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# Vacuum expectation value of the stress tensor in an arbitrary curved background: The covariant point-separation method*

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A method known as covariant geodesic point separation is developed to calculate the vacuum expectation value of the stress tensor for a massive scalar field in an arbitrary gravitational field. The vacuum expectation value will diverge because the stress-tensor operator is constructed from products of field operators evaluated at the same space-time point. To remedy this problem, one of the field operators is taken to a nearby point. The resultant vacuum expectation value is finite and may be expressed in terms of the Hadamard elementary function. This function is calculated using a curved-space generalization of Schwinger's proper-time method for calculating the Feynman Green's function. The expression for the Hadamard function is written in terms of the biscalar of geodetic interval which gives a measure of the square of the geodesic distance between the separated points. Next, using a covariant expansion in terms of the tangent to the geodesic, the stress tensor may be expanded in powers of the length of the geodesic. Covariant expressions for each divergent term and for certain terms in the finite portion of the vacuum expectation value of the stress tensor are found. The properties, uses, and limitations of the results are discussed.

## I. INTRODUCTION

In the past few years the study of quantum field theory in curved space-times has grown at a rapid rate. This growth has been stimulated partly by investigations of particle production in the gravitational fields of black holes and various cosmological models and partly because it seems natural to consider situations where the gravitational field is fixed before proceeding to more difficult problems in a full quantum theory of gravity.

Consider some quantum scalar field propagating on a fixed curved background. If the background gravitational field is strong and time-varying, particles may be produced. (In some cases it is better to talk about the flux of energy rather than particles. A particle is not always a well-defined concept in a curved-space setting.) We may calculate the vacuum expectation value (VEV) of the stress tensor for this flux and use it as the source in Einstein's field equations. The new field equations are then solved (exactly, if possible; numerically otherwise). This will give a new (semiclassical since the gravitational field has not been quantized) approximation to the metric structure of the space-time.

Unfortunately, as is almost always the case, the VEV of the stress tensor diverges and hence some method of regularization must be found. The method of regularization one chooses to use depends on how the VEV of the stress tensor is originally calculated or, as is often the case, on one's personal taste, that is, on which method seems more physical to the individual doing the calculation.

When calculating a VEV of the stress tensor, one must normally do a separate calculation for each background geometry. Usually, some complete set of mode functions is found by solving the scalar field equations and is used to express the scalar field operator in terms of creation and annihilation operators. The stress tensor is written in terms of these field operators and its VEV taken. The result is a sum over products of the mode functions and their derivatives (see Sec. VII). If possible, the sums are done and the pieces which diverge are isolated and disposed of in some fashion.

In many interesting cases, such as the Schwarzschild metric, the mode functions cannot be written in terms of known functions, or, if the functions are known, the integrals or sums which appear cannot be evaluated. One must normally resort to approximations which rarely give all the desired information.

The work presented here will focus on the problem of calculating the VEV of the stress tensor for a massive scalar field in an arbitrary curved background in a covariant manner without resorting to mode sums. Using a method proposed by DeWitt,[1] known as geodesic point separation, covariant expressions for the divergences in the VEV of the stress tensor are found. Also, valuable, but incomplete, information (no knowledge of real particle production is found) about the finite portions of the VEV will be given.

The point-separation procedure is presented in Secs. II through VI. In Sec. II, we define the stress-tensor operator and note that it is constructed out of products of field operators evaluated at the same space-time point. It is this fact which causes the VEV of the stress tensor to diverge. To avoid these divergent quantities, one operator in each product is moved to a nearby point. The point-separated object which results is expressed in terms of the so-called Hadamard elementary function.

Section III presents the Schwinger-DeWitt coordinate-space method for calculating the Feynman Green's function, and from it Hadamard's function. It is found that through the use of biscalars, the Hadamard function may be written in terms of the distance along the geodesic between the separated points and purely geometrical quantities constructed out of the Riemann tensor.

Sections IV and V are a summary of the properties of bitensors. We find that the two-point functions in the Hadamard function may be written as functions of one of the points and a tangent vector to the geodesic between the points by using covariant expansions in terms of the tangent vector.

In Sec. VI, the information derived in the preceding sections is brought together to form the VEV of the stress tensor. The results show that as the length of the tangent vector goes to zero (that is, when the points coincide) there will be quartic, quadratic, logarithmic, and linear divergences in general. There are also finite tangent-vector-dependent and -independent terms.

The last section discusses the properties, uses, and limitations of the results of Sec. VI. We see that there are serious problems when one attempts to use the results in the limit of zero scalar field mass.

In an appendix, we present the proof of a valuable theorem on bitensors introduced in Sec. IV.

## II. THE POINT-SEPARATED STRESS TENSOR

The action functional for a scalar field in a curved background is[2]

$$S[\phi]=-\tfrac{1}{2}\int g^{1/2}(\phi_{;\mu}\phi^{;\mu}+\xi R\phi^2+m^2\phi^2)\,d^4x\,, \quad (2.1)$$

where $\phi(x)$ is the scalar field, $g$ is minus the determinant of the background metric, $g_{\mu\nu}$, $R$ is the curvature scalar, $m$ is the scalar field's mass, and $\xi$ is some constant which is $\frac{1}{6}$ for a conformal scalar field and 0 for an ordinary scalar field. Varying $\phi$ infinitesimally in Eq. (2.1), we obtain the scalar field equations

$$0=\frac{\delta S}{\delta\phi}=-g^{1/2}[\phi_{;\mu}{}^{\mu}-(\xi R+m^2)\phi]\,, \quad (2.2)$$

where $\delta/\delta\phi$ indicates functional differentiation.

The classical stress-tensor density[3] is defined as

$$T^{\mu\nu}\equiv 2\frac{\delta S}{\delta g_{\mu\nu}}\,. \quad (2.3)$$

Using

$$\delta g^{\mu\nu}=-g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}\,,$$

$$\delta g^{1/2}=\tfrac{1}{2}g^{1/2}g^{\alpha\beta}\delta g_{\alpha\beta}\,,$$

and

$$\delta R=-R^{\alpha\beta}\delta g_{\alpha\beta}+g^{\alpha\beta}g^{\mu\nu}(-\delta g_{\alpha\beta;\mu\nu}+\delta g_{\alpha\mu;\beta\nu})\,,$$

we find that

$$\begin{aligned}T^{\mu\nu}=g^{1/2}\{&\tfrac{1}{2}(1-2\xi)[\phi^{;\mu},\phi^{;\nu}]_+ +\tfrac{1}{2}(2\xi-\tfrac{1}{2})g^{\mu\nu}[\phi_{;\sigma},\phi^{;\sigma}]_+\\&-\xi[\phi^{;\mu\nu},\phi]_+ +\xi g^{\mu\nu}[\phi_{;\sigma}{}^{\sigma},\phi]_+\\&+\tfrac{1}{2}\xi(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R)[\phi,\phi]_+ -\tfrac{1}{4}m^2g^{\mu\nu}[\phi,\phi]_+\}\,,\end{aligned} \quad (2.4)$$

where $[\;,\;]_+$ is the anticommutator. $T^{\mu\nu}$ is symmetric and by virtue of the field equations (2.2) satisfies $T^{\mu\nu}{}_{;\nu}=0$. Also, $T^{\mu}{}_{\mu}=0$ when $m=0$, $\xi=\frac{1}{6}$, and Eq. (2.2) holds. The transition from classical to quantum fields is made by replacing the classical field $\phi$ by a field operator $\underline{\phi}$. We then note that Eq. (2.4) is constructed from products of field operators or their derivatives at the same spacetime point. These quantities diverge when their vacuum expectation value is taken. With a little foresight, we rewrite the fourth term in Eq. (2.4) as

$$[\phi_{;\sigma}{}^{\sigma},\phi]_+=\tfrac{1}{4}[\phi_{;\sigma}{}^{\sigma},\phi]_+ +\tfrac{3}{4}(\xi R+m^2)[\phi,\phi]_+\,, \quad (2.5)$$

where we have employed the field equations (2.2). This change makes Eq. (2.4) manifestly trace-free [independent of Eq. (2.2)] when $\xi=\frac{1}{6}$ and $m=0$.

The systems we consider will be required to have some initial in-region and final out-region with vacuum states $|\text{in},\text{vac}\rangle$ and $|\text{out},\text{vac}\rangle$, respectively.[4] All dynamics occurs in the region separating the in and out regions. As DeWitt (Ref. 1) has pointed out, all information on the divergences in the expectation values $\langle\text{in},\text{vac}|\underline{T}^{\mu\nu}|\text{in},\text{vac}\rangle$ and $\langle\text{out},\text{vac}|\underline{T}^{\mu\nu}|\text{out},\text{vac}\rangle$ may be found by studying

$$\langle\underline{T}^{\mu\nu}\rangle_{\text{div}}\equiv\frac{\langle\text{out},\text{vac}|\underline{T}^{\mu\nu}|\text{in},\text{vac}\rangle}{\langle\text{out},\text{vac}|\text{in},\text{vac}\rangle}\,. \quad (2.6)$$

Terms such as

$$\frac{\langle\text{out},\text{vac}|[\underline{\phi}(x),\underline{\phi}(x)]_+|\text{in},\text{vac}\rangle}{\langle\text{out},\text{vac}|\text{in},\text{vac}\rangle}$$

appear and are divergent. The point-separation procedure consists of replacing one of the $\underline{\phi}(x)$ by $\underline{\phi}(x')$, where $x'$ is some point near $x$. The finite quantity obtained is called the Hadamard elementary function, $G^{(1)}(x,x')$. We have the definition

$$G^{(1)}(x,x') \equiv \frac{\langle \text{out, vac} | [\underline{\phi}(x), \underline{\phi}(x')]_+ | \text{in, vac} \rangle}{\langle \text{out, vac} | \text{in, vac} \rangle} \tag{2.7a}$$

as well as[5]

$$G^{(1);\mu\nu} = \frac{\langle \text{out, vac} | [\underline{\phi}^{;\mu\nu}, \underline{\phi}']_+ | \text{in, vac} \rangle}{\langle \text{out, vac} | \text{in, vac} \rangle}, \tag{2.7b}$$

$$G^{(1);\mu\nu'} = \frac{\langle \text{out, vac} | [\underline{\phi}^{;\mu}, \underline{\phi}^{;\nu'}]_+ | \text{in, vac} \rangle}{\langle \text{out, vac} | \text{in, vac} \rangle}, \tag{2.7c}$$

$$G^{(1);\mu'\nu'} = \frac{\langle \text{out, vac} | [\underline{\phi}, \underline{\phi}^{;\mu'\nu'}]_+ | \text{in, vac} \rangle}{\langle \text{out, vac} | \text{in, vac} \rangle}, \tag{2.7d}$$

where the prime on the indices indicates that the derivative is taken at the point $x'$. Note that $G^{(1)}$ and its derivatives are bitensors; they are functions which transform as tensors at two different points. For example, $G^{(1);\mu\nu'}$ transforms like the product of two vectors, one at $x$, the other at $x'$, $A^\mu(x)B^\nu(x')$.

