on the restrictions on the BD coupling parameter ω. For $2\omega + 3 > 0$, the BD Maxwell field equations admit a class of electrostatic solutions which are generated from the known solutions to Einstein vacuum fields. For $\omega = -\frac{3}{2}$, the solutions of these equations are conformal to the Majumdar-Papapetrou class of solutions with the conformal factor given by ϕ^{-1}, where $\ln\phi$ satisfies the Laplace equation. We observe that (2.13) plays a key role in obtaining the exact solutions of the highly nonlinear field equations. To cite an interesting consequence of (2.13), it should be mentioned here that its validity in the interior of the static charged dust leads to the ratio of charge to mass density as

$$\frac{\sigma}{\rho} = \pm \phi^{-1/2}.$$

This suggests a possibility of explaining the structure of a finite electron in the domain of the BD theory of gravitation.[12] However, the solutions obtained are to be studied further in the light of the suggestions given by Penrose. This requires us to find if any of these solutions possesses an event horizon. This aspect of the study is in progress and will be reported later.

ACKNOWLEDGMENTS

The authors are grateful to Professor J. R. Rao and Professor G. Bandyopadhyay for their constant encouragement and inspiration.

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