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Generation of harmonics in gravitational waves

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Exact plane-wave solutions of the Einstein equations which correspond to monochromatic superpositions of plane waves in the linearized theory are studied. The metric involves solutions of the Mathieu equation, and it is shown that it can be interpreted in terms of harmonics of the linearized solution.

INTRODUCTION

An essential feature of general relativity is the nonlinearity of the Einstein equations. While the theory certainly predicts the existence of gravitational waves