We now take the first $\phi$ in each bracket in Eq. (2.4) [modified by Eq. (2.5)] to $x'$, take the result, and sandwich it between vacuum states as in Eq. (2.6). We then do the same for the second $\phi$ in each bracket and average the two results.[6] Using Eqs. (2.7), we get

$$\begin{aligned}\langle \underline{T}^{\mu\nu} \rangle_{\text{div}} = \lim_{x' \to x} g^{1/2} [ & \tfrac{1}{2}(\tfrac{1}{2} - \xi)(G^{(1);\mu'\nu} + G^{(1);\mu\nu'}) + (\xi - \tfrac{1}{4}) g^{\mu\nu} G^{(1)}{}_{;\sigma}{}^{\sigma'} \\ & - \tfrac{1}{2}\xi(G^{(1);\mu\nu} + G^{(1);\mu'\nu'}) + \tfrac{1}{8}\xi g^{\mu\nu}(G^{(1)}{}_{;\sigma}{}^{\sigma} + G^{(1)}{}_{;\sigma'}{}^{\sigma'}) + \tfrac{3}{4}\xi g^{\mu\nu}(\xi R + m^2) G^{(1)} \\ & + \tfrac{1}{2}\xi(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) G^{(1)} - \tfrac{1}{4} m^2 g^{\mu\nu} G^{(1)} ] .\end{aligned} \tag{2.8}$$

Equation (2.8) is a purely formal expression. Terms such as $G^{(1);\mu\nu} + G^{(1);\mu'\nu'}$ are meaningless. Each term transforms differently as a bitensor, so they cannot be added. These terms will be given a well-defined meaning in Sec. V. $\langle \underline{T}^{\mu\nu} \rangle_{\text{div}}$ is symmetric and, when $\xi = \frac{1}{6}$ and $m = 0$, trace-free. However, a "conservation equation" such as $\langle \underline{T}^{\mu\nu} \rangle_{\text{div};\nu} = 0$ has no meaning. $\langle \underline{T}^{\mu\nu} \rangle_{\text{div}}$ is not a stress tensor in the strict sense since it is a bitensor rather than a tensor at one point. In the end, of course, the object we construct using $\langle \underline{T}^{\mu\nu} \rangle_{\text{div}}$ and put in Einstein's equations as a source will satisfy conservation equations.

## III. GREEN'S FUNCTIONS

Because we are doing calculations in a curved-space setting, all of the procedures we adopt will be carried out in a fully covariant (as opposed to merely Lorentz-covariant) manner. This requires that we use coordinate-space methods. The method chosen here is DeWitt's[7] curved-space generalization of Schwinger's[8] proper-time technique for finding the Feynman Green's function, $G(x,x')$. Using the relation

$$G(x,x') = \overline{G}(x,x') - \tfrac{1}{2} i G^{(1)}(x,x'), \tag{3.1}$$

where $\overline{G}$ is the principal-value function (equal to one-half the sum of the advanced and retarded Green's functions), $G^{(1)}$ may be found by studying $G$. The Feynman function satisfies

$$\int F(x,x'')G(x'',x')\,d^4x'' = -\delta(x,x'), \tag{3.2}$$

where $F(x,x'')$ is some differential operator.

We introduce an abstract Hilbert space whose basis vectors $|x\rangle, |x'\rangle, \ldots$ are eigenvectors of a coordinate operator $\underline{x}^\mu$, whose eigenvalues are the coordinates themselves, i.e.,

$$\underline{x}^\mu |x'\rangle = x^{\mu'} |x'\rangle .$$

This allows us to write Eq. (3.2) in matrix form

$$\underline{F}\,\underline{G} = -\underline{1}, \tag{3.3}$$

where

$$G(x,x') = \langle x | \underline{G} | x' \rangle ,$$

$$F(x,x') = \langle x | \underline{F} | x' \rangle ,$$

and

$$\delta(x,x') = \langle x | \underline{1} | x' \rangle = \langle x | x' \rangle .$$

We may assure ourselves that $G$ is the Feynman function by adding a small positive imaginary number, $i0_+$, to $F(x,x')$. Formally,

$$\underline{g}^{1/4}\underline{G}\,\underline{g}^{1/4} = -\frac{1}{\underline{g}^{1/4}\underline{F}\,\underline{g}^{-1/4} + i0_+} .$$

The $\underline{g}^{1/4} = g^{1/4}(\underline{x})$ factors are added to maintain the transformation properties of the matrix elements of $G$. Now using

$$\int_0^\infty e^{-i(z - i0_+)s}\,ds = \frac{1}{i(z - i0_+)} ,$$

we have (dropping the $i0_+$'s for the moment, but remembering that we will need to reinsert them later)

$$\underline{g}^{1/4}\underline{G}\,\underline{g}^{1/4}=i\int_0^\infty \exp(i\underline{g}^{-1/4}\underline{F}\,\underline{g}^{-1/4}s)\,ds\,,$$

or after taking matrix elements,

$$G(x,x')=i\int_0^\infty g^{-1/4}(x)\langle x|\exp(i\underline{g}^{-1/4}\underline{F}\,\underline{g}^{-1/4}s)|x'\rangle \times g^{-1/4}(x')\,ds. \qquad (3.4)$$

We define the "transition amplitude"

$$\langle x,s|x',0\rangle \equiv \langle x|\exp(i\underline{g}^{-1/4}\underline{F}\,\underline{g}^{-1/4}s)|x'\rangle\,,$$

which when operated on by $i\partial/\partial s$ gives

$$i\frac{\partial}{\partial s}\langle x,s|x',0\rangle=-F\langle x,s|x',0\rangle\,. \qquad (3.5)$$

For a scalar field

$$F=\nabla_\mu\nabla^\mu-\xi R-m^2\,,$$

where $\nabla^\mu$ is the covariant derivative operator. When $s=0$,

$$\langle x,0|x',0\rangle=\langle x|x'\rangle=\delta(x,x')\,. \qquad (3.6)$$

Thus $\langle x,s|x',0\rangle$ satisfies a Schrödinger-type equation with the boundary condition (3.6).

In the context of a WKB expansion, which is sufficient to give all the divergences, Eq. (3.5) has a solution of the form

$$\langle x,s|x',0\rangle=-\frac{i}{(4\pi)^2}\frac{D^{1/2}(x,x')}{s^2}\times\exp\left[i\frac{\sigma(x,x')}{2s}-im^2s\right]\Omega(x,x',s)\,, \qquad (3.7)$$

where $\Omega(x,x',s)$ is some function to be determined. The boundary condition (3.6) forces $\Omega(x,x',0)$ to equal unity. The quantity $\sigma(x,x')$ is the biscalar of a geodetic interval[9] which is equal to one-half the square of the geodesic distance between $x$ and $x'$. The VanVleck-Morette determinant, $D(x,x')$, is defined by (Ref. 7)

$$D(x,x')\equiv-\det(-\sigma_{;\mu\nu'})\,. \qquad (3.8)$$

Geodesic theory (Ref. 7) gives us

$$\sigma(x,x')=\tfrac{1}{2}\sigma^{;\mu}\sigma_{;\mu} \qquad (3.9)$$

and

$$D^{-1}(D\sigma^{;\mu})_{;\mu}=4\,. \qquad (3.10)$$

Now substituting Eq. (3.7) into (3.5), we obtain a differential equation for $\Omega$,

$$i\frac{\partial}{\partial s}\Omega+\frac{i}{s}\Omega^{;\mu}\sigma_{;\mu}=-D^{-1/2}(D^{1/2}\Omega)_{;\mu}{}^{\mu}+\xi R\Omega\,.$$

We try to solve this equation by a power series in $(is)$,

$$\Omega(x,x',s)=\sum_{n=0}^{\infty}a_n(x,x')(is)^n\,,$$

which results in a set of recursion relations for the $a_n$'s,

$$a_{0;\mu}\sigma^{;\mu}=0 \qquad (3.11)$$

and

$$\sigma^{;\mu}a_{n+1;\mu}+(n+1)a_{n+1}=\Delta^{-1/2}(\Delta^{1/2}a_n)_{;\mu}{}^{\mu}-\xi Ra_n\,, \qquad (3.12)$$

where $\Delta(x,x')\equiv g^{-1/2}(x)D(x,x')g^{-1/2}(x')$. When $s=0$,

$$\Omega(x,x',0)=a_0(x,x')=1\,, \qquad (3.13)$$

for all $x$ and $x'$. Equation (3.11) is satisfied immediately. As we shall see, we will not need to solve the recursion relation for the $a_n$'s, but will use Eq. (3.12) to find the $a_n$'s and their derivatives in the limit that $x'$ coincides with $x$.

Now substitute Eq. (3.7) into (3.4) to get

$$G(x,x')=\frac{\Delta^{1/2}}{(4\pi)^2}\int_0^\infty\frac{1}{s^2}\exp\left[-i\left(m^2s-\frac{\sigma}{2s}\right)\right]\times\sum_{n=0}^{\infty}a_n(is)^n\,ds\,. \qquad (3.14)$$

Exchanging the summation and integration, we have

$$G(x,x')=\frac{\Delta^{1/2}}{(4\pi)^2}\sum_{n=0}^{\infty}a_n\left(-\frac{\partial}{\partial m^2}\right)^n\times\int_0^\infty\frac{1}{s^2}\exp\left[-i\left(m^2s-\frac{\sigma}{2s}\right)\right]ds\,.$$

Using[10]

$$\frac{1}{(4\pi)^2}\int_0^\infty\frac{1}{s^2}\exp\left[-i\left(m^2s-\frac{\sigma}{2s}\right)\right]ds=-\frac{m^2}{8\pi}\frac{H_1^{(2)}((-2m^2\sigma)^{1/2})}{(-2m^2\sigma)^{1/2}}\,,$$

we find that

$$G(x,x')=-\frac{\Delta^{1/2}}{8\pi}\sum_{n=0}^{\infty}a_n\left(-\frac{\partial}{\partial m^2}\right)^n\times\frac{m^2H_1^{(2)}((-2m^2\sigma)^{1/2})}{(-2m^2\sigma)^{1/2}}\,, \qquad (3.15)$$

where $H_1^{(2)}$ is the Hankel function of the second kind of order one.

We may write $H_1^{(2)}$ as an asymptotic series,

$$\frac{m^2 H_1^{(2)}((-2m^2\sigma)^{1/2})}{(-2m^2\sigma)^{1/2}} = \frac{1}{\pi i}\left(\frac{1}{\sigma + i0_+} + 2m^2\left\{[\gamma + \tfrac{1}{2}\ln(\tfrac{1}{2}m^2) + \tfrac{1}{2}\ln(\sigma + i0_+)]\left[\tfrac{1}{2} + \frac{2m^2\sigma}{2^2\times 4} + \frac{(2m^2\sigma)^2}{2^2\times 4^2\times 6} + \cdots\right]\right.\right.$$
$$\left.\left. - \tfrac{1}{4} - \frac{2m^2\sigma}{2^2\times 4}(1+\tfrac{1}{4}) - \frac{(2m^2\sigma)^2}{2^2\times 4^2\times 6}(1+\tfrac{1}{2}+\tfrac{1}{6}) - \cdots\right\} - \cdots\right), \tag{3.16}$$

where we have reinserted $i0_+$'s. The constant $\gamma$ is Euler's constant. We now substitute Eq. (3.16) into (3.15) and carry out the differentiations and summation. Using

$$\frac{1}{\sigma + i0_+} = \frac{1}{\sigma} - \pi i \delta(\sigma), \quad \ln(\sigma + i0_+) = \ln|\sigma| + \pi i\,\theta(-\sigma),$$

where

$$\theta(-\sigma) = \begin{cases} 1, & \sigma < 0 \\ 0, & \sigma > 0 \end{cases}$$

and Eq. (3.1), we obtain an expansion for $G^{(1)}$:

$$G^{(1)}(x,x') = \frac{\Delta^{1/2}}{4\pi^2}\left\{a_0\left[\frac{1}{\sigma} + m^2(\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma|)(1 + \tfrac{1}{4}m^2\sigma + \cdots) - \tfrac{1}{2}m^2 - \tfrac{5}{16}m^2\sigma + \cdots\right]\right.$$
$$- a_1[(\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma|)(1 + \tfrac{1}{2}m^2\sigma + \cdots) - \tfrac{1}{2}m^2\sigma - \cdots]$$
$$+ a_2\sigma[(\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma|)(\tfrac{1}{2} + \tfrac{1}{8}m^2\sigma + \cdots) - \tfrac{1}{4} - \cdots] + \cdots$$
$$\left. + \frac{1}{2m^2}[a_2 + \cdots] + \frac{1}{2m^4}[a_3 + \cdots] + \cdots\right\}. \tag{3.17}$$

Equation (3.17) includes only those terms which will contribute to the divergences and certain finite terms in $\langle T^{\mu\nu}\rangle_{\text{div}}$. The divergences in $G^{(1)}(x,x')$ appear as $\sigma^{-1}$ and $\ln|\tfrac{1}{2}m^2\sigma|$ terms which blow up when $\sigma \to 0$ as $x' \to x$.

## IV. BITENSORS

In Sec. II, we introduced the term bitensor. A general bitensor,

$$T_{\alpha_1 \cdots \alpha_n \beta_1' \cdots \beta_m'},$$

transforms like the product of two tensors, one at each space-time point,

$$A_{\alpha_1 \cdots \alpha_n}(x)\, B_{\beta_1 \cdots \beta_m}(x').$$

So far we have introduced the biscalars, $\sigma(x,x')$, $\Delta^{1/2}(x,x')$, and $a_n(x,x')$. We will also need to study the bivector, $g^{\mu}{}_{\nu'}$, which is called the bivector of parallel displacement. This object, when acting on a vector $A^{\nu'}$ at $x'$, gives the vector $\bar{A}^{\mu}$, which is obtained by parallel transport of $A^{\nu'}$ to $x$ along the geodesic connecting $x$ and $x'$. So[11]

$$\bar{A}^{\mu} = g^{\mu}{}_{\nu'} A^{\nu'}.$$

We can find the properties of $g^{\mu}{}_{\nu'}$ by studying the action of $g^{\mu}{}_{\nu'}$ on $\sigma^{;\nu'}$, which is tangent to the geodesic at $x'$, has length equal to the geodesic distance between $x$ and $x'$, and is oriented in the $x \to x'$ direction. We have

$$-\sigma^{;\mu} = g^{\mu}{}_{\nu'}\sigma^{;\nu'}. \tag{4.1}$$

When a tangent vector is parallel transported, it remains tangent and keeps the same length. So the action of $g^{\mu}{}_{\nu'}$ on $\sigma^{;\nu'}$ must give $-\sigma^{;\mu}$, which is tangent to the geodesic at $x$ and has the same length as $\sigma^{;\nu'}$. The minus sign comes from the fact that $\sigma^{;\mu}$ is oriented in the $x' \to x$ direction.

The key property of bitensors is the coincidence limit, defined by

$$[T_{\alpha_1 \cdots \alpha_n \beta_1' \cdots \beta_m'}] = \lim_{x' \to x} T_{\alpha_1 \cdots \alpha_n \beta_1' \cdots \beta_m'},$$

where we have adopted Synge's bracket notation (see Ref. 9).

We will start by finding the coincidence limits of $\sigma(x,x')$ and its derivatives. As $x'$ approaches $x$, the length of the geodesic goes to zero, so that by definition

$$[\sigma] = 0 \tag{4.2a}$$

and

$$[\sigma_{;\mu}] = 0. \tag{4.2b}$$

Turning to Eq. (3.9) and differentiating at $x$ repeatedly, we find

$$\sigma_{;\mu} = \sigma^{;\rho}\sigma_{;\rho\mu}\,, \tag{4.3a}$$

$$\sigma_{;\mu\nu} = \sigma^{;\rho}{}_{\nu}\sigma_{;\rho\mu} + \sigma^{;\rho}\sigma_{;\rho\mu\nu}\,, \tag{4.3b}$$

$$\sigma_{;\mu\nu\sigma} = \sigma^{;\rho}{}_{\nu\sigma}\sigma_{;\rho\mu} + \sigma^{;\rho}{}_{\nu}\sigma_{;\rho\mu\sigma} + \sigma^{;\rho}{}_{\sigma}\sigma_{;\rho\mu\nu} + \sigma^{;\rho}\sigma_{;\rho\mu\nu\sigma}\,, \tag{4.3c}$$

and so forth. From Eq. (4.2b) and Eq. (4.3b), we get

$$[\sigma_{;\mu\nu}] = g_{\mu\nu}\,. \tag{4.4a}$$

Taking the coincidence limits of Eq. (4.3c) and the higher-derivative equations, we have

$$[\sigma_{;\mu\nu\sigma}] = 0\,, \tag{4.4b}$$

$$[\sigma_{;\mu\nu\sigma\tau}] = S_{\mu\nu\sigma\tau} \equiv -\tfrac{1}{3}(R_{\mu\sigma\nu\tau} + R_{\mu\tau\nu\sigma})\,, \tag{4.4c}$$

$$[\sigma_{;\mu\nu\sigma\tau\rho}] = \tfrac{3}{4}(S_{\mu\nu\sigma\tau;\rho} + S_{\mu\nu\tau\rho;\sigma} + S_{\mu\nu\rho\sigma;\tau})\,, \tag{4.4d}$$

and a six-derivative limit which has 36 terms involving $S_{\mu\nu\sigma\tau}$.[12]

Next we look at $\Delta^{1/2}$. From Eq. (4.1), we get

$$\sigma_{;\mu\nu'} = -g_{\nu'\sigma;\mu}\sigma^{;\sigma} - g_{\nu'\sigma}\sigma^{;\sigma}{}_{\mu}\,,$$

which, when we note that $[g_{\mu\nu'}] = g_{\mu\nu}$, gives

$$[\sigma_{;\mu\nu'}] = -g_{\mu\nu}\,. \tag{4.5}$$

Definition (3.8) then says that

$$[D] = -\det(-[\sigma_{;\mu\nu'}]) = -\det g_{\mu\nu} \equiv g(x)\,.$$

Now we have

$$[\Delta(x,x')] = [g^{-1/2}(x)D(x,x')g^{-1/2}(x')] = 1\,,$$

or

$$[\Delta^{1/2}] = 1\,. \tag{4.6}$$

Equation (3.10) may be written as

$$\Delta^{-1}(\Delta\sigma^{;\mu})_{;\mu} = 4$$

or (4.7)

$$4\Delta^{1/2} = 2\Delta^{1/2}{}_{;\rho}\sigma^{;\rho} + \Delta^{1/2}\sigma^{;\rho}{}_{\rho}\,.$$

Once again, we differentiate Eq. (4.7) repeatedly, take the coincidence limit of each equation, and then use the $\sigma$-coincidence limits. This gives

$$[\Delta^{1/2}{}_{;\alpha}] = 0\,, \tag{4.8a}$$

$$[\Delta^{1/2}{}_{;\alpha\beta}] = \tfrac{1}{6}R_{\alpha\beta}\,, \tag{4.8b}$$

$$[\Delta^{1/2}{}_{;\alpha\beta\gamma}] = \tfrac{1}{12}(R_{\alpha\beta;\gamma} + R_{\alpha\gamma;\beta} + R_{\beta\gamma;\alpha})\,, \tag{4.8c}$$

$$\begin{aligned}[\Delta^{1/2}{}_{;\alpha\beta\gamma\delta}] = -\tfrac{1}{8}\big([\sigma^{;\rho}{}_{\rho\alpha\beta\gamma\delta}] &- \tfrac{1}{3}R_{\alpha\rho}R^{\rho}{}_{\beta\gamma\delta} - \tfrac{1}{3}R_{\beta\rho}R^{\rho}{}_{\alpha\gamma\delta} \\ &- \tfrac{1}{3}R_{\gamma\rho}R^{\rho}{}_{\alpha\beta\delta} - \tfrac{1}{3}R_{\delta\rho}R^{\rho}{}_{\alpha\beta\gamma} + \tfrac{1}{3}R_{\alpha\rho}S^{\rho}{}_{\beta\gamma\delta} \\ &+ \tfrac{1}{3}R_{\beta\rho}S^{\rho}{}_{\alpha\gamma\delta} + \tfrac{1}{3}R_{\gamma\rho}S^{\rho}{}_{\alpha\beta\delta} + \tfrac{1}{3}R_{\delta\rho}S^{\rho}{}_{\alpha\beta\gamma} \\ &- \tfrac{2}{9}R_{\alpha\beta}R_{\gamma\delta} - \tfrac{2}{9}R_{\alpha\gamma}R_{\beta\delta} - \tfrac{2}{9}R_{\alpha\delta}R_{\beta\gamma}\big)\,.\end{aligned} \tag{4.8d}$$

For $g^{\mu}{}_{\nu'}$, we use Eq. (4.1) and $[g^{\mu}{}_{\nu'}] = \delta^{\mu}{}_{\nu}$ to obtain

$$[g^{\mu}{}_{\nu';\alpha}] = 0\,, \tag{4.9a}$$

$$[g^{\mu}{}_{\nu';\alpha\beta}] = -\tfrac{1}{2}R^{\mu}{}_{\nu\alpha\beta}\,, \tag{4.9b}$$

and higher derivatives whose most important property is

$$[g^{\mu}{}_{\nu';\alpha\beta\cdots}]\sigma^{;\alpha}\sigma^{;\beta}\cdots = 0\,, \tag{4.9c}$$

where the dots denote any number of unprimed indices.

Finally, we will need the coincidence limits of the $a_n$'s and their first few derivatives. We already know that

$$a_0(x,x') = 1\,,$$

which gives

$$[a_0] = 1 \tag{4.10a}$$

and

$$[a_{0;\alpha\beta\cdots}] = 0\,. \tag{4.10b}$$

These limits and those of $\sigma$ and $\Delta^{1/2}$ allow us to use the recursion relations (3.12) to show that

$$[a_1] = (\tfrac{1}{6} - \xi)R\,, \tag{4.11a}$$

$$[a_{1;\mu}] = \tfrac{1}{2}(\tfrac{1}{6} - \xi)R_{;\mu}\,, \tag{4.11b}$$

$$[a_{1;\mu\nu}] = (\tfrac{1}{20} - \tfrac{1}{3}\xi)R_{;\mu\nu} + \tfrac{1}{60}R_{\mu\nu;\rho}{}^{\rho} + \tfrac{1}{90}R^{\rho\tau}R_{\rho\mu\tau\nu} - \tfrac{1}{45}R_{\mu\rho}R^{\rho}{}_{\nu} + \tfrac{1}{90}R^{\rho\kappa\tau}{}_{\mu}R_{\rho\kappa\tau\nu}\,, \tag{4.11c}$$

and

$$[a_2] = -\tfrac{1}{180}R^{\rho\tau}R_{\rho\tau} + \tfrac{1}{180}R^{\rho\tau\kappa\iota}R_{\rho\tau\kappa\iota} + \tfrac{1}{6}(\tfrac{1}{5} - \xi)R_{;\rho}{}^{\rho} + \tfrac{1}{2}(\tfrac{1}{6} - \xi)^2R^2\,. \tag{4.11d}$$

In later calculations, we will need coincidence limits of biscalars with primed derivatives. These may be found most easily by using a generalization of a theorem proved by Synge (Ref. 9) originally for $\sigma(x,x')$ only. The general theorem, which we prove in the Appendix, is

$$[T_{\alpha_1\cdots\alpha_n\beta_1'\cdots\beta_m';\mu'}] = -[T_{\alpha_1\cdots\alpha_n\beta_1'\cdots\beta_m';\mu}] + [T_{\alpha_1\cdots\alpha_n\beta_1'\cdots\beta_m'}]_{;\mu}\,, \tag{4.12}$$

where

$$T_{\alpha_1\cdots\alpha_n\beta_1'\cdots\beta_m'}$$

is any bitensor whose coincidence limit and derivative coincidence limits exist. Consider $[\sigma_{;\mu\nu\sigma'\tau'}]$:

$$\begin{aligned}[\sigma_{;\mu\nu\sigma'\tau'}] &= -[\sigma_{;\mu\nu\sigma'\tau}] + [\sigma_{;\mu\nu\sigma'}]_{;\tau} \\ &= -(-[\sigma_{;\mu\nu\tau\sigma}] + [\sigma_{;\mu\nu\tau}]_{;\sigma}) \\ &\quad + (-[\sigma_{;\mu\nu\sigma}] + [\sigma_{;\mu\nu}]_{;\sigma})_{;\tau} \\ &= [\sigma_{;\mu\nu\tau\sigma}] = S_{\mu\nu\tau\sigma} = S_{\mu\nu\sigma\tau}\,,\end{aligned}$$

and $[g^{\alpha}{}_{\beta';\mu\nu'}]$:

$$\begin{aligned}[g^{\alpha}{}_{\beta';\mu\nu'}] &= -[g^{\alpha}{}_{\beta';\mu\nu}] + [g^{\alpha}{}_{\beta';\mu}]_{;\nu} \\ &= \tfrac{1}{2}R^{\alpha}{}_{\beta\mu\nu}\,.\end{aligned}$$

This technique saves a great deal of time when a large number of primed indices is involved. Once we have the coincidence limits of quantities with no primes, a simple application of the theorem gives all other primed-unprimed combinations.

## V. COVARIANT EXPANSIONS

In the expression (2.8) for $\langle \underline{T^{\mu\nu}} \rangle_{\text{div}}$, the $G^{(1)}(x,x')$ quantities diverge in the limit $x' \to x$. We can avoid this problem by simply not taking the coincidence limit. This presents us with a nonsensical equation. On the left side of Eq. (2.8), we have a tensor at $x$, while on the right, a mixture of quantities with various transformation properties. We need to express the right-hand side in terms of quantities at $x$ with the correct transformation properties. At the same time, we want to isolate, in a covariant manner, those quantities which diverge when $x' \to x$. We accomplish this by expanding all bitensors in terms of functions at $x$ and the tangent vector, $\sigma^{;\mu} \equiv \sigma^{\mu}$.

Consider some bitensor $T_{\alpha_1\cdots\alpha_n\beta_1'\cdots\beta_m'}$ which, along with its derivatives, has a known finite coincidence limit. We note that an expansion such as

$$T_{\alpha_1\cdots\alpha_n\beta_1'\cdots\beta_m'} = t_{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m}(x) + t_{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m\rho}(x)\sigma^{\rho} + \cdots \quad (5.1)$$

is meaningless because again the two sides have different transformation properties. We must transform

$$T_{\alpha_1\cdots\alpha_n\beta_1'\cdots\beta_m'}$$

into a tensor at $x$ and then try to expand it. We do this by using $g^{\alpha}{}_{\beta'}$, constructing

$$\overline{T}_{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m} = g_{\beta_1}{}^{\rho_1'}\cdots g_{\beta_m}{}^{\rho_m'} T_{\alpha_1\cdots\alpha_n\rho_1'\cdots\rho_m'}\,,$$

and then expanding as in Eq. (5.1).

To avoid writing so many indices, we will find the expansion for a bivector $T_{\alpha\beta'}$. The method we use is applicable to any bitensor with known coincidence limits. Suppose we know the coincidence limits of $T_{\alpha\beta'}$ and its derivatives. We assume that

$$\overline{T}_{\alpha\beta} \equiv g_{\beta}{}^{\rho'} T_{\alpha\rho'} = t_{\alpha\beta} + t_{\alpha\beta\mu}\sigma^{\mu} + \frac{1}{2!}\, t_{\alpha\beta\mu\nu}\sigma^{\mu}\sigma^{\nu} + \cdots\,, \quad (5.2)$$

where the $t$ coefficients are functions of $x$ only. Differentiate Eq. (5.2) repeatedly and take coincidence limits of each equation. We find that

$$\begin{aligned}[\overline{T}_{\alpha\beta}] &= [g_{\beta}{}^{\rho'} T_{\alpha\rho'}] = t_{\alpha\beta}\,, \\ [\overline{T}_{\alpha\beta;\mu}] &= [g_{\beta}{}^{\rho'}{}_{;\mu} T_{\alpha\rho'} + g_{\beta}{}^{\rho'} T_{\alpha\rho';\mu}] \\ &= t_{\alpha\beta;\mu} + t_{\alpha\beta\mu}\,,\end{aligned}$$

and so forth. Using the properties of $g^{\alpha}{}_{\beta'}$, we obtain

$$\begin{aligned} t_{\alpha\beta} &= [T_{\alpha\beta'}]\,, \\ t_{\alpha\beta\mu} &= [T_{\alpha\beta';\mu}] - t_{\alpha\beta;\mu}\,, \\ t_{\alpha\beta\mu\nu} &= [T_{\alpha\beta';\mu\nu}] - t_{\alpha\beta;\mu\nu} - t_{\alpha\beta\mu;\nu} \\ &\quad - t_{\alpha\beta\nu;\mu} + [g_{\alpha\beta';\mu\nu}] \text{ terms}, \\ t_{\alpha\beta\mu\nu\sigma} &= [T_{\alpha\beta';\mu\nu\sigma}] - t_{\alpha\beta;\mu\nu\sigma} - t_{\alpha\beta\mu;\nu\sigma} \\ &\quad - t_{\alpha\beta\nu;\mu\sigma} - t_{\alpha\beta\sigma;\mu\nu} - t_{\alpha\beta\mu\nu;\sigma} \\ &\quad - t_{\alpha\beta\nu\sigma;\mu} - t_{\alpha\beta\sigma\mu;\nu} - t_{\alpha\beta\rho}S^{\rho}{}_{\mu\nu\sigma} \\ &\quad + [g_{\alpha\beta';\mu\nu\sigma}] \text{ terms}\,,\end{aligned}$$

and very long higher-coefficient expressions. Owing to Eq. (4.9c), the $[g_{\alpha\beta';\mu\nu\ldots}]$ terms vanish when the coefficients are put into Eq. (5.2), so we ignore them. To illustrate the use of these expressions, we will calculate the expansion for $\overline{\sigma}_{\alpha\beta} = g_{\beta}{}^{\rho'}\sigma_{;\alpha\rho'}$. We have

$$\begin{aligned} t_{\alpha\beta} &= [\sigma_{;\alpha\beta'}] = -g_{\alpha\beta}\,, \\ t_{\alpha\beta\mu} &= [\sigma_{;\alpha\beta'\mu}] + g_{\alpha\beta;\mu} = 0\,, \\ t_{\alpha\beta\mu\nu} &= [\sigma_{;\alpha\beta'\mu\nu}] + g_{\alpha\beta;\mu\nu} = -S_{\alpha\mu\nu\beta}\,,\end{aligned}$$

and

$$\begin{aligned} t_{\alpha\beta\mu\nu\sigma} &= [\sigma_{;\alpha\beta'\mu\nu\sigma}] + g_{\alpha\beta;\mu\nu\sigma} + S_{\alpha\mu\nu\beta;\sigma} + S_{\alpha\nu\sigma\beta;\mu} + S_{\alpha\sigma\mu\beta;\nu} \\ &= -\tfrac{3}{4}(S_{\alpha\mu\nu\sigma;\beta} + S_{\alpha\mu\sigma\beta;\nu} + S_{\alpha\mu\beta\nu;\sigma}) + S_{\alpha\mu\nu\sigma;\beta} + S_{\alpha\mu\nu\beta;\sigma} + S_{\alpha\nu\sigma\beta;\mu} + S_{\alpha\sigma\mu\beta;\nu}\,,\end{aligned}$$

where we have used Eqs. (4.2)–(4.4), (4.12), and $g_{\mu\nu;\sigma}=0$. Putting these and higher terms into Eq. (5.2), we get

$$\bar{\sigma}_{\alpha\beta}=-g_{\alpha\beta}-\tfrac{1}{6}R_{\alpha\mu\beta\nu}\sigma^\mu\sigma^\nu+\tfrac{1}{12}R_{\alpha\mu\beta\nu;\sigma}\sigma^\mu\sigma^\nu\sigma^\sigma-(\tfrac{1}{40}R_{\alpha\mu\beta\nu;\sigma\tau}+\tfrac{7}{360}R^{\kappa}{}_{\mu\alpha\nu}R_{\kappa\sigma\beta\tau})\sigma^\mu\sigma^\nu\sigma^\sigma\sigma^\tau+\cdots\,. \tag{5.3}$$

Applying this method to the other bitensors, we get

$$\sigma=\tfrac{1}{2}g_{\alpha\beta}\sigma^\alpha\sigma^\beta,$$

$$\bar{\sigma}^\mu\equiv g^{\mu}{}_{\nu'}\sigma^{;\nu'}=-\sigma^\mu,$$

$$\sigma^{;\mu\nu}=g^{\mu\nu}-\tfrac{1}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,\sigma^\alpha\sigma^\beta+\tfrac{1}{12}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\gamma}\sigma^\alpha\sigma^\beta\sigma^\gamma-(\tfrac{1}{60}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\gamma\delta}+\tfrac{1}{45}R^{\rho}{}_{\alpha}{}^{\mu}{}_{\beta}R_{\rho\gamma}{}^{\nu}{}_{\delta})\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta+\cdots\,,$$

$$\Delta^{1/2}=1+\tfrac{1}{12}R_{\alpha\beta}\sigma^\alpha\sigma^\beta-\tfrac{1}{24}R_{\alpha\beta;\gamma}\sigma^\alpha\sigma^\beta\sigma^\gamma+(\tfrac{1}{288}R_{\alpha\beta}R_{\gamma\delta}+\tfrac{1}{360}R^{\rho}{}_{\alpha}{}^{\tau}{}_{\beta}R_{\rho\gamma\tau\delta}+\tfrac{1}{80}R_{\alpha\beta;\gamma\delta})\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta+\cdots\,,$$

$$\Delta^{1/2;\mu}=\tfrac{1}{6}R^{\mu}{}_{\alpha}\sigma^\alpha-\tfrac{1}{24}(2R^{\mu}{}_{\alpha;\beta}-R_{\alpha\beta}{}^{;\mu})\sigma^\alpha\sigma^\beta+(\tfrac{1}{40}R^{\mu}{}_{\alpha;\beta\gamma}-\tfrac{1}{60}R_{\alpha\beta}{}^{;\mu}{}_{\gamma}+\tfrac{1}{90}R^{\rho\mu\tau}{}_{\alpha}R_{\rho\beta\tau\gamma}+\tfrac{1}{72}R^{\mu}{}_{\alpha}R_{\beta\gamma}$$
$$+\tfrac{1}{360}R_{\alpha}{}^{\rho}R_{\rho\beta}{}^{\mu}{}_{\gamma})\sigma^\alpha\sigma^\beta\sigma^\gamma+\cdots\,,$$

$$\Delta^{1/2;\mu\nu}=\tfrac{1}{6}R^{\mu\nu}+\tfrac{1}{12}(2R^{(\mu}{}_{\alpha}{}^{;\nu)}-R^{\mu\nu}{}_{;\alpha})\sigma^\alpha$$
$$+(\tfrac{1}{40}R^{\mu\nu}{}_{;\alpha\beta}+\tfrac{1}{40}R_{\alpha\beta}{}^{;(\mu\nu)}-\tfrac{1}{15}R^{(\mu}{}_{\alpha}{}^{;\nu)}{}_{\beta}+\tfrac{1}{72}R_{\alpha\beta}R^{\mu\nu}+\tfrac{1}{36}R^{\mu}{}_{\alpha}R^{\nu}{}_{\beta}+\tfrac{1}{180}R^{(\mu}{}_{\rho}R^{\nu)}{}_{\alpha}{}^{\rho}{}_{\beta}+\tfrac{1}{90}R^{\mu\rho\nu\tau}R_{\rho\alpha\tau\beta}$$
$$+\tfrac{1}{90}R^{\rho}{}_{\alpha}{}^{\tau(\mu}R_{\tau}{}^{\nu)}{}_{\rho\beta}+\tfrac{1}{90}R^{\rho}{}_{\alpha}{}^{\tau(\mu}R_{\rho}{}^{\nu)}{}_{\tau\beta}+\tfrac{11}{180}R_{\alpha\rho}R^{\rho(\mu\nu)}{}_{\beta})\sigma^\alpha\sigma^\beta+\cdots\,,$$

$$a_1=(\tfrac{1}{6}-\xi)R-\tfrac{1}{2}(\tfrac{1}{6}-\xi)R_{;\alpha}\sigma^\alpha+[-\tfrac{1}{90}R_{\alpha\rho}R^{\rho}{}_{\beta}+\tfrac{1}{180}R^{\rho\tau}R_{\rho\alpha\tau\beta}+\tfrac{1}{180}R_{\rho\tau\kappa\alpha}R^{\rho\tau\kappa}{}_{\beta}+\tfrac{1}{120}R_{\alpha\beta;\rho}{}^{\rho}+(\tfrac{1}{40}-\tfrac{1}{6}\xi)R_{;\alpha\beta}]\sigma^\alpha\sigma^\beta+\cdots\,,$$

$$a_1{}^{;\mu}=\tfrac{1}{2}(\tfrac{1}{6}-\xi)R^{;\mu}+[-\tfrac{1}{45}R^{\mu}{}_{\rho}R^{\rho}{}_{\alpha}+\tfrac{1}{90}R^{\rho\tau}R^{\mu}{}_{\rho\alpha\tau}+\tfrac{1}{90}R_{\rho\tau\kappa}{}^{\mu}R^{\rho\tau\kappa}{}_{\alpha}+\tfrac{1}{60}R^{\mu}{}_{\alpha;\rho}{}^{\rho}+\tfrac{1}{6}(\xi-\tfrac{1}{5})R^{;\mu}{}_{\alpha}]\sigma^\alpha+\cdots\,,$$

$$a_1{}^{;\mu\nu}=-\tfrac{1}{45}R^{\mu}{}_{\rho}R^{\nu\rho}+\tfrac{1}{90}R^{\rho\tau}R^{\mu}{}_{\rho}{}^{\nu}{}_{\tau}+\tfrac{1}{90}R_{\rho\tau\kappa}{}^{\mu}R^{\rho\tau\kappa\nu}+\tfrac{1}{60}R^{\mu\nu}{}_{;\rho}{}^{\rho}+(\tfrac{1}{20}-\tfrac{1}{3}\xi)R^{;\mu\nu}+\cdots\,,$$

and

$$a_2=-\tfrac{1}{180}R^{\rho\tau}R_{\rho\tau}+\tfrac{1}{180}R^{\rho\tau\kappa\iota}R_{\rho\tau\kappa\iota}+\tfrac{1}{6}(\tfrac{1}{5}-\xi)R_{;\rho}{}^{\rho}+\tfrac{1}{2}(\tfrac{1}{6}-\xi)^2R^2+\cdots\,.$$

The primed derivative expansions can be obtained by differentiating the series above and using Eq. (5.3).

Finally, we will need to expand objects such as $(\sigma^{-1})^{;\mu\nu}$ which diverge in the coincidence limit so that we cannot use Eq. (5.2) directly. However, if we carry out the differentiations

$$(\sigma^{-1})^{;\mu\nu}=2\sigma^{-3}\sigma^{;\mu}\sigma^{;\nu}-\sigma^{-2}\sigma^{;\mu\nu},$$

we can expand $\sigma^{;\mu\nu}$ to get

$$(\sigma^{-1})^{;\mu\nu}=-\frac{4}{(\sigma^\rho\sigma_\rho)^2}\left[g^{\mu\nu}-4\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right]+\tfrac{4}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)^2}-\tfrac{1}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\gamma}\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma}{(\sigma^\rho\sigma_\rho)^2}+4(\tfrac{1}{60}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\gamma\delta}+\tfrac{1}{45}R^{\rho}{}_{\alpha}{}^{\mu}{}_{\beta}R_{\rho\gamma}{}^{\nu}{}_{\delta})\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta}{(\sigma^\rho\sigma_\rho)^2}\,.$$

We now have all the information we will need to find $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$.

## VI. THE RESULTS

We begin the expansion of the right-hand side of Eq. (2.8) by considering $G^{(1)}(x,x')$ in Eq. (3.17). Substituting the series for $\sigma$, $\Delta^{1/2}$, $a_1$, and $a_2$, and collecting together terms in like powers of $\sigma^\mu$, we get

$$4\pi^2G^{(1)}(x,x')=\frac{2}{(\sigma^\rho\sigma_\rho)}+[m^2-(\tfrac{1}{6}-\xi)R][\gamma+\tfrac{1}{2}\ln|\tfrac{1}{4}m^2(\sigma^\rho\sigma_\rho)|]-\tfrac{1}{2}m^2$$
$$+\tfrac{1}{6}R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)}+\frac{1}{2m^2}[\tfrac{1}{2}(\tfrac{1}{6}-\xi)^2R^2-\tfrac{1}{180}R^{\rho\tau}R_{\rho\tau}+\tfrac{1}{180}R^{\rho\tau\kappa\iota}R_{\rho\tau\kappa\iota}+\tfrac{1}{6}(\tfrac{1}{5}-\xi)R_{;\rho}{}^{\rho}]+O(1/m^4). \tag{6.1}$$

$G^{(1)}$ has a quadratic and logarithmic divergence as well as direction-dependent ($\sigma^\mu$-dependent) finite terms and finite terms with no $\sigma^\mu$ dependence at all. $O(1/m^4)$ implies that there will also be finite terms proportional to $1/m^4$, $1/m^6$, and so forth.

We now differentiate Eq. (3.17) to form $G^{(1);\mu\nu}$, $G^{(1);\mu'\nu}$, $G^{(1);\mu\nu'}$, and $G^{(1);\mu'\nu'}$. In the expressions we obtain, we substitute the series expansion from Sec. V. Finally, after much algebra, we form $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ using Eq. (2.8) and collect equal powers of $\sigma^\mu$. The results are

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm quartic}=\frac{g^{1/2}}{2\pi^2}\frac{1}{(\sigma^\rho\sigma_\rho)^2}\left[g^{\mu\nu}-4\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right], \tag{6.2}$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{quadratic}} = \frac{g^{1/2}}{4\pi^2}\,\frac{1}{(\sigma^\rho\sigma_\rho)}\left(\left\{\tfrac{2}{3}R^{(\mu}{}_\alpha\frac{\sigma^{\nu)}\sigma^\alpha}{(\sigma^\rho\sigma_\rho)} - \tfrac{2}{3}R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)^2} - \tfrac{1}{2}m^2\left[g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right]\right\}\right.$$
$$\left. - (\tfrac{1}{6}-\xi)\left\{R^{\mu\nu} - \tfrac{1}{2}R\left[g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right] - 2R^\mu{}_\alpha{}^\nu{}_\beta\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)} + 2R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)}g^{\mu\nu}\right\}\right), \tag{6.3}$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{logarithmic}} = \frac{g^{1/2}}{4\pi^2}\{[\tfrac{1}{60}(R^{\rho\mu\tau\nu}R_{\rho\tau} - \tfrac{1}{4}R^{\rho\tau}R_{\rho\tau}g^{\mu\nu}) - \tfrac{1}{180}R(R^{\mu\nu} - \tfrac{1}{4}R\,g^{\mu\nu}) + \tfrac{1}{120}R^{\mu\nu}{}_{;\rho}{}^\rho - \tfrac{1}{360}R^{;\mu\nu} - \tfrac{1}{720}R_{;\rho}{}^\rho g^{\mu\nu} - \tfrac{1}{8}m^4 g^{\mu\nu}]$$
$$- \tfrac{1}{2}(\tfrac{1}{6}-\xi)[m^2(R^{\mu\nu} - \tfrac{1}{2}R\,g^{\mu\nu})] - \tfrac{1}{4}(\tfrac{1}{6}-\xi)^2[-2R(R^{\mu\nu} - \tfrac{1}{4}R\,g^{\mu\nu}) + 2R^{;\mu\nu} - 2R_{;\rho}{}^\rho g^{\mu\nu}]\}$$
$$\times[\gamma + \tfrac{1}{2}\ln|\tfrac{1}{4}m^2(\sigma^\rho\sigma_\rho)|], \tag{6.4}$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{linear}} = \frac{g^{1/2}}{4\pi^2}\left(\left\{\tfrac{1}{12}(R^{\mu\nu} - \tfrac{1}{4}R\,g^{\mu\nu})_{;\alpha}\frac{\sigma^\alpha}{(\sigma^\rho\sigma_\rho)} - \tfrac{1}{3}(R^{(\mu}{}_\alpha g^{\nu)}{}_\beta - \tfrac{1}{4}R_{\alpha\beta}g^{\mu\nu})_{;\gamma}\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma}{(\sigma^\rho\sigma_\rho)^2} - \tfrac{1}{12}R_{\alpha\beta;\gamma}\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma}{(\sigma^\rho\sigma_\rho)^2}\left[g^{\mu\nu} - 4\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right]\right\}\right.$$
$$\left. - (\tfrac{1}{6}-\xi)\left\{(R^\mu{}_\alpha{}^\nu{}_\beta - R_{\alpha\beta}g^{\mu\nu})_{;\gamma}\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma}{(\sigma^\rho\sigma_\rho)^2} + \tfrac{1}{4}R_{;\alpha}\frac{\sigma^\alpha}{(\sigma^\rho\sigma_\rho)}\left[g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right]\right\} + (\tfrac{1}{6}-\xi)^2\left[\tfrac{3}{4}R_{;\alpha}\frac{\sigma^\alpha}{(\sigma^\rho\sigma_\rho)}g^{\mu\nu}\right]\right), \tag{6.5}$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{finite}} = -\frac{g^{1/2}}{8\pi^2}\left(\left\{-\tfrac{3}{16}m^4\left[g^{\mu\nu} - \frac{4}{3}\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right] + \tfrac{1}{360}(R^{\rho\tau\kappa\iota}R_{\rho\tau\kappa\iota} - R^{\rho\tau}R_{\rho\tau} + R_{;\rho}{}^\rho)\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right\}\right.$$
$$+ (\tfrac{1}{6}-\xi)\left\{\tfrac{1}{4}m^2R\left[g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right] + \tfrac{1}{12}R_{;\rho}{}^\rho\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)} - \tfrac{1}{2}m^2R^{\mu\nu}\right\} + (\tfrac{1}{6}-\xi)^2\left[\tfrac{1}{4}R^2\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right]$$
$$+\left\{(\tfrac{1}{20}R^{\mu\nu}{}_{;\alpha\beta} + \tfrac{1}{30}R^{(\mu}{}_{\alpha;}{}^{\nu)}{}_\beta - \tfrac{1}{48}R_{;\alpha\beta}g^{\mu\nu} - \tfrac{1}{180}R_{\alpha\beta}{}^{;(\mu\nu)} + \tfrac{1}{72}R_{\alpha\beta;\rho}{}^\rho g^{\mu\nu} + \tfrac{1}{18}R^\mu{}_\alpha R^\nu{}_\beta - \tfrac{1}{36}R_\alpha{}^\rho R_{\rho\beta}g^{\mu\nu}\right.$$
$$+ \tfrac{1}{90}R_{\alpha\rho}R^{\rho(\mu\nu)}{}_\beta + \tfrac{1}{90}R^{(\mu}{}_\rho R^{\nu)}{}_\alpha{}^\rho{}_\beta + \tfrac{1}{45}R^{\mu\rho\nu\tau}R_{\rho\alpha\tau\beta} + \tfrac{1}{45}R^\mu{}_{\tau\rho\alpha}R^{\nu\rho\tau}{}_\beta + \tfrac{1}{45}R^\mu{}_{\rho\tau\alpha}R^{\nu\rho\tau}{}_\beta)\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)}$$
$$- 2(-\tfrac{1}{45}R_\alpha{}^\rho R^{(\mu}{}_\rho + \tfrac{1}{90}R^{\rho\tau}R^{(\mu}{}_{\rho\alpha\tau} + \tfrac{1}{90}R^{\rho\tau\kappa}{}_\alpha R_{\rho\tau\kappa}{}^{(\mu} + \tfrac{1}{60}R^{(\mu}{}_{\alpha;\rho}{}^\rho - \tfrac{1}{180}R^{;(\mu}{}_\alpha)\frac{\sigma^{\nu)}\sigma^\alpha}{(\sigma^\rho\sigma_\rho)}$$
$$- (-\tfrac{1}{90}R_{\alpha\rho}R^\rho{}_\beta + \tfrac{1}{180}R^{\rho\tau}R_{\rho\alpha\tau\beta} + \tfrac{1}{180}R_{\rho\tau\kappa\alpha}R^{\rho\tau\kappa}{}_\beta + \tfrac{1}{120}R_{\alpha\beta;\rho}{}^\rho - \tfrac{1}{360}R_{;\alpha\beta})\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)}\left[g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right]$$
$$\left. + \tfrac{1}{6}m^2\left[2R^{(\mu}{}_\alpha\frac{\sigma^{\nu)}\sigma^\alpha}{(\sigma^\rho\sigma_\rho)} - R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)^2}\right]\right\}$$
$$+ (\tfrac{1}{6}-\xi)\left\{-\tfrac{1}{3}(RR^{(\mu}{}_\alpha + \tfrac{1}{2}R^{;(\mu}{}_\alpha)\frac{\sigma^{\nu)}\sigma^\alpha}{(\sigma^\rho\sigma_\rho)} + \left[\tfrac{1}{6}RR_{\alpha\beta}\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)} - \tfrac{1}{6}R_{;\alpha\beta}\left(g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right)\right.\right.$$
$$+ \tfrac{1}{3}R_\alpha{}^\rho R^{(\mu}{}_\rho{}^{\nu)}{}_\beta - \tfrac{1}{6}R_{\alpha\beta}{}^{;(\mu\nu)} + \tfrac{1}{6}R_{\alpha\beta;\rho}{}^\rho g^{\mu\nu} - \tfrac{1}{3}R_\alpha{}^\rho R_{\rho\beta}g^{\mu\nu}$$
$$\left.\left. + \tfrac{1}{6}R^{\mu\nu}R_{\alpha\beta} + m^2(R^\mu{}_\alpha{}^\nu{}_\beta - R_{\alpha\beta}g^{\mu\nu})\right]\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)}\right\}$$
$$- (\tfrac{1}{6}-\xi)^2\left[R(R^\mu{}_\alpha{}^\nu{}_\beta - R_{\alpha\beta}g^{\mu\nu})\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)} - \tfrac{3}{4}R_{;\alpha\beta}\frac{\sigma^\alpha\sigma^\beta}{(\sigma^\rho\sigma_\rho)}g^{\mu\nu}\right]$$
$$+ \left\{(-\tfrac{1}{5}R^{(\mu}{}_{\alpha;\beta\gamma} - \tfrac{1}{30}R_{\alpha\beta}{}^{;(\mu}{}_\gamma - \tfrac{4}{45}R^\tau{}_\alpha{}^{\rho(\mu}R_{\rho\beta\tau\gamma} - \tfrac{1}{9}R^{(\mu}{}_\alpha R_{\beta\gamma} - \tfrac{1}{45}R_\alpha{}^\rho R_{\rho\beta}{}^{(\mu}{}_\gamma)\frac{\sigma^{\nu)}\sigma^\alpha\sigma^\beta\sigma^\gamma}{(\sigma^\rho\sigma_\rho)^2}\right.$$
$$- (\tfrac{1}{72}R_{\alpha\beta}R_{\gamma\delta} + \tfrac{1}{90}R^\rho{}_\alpha{}^\tau{}_\beta R_{\rho\gamma\tau\delta} + \tfrac{1}{20}R_{\alpha\beta;\gamma\delta})\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta}{(\sigma^\rho\sigma_\rho)^2}\left[g^{\mu\nu} - 4\frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)}\right]$$
$$\left. + (\tfrac{1}{90}R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma\delta} - \tfrac{1}{45}R^\rho{}_\alpha{}^\mu{}_\beta R_{\rho\gamma}{}^\nu{}_\delta + \tfrac{1}{18}R_{\alpha\beta;\gamma\delta}g^{\mu\nu} + \tfrac{1}{36}R_{\alpha\beta}R_{\gamma\delta}g^{\mu\nu} + \tfrac{1}{36}R^\rho{}_\alpha{}^\tau{}_\beta R_{\rho\gamma\tau\delta}g^{\mu\nu})\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta}{(\sigma^\rho\sigma_\rho)^2}\right\}$$
$$- (\tfrac{1}{6}-\xi)\left[(\tfrac{2}{3}R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma\delta} + \tfrac{1}{3}R^\rho{}_\alpha{}^\mu{}_\beta R_{\rho\gamma}{}^\nu{}_\delta + \tfrac{1}{3}R_{\alpha\beta}R^\mu{}_\gamma{}^\nu{}_\delta - \tfrac{2}{3}R_{\alpha\beta;\gamma\delta}g^{\mu\nu} - \tfrac{1}{3}R_{\alpha\beta}R_{\gamma\delta}g^{\mu\nu} - \tfrac{1}{3}R^\rho{}_\alpha{}^\tau{}_\beta R_{\rho\gamma\tau\delta}g^{\mu\nu})\frac{\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta}{(\sigma^\rho\sigma_\rho)^2}\right]$$
$$\left. + O\left(\frac{1}{m^2}\right)\right). \tag{6.6}$$

The various orders in $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ have been written so that the conformal scalar field's stress tensor may be found easily. It consists of those terms independent of $(\frac{1}{6}-\xi)$. In the finite term, when the trace is taken we will get terms with no $\sigma^\mu$'s, terms with $\sigma^\alpha\sigma^\beta/(\sigma^\rho\sigma_\rho)$ dependence, and terms with $\sigma^\alpha\sigma^\beta\sigma^\gamma\sigma^\delta/(\sigma^\rho\sigma_\rho)^2$ dependence. $\langle \underline{T}^{\mu\nu}\rangle_{\rm finite}$ has been written so that those terms which give each type when the trace is taken are grouped together. As many symmetrizations as possible have been done to shorten the lengthy expressions. We have also grouped terms so that as many trace-free combinations as possible are presented. Finally, those terms of order $1/m^2$ and higher in $\langle \underline{T}^{\mu\nu}\rangle_{\rm finite}$ are $\sigma^\mu$-independent.

## VII. DISCUSSION

Now that we have $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$, how do we use it? Suppose we are given some background gravitational field and are able to find some complete set of mode functions,[13] $u_i(x)$, by solving the scalar field equations with some given boundary conditions. The field operator, $\underline{\phi}$, is written in terms of these functions as

$$\underline{\phi}(x)=\sum_i \left[\underline{a}_i u_i(x)+\underline{a}_i^* u_i^*(x)\right], \tag{7.1}$$

where $\underline{a}_i$ and $\underline{a}_i^*$ are creation and annihilation operators. The choice of a particular set of mode functions in, say, the in-region will define the $|\text{in, vac}\rangle$ state. Using Eq. (7.1) and the fact that $\underline{T}^{\mu\nu}$ is constructed from products of the form $\underline{\phi}^2$, we find that

$$\langle \text{in, vac}|\underline{T}^{\mu\nu}|\text{in, vac}\rangle=\sum_i T^{\mu\nu}(u_i(x),u_i^*(x)),$$

where the summation may include integrations if the $i$ index is continuous.

DeWitt (Ref. 1) has shown that

$$\langle \text{in, vac}|\underline{T}^{\mu\nu}|\text{in, vac}\rangle=\langle \underline{T}^{\mu\nu}\rangle_{\rm div}+\text{finite terms}.$$

Hence, any divergences in $\langle \text{in, vac}|\underline{T}^{\mu\nu}|\text{in, vac}\rangle$ also appear in $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$.

We take Eq. (7.1) and perform a point separation on the $u_i(x)$ and $u_i^*(x)$ in the mode sum and then express the $x'$-dependent quantities in terms of the tangent vector at $x$ to the geodesic between $x$ and $x'$. All of this is done in some convenient coordinate system so the explicit form of the result will be highly coordinate-dependent. It is normally extremely difficult to put the various divergent and direction-dependent terms into any covariant form. It is at this point where results (6.2) through (6.6) become important. One takes these expressions and specializes them to the particular metric one has chosen. Unless errors have been made, the mode sums for the divergent and direction-dependent finite parts (with one exception to be explained below) should be identical to the answer obtained by the general method.

At this point, we can ask a practical question: How do we choose $x'$ most efficiently so as to shorten what is almost always a long calculation? There are two possible ways to choose $x'$ given $x$. First, there may be some natural vector such as a Killing vector which when plugged into the general expression for $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ in Sec. VI gives a simple expression for each order. The point $x$ and the vector at $x$ will define a geodesic and if the length of the vector (chosen to be non-null) is given, some point $x'$ along the geodesic will be fixed. This is the point we use to separate the mode sum. However, this choice may be a bad one since even though it gives simple results, it may make the mode sums very hard to do. So a second method is to choose a convenient point $x'$ which permits one to do the mode sums. The result may not be as simple as in the first method but at least an answer has been found. Details of the application of the results in this paper to practical calculations will be given in a future paper.

Suppose we consider the Schwarzschild metric,

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2),$$

where $R_{\mu\nu}=0$. Calculating the divergences in Eqs. (6.2)–(6.5) for a massless conformal scalar field, we find that there is only a quartic divergence. If we choose the vector $\sigma^\mu=(\epsilon(1-2M/r)^{-1/2},0,0,0)$ so that $\sigma^\rho\sigma_\rho=-\epsilon^2$, then the quartic term is

$$\langle \underline{T}^\mu{}_\nu\rangle_{\rm quartic}=\frac{3g^{1/2}}{2\pi^2\epsilon^4}\begin{bmatrix}-1&0&0&0\\0&\frac{1}{3}&0&0\\0&0&\frac{1}{3}&0\\0&0&0&\frac{1}{3}\end{bmatrix}, \tag{7.2}$$

which is identical to the usual zero-point fluctuation term found in the stress tensor for a scalar field in flat space with Minkowski coordinates. There is also a finite term which will be discussed elsewhere. Had we chosen a different vector, the meaning of the quartic term might not have been so transparent.

The point-separation method has recently been used by Davies and Fulling[14] for calculating, via mode sums, the VEV of the stress tensor for a massless conformal scalar field in a spatially flat Robertson-Walker background. They report agreement with all divergent terms. Agreement with the finite direction-dependent terms is not complete because of problems in the massless case to be discussed below.

Now let us investigate some of the general properties of the results for $\langle T^{\mu\nu}\rangle_{\text{div}}$. If we take the trace of $\langle T^{\mu\nu}\rangle_{\text{div}}$ and set $\xi=\frac{1}{6}$ and $m=0$, we find that it is trace-free except for the finite term. There we get a trace

$$\langle T^{\mu}{}_{\mu}\rangle_{\text{finite}} = -\frac{g^{1/2}}{2880\,\pi^2}\,(R^{\rho\tau\kappa\iota}R_{\rho\tau\kappa\iota} - R^{\rho\tau}R_{\rho\tau} + R_{;\rho}{}^{\rho}). \tag{7.3}$$

Something is wrong. We started with a manifestly trace-free object (even when $x'\neq x$) and now we have an object whose trace is nonzero when $\xi=\frac{1}{6}$ and $m=0$. Looking back we find that the trace above arises from a $1/m^2$ term in $G^{(1)}$. Clearly, when $m=0$, this term diverges. Looking at the integral for $H_1^{(2)}$, we see that when $m=0$, the integral is undefined. The asymptotic expansion in powers of $1/m^2$ is not valid. Also, we see that the trace in Eq. (7.3) comes from a direction-dependent term in $\langle T^{\mu\nu}\rangle_{\text{finite}}$, namely,

$$(R^{\rho\tau\kappa\iota}R_{\rho\tau\kappa\iota} - R^{\rho\tau}R_{\rho\tau} + R_{;\rho}{}^{\rho})\frac{\sigma^{\mu}\sigma^{\nu}}{(\sigma^{\rho}\sigma_{\rho})}\,.$$

This term is not correct in the massless case, so some of the direction-dependent terms in $\langle T^{\mu\nu}\rangle_{\text{div}}$ will not be found in $\langle \text{in},\text{vac}|T^{\mu\nu}|\text{in},\text{vac}\rangle$. We must take care to remember this when we do a calculation.

The massless limit also presents problems for the direction-independent terms in the finite part of $\langle T^{\mu\nu}\rangle_{\text{div}}$ and in the logarithmic term. The terms of order $1/m^2$ and higher are divergent in the massless limit and cannot be used. The logarithmic term also diverges when $m=0$, except in certain cases such as for the massless conformal field in a conformally flat background (as in the Fulling and Davies calculation) or when the background field satisfies the vacuum field equations, $R_{\mu\nu}=0$. In each of these cases, the logarithmic term is zero.

There is another problem which we must resolve. In Eq. (6.3), for example, we find terms of the form

$$R_{\alpha\beta}\frac{\sigma^{\alpha}\sigma^{\beta}}{(\sigma^{\rho}\sigma_{\rho})}\,,$$

which may be written as

$$(R_{\alpha\beta} - \tfrac{1}{2}Rg_{\alpha\beta})\frac{\sigma^{\alpha}\sigma^{\beta}}{(\sigma^{\rho}\sigma_{\rho})} + \tfrac{1}{2}R\,.$$

We see that some direction-dependent terms can be written as the sum of other direction-dependent terms and direction-independent terms. How do we determine which way to write such terms? We will fix these terms by demanding that the direction-independent terms satisfy the covariant conservation equations, $\langle T^{\mu\nu}\rangle_{;\nu}=0$. This restriction is reasonable since the direction-independent terms remain when the length of the tangent vector goes to zero. The divergent terms have been regularized away in some fashion and so we expect what remains to satisfy the conservation equations. If we look at the direction-independent terms in the results in Sec. VI, we see that they do satisfy the conservation equations already so no separation of the direction-dependent terms is called for.

Next we see that the linear term is completely direction-dependent. We can eliminate the linear term altogether if we average over a separation in the $\sigma^{\mu}$ direction and one in the $-\sigma^{\mu}$ direction. An alternative method is to originally separate one $\phi(x)$ to $\phi(x')$ and the other $\phi(x)$ to $\phi(x'')$ along the geodesic between $x$ and $x'$ an equal distance in the opposite direction. This "symmetric" point separation eliminates the linear term that appears in the "asymmetric" method described in Sec. II. The two methods will give different finite direction-dependent results. This difference will be discussed in Ref. 14.

Finally, we look once again at the logarithmic term. It is obvious that we could absorb Euler's constant term into the logarithm. What looked like a finite term is now part of the logarithmic divergence. Actually, it is possible to add any multiple of the coefficient of the logarithmic term into the logarithmic divergence as long as we subtract it from the finite term. This ambiguity is well known. It simply corresponds to the ambiguity that is always present when a logarithmically divergent quantity appears in a theory. Only in the cases where the logarithmic term vanishes (as in the cases discussed earlier) will this problem not present itself.


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## APPENDIX

We will prove the generalization of Synge's theorem stated in Sec. IV.

*Theorem.*

$$[T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m ; \mu'}] = -[T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m ; \mu}] + [T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m}]_{;\mu} ,$$

where

$$T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m}$$

is any bitensor whose unprimed derivative coincidence limits are known.

*Proof.* We are given

$$[\sigma_{;\mu}] = [\sigma_{;\mu'}] = 0 ,$$

$$[\sigma_{;\mu\nu}] = [\sigma_{;\mu'\nu'}] = [-\sigma_{;\mu\nu'}] = g_{\mu\nu} ,$$

and $\qquad$ (A1)

$$[g^{\alpha}{}_{\beta'}] = \delta^{\alpha}{}_{\beta}, \quad [g^{\alpha}{}_{\beta';\mu}] = [g^{\alpha}{}_{\beta';\mu'}] = 0 .$$

We homogenize $T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m}$,

$$\overline{T}_{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m} = g_{\beta_1}{}^{\rho'_1} \cdots g_{\beta_m}{}^{\rho'_m} T_{\alpha_1 \cdots \alpha_n \rho'_1 \cdots \rho'_m} , \tag{A2}$$

and expand

$$\overline{T}_{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m} = t_{\alpha_1 \cdots \beta_m} + t_{\alpha_1 \cdots \beta_m \rho} \sigma^{\rho} + \cdots , \tag{A3}$$

where the $t$'s are functions of $x$ only. Now differentiate Eq. (A3) with respect to $x^{\mu}$,

$$\overline{T}_{\alpha_1 \cdots \beta_m ; \mu} = t_{\alpha_1 \cdots \beta_m ; \mu} + t_{\alpha_1 \cdots \beta_m \rho ; \mu} \sigma^{\rho} + t_{\alpha_1 \cdots \beta_m \rho} \sigma^{\rho}{}_{\mu} + \cdots ,$$

and take the coincidence limits using properties (A1) to get

$$[\overline{T}_{\alpha_1 \cdots \beta_m ; \mu}] = t_{\alpha_1 \cdots \beta_m ; \mu} + t_{\alpha_1 \cdots \beta_m \mu} .$$

Doing the same for $x^{\mu'}$, we find

$$[\overline{T}_{\alpha_1 \cdots \beta_m ; \mu'}] = -t_{\alpha_1 \cdots \beta_m \mu} .$$

Thus

$$[\overline{T}_{\alpha_1 \cdots \beta_m ; \mu'}] + [\overline{T}_{\alpha_1 \cdots \beta_m ; \mu}] = [\overline{T}_{\alpha_1 \cdots \beta_m}]_{;\mu} , \tag{A4}$$

since $t_{\alpha_1 \cdots \beta_m} = [\overline{T}_{\alpha_1 \cdots \beta_m}]$. Substituting definition (A2) into (A4) and using the $g^{\alpha}{}_{\beta'}$ properties in (A1), we have

$$[T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m ; \mu'}] = -[T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m ; \mu}] + [T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m}]_{;\mu} ,$$

which is the result we wanted.

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[1]B. S. DeWitt, Phys. Rep. 19C, 295 (1975).

[2]We use the sign conventions of C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[3]A tensor density naturally arises when the action (2.1) is varied with respect to the metric. We shall use the density throughout this paper but will omit the word density in the text.

[4]See B. S. DeWitt (Ref. 1) for a detailed discussion of vacuum states in curved spaces. He shows that existence of the in and out regions is not essential to the final results.

[5]We could write $G^{(1);\nu'\mu}$ since indices at different points commute freely.

[6]This symmetrization is done so that the expression for $\langle \underline{T}^{\mu\nu} \rangle_{\rm div}$ is symmetric when the points are separated.

[7]B. S. DeWitt, *The Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965), pp. 147–159.

[8]J. Schwinger, Phys. Rev. 82, 664 (1951).

[9]J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960). Synge calls $\sigma(x,x')$ the world function and writes it as $\Omega(x,x')$.

[10]G. N. Watson, *Theory of Bessel Functions* (Cambridge Univ. Press, New York, 1944). See pp. 178 and 179.

[11]Bars over a tensor denote a quantity obtained from an object at $x'$ by parallel transport.

[12]S. M. Christensen, Ph.D. dissertation, University of Texas at Austin, 1975 (unpublished). This contains all necessary coincidence limits and expansions as well as shortcut calculational techniques.

[13]See Ref. 1 for details of how these mode functions are chosen.

[14]This work will be reported in a paper by P. C. W. Davies, S. A. Fulling, S. M. Christensen, and T. S. Bunch, in preparation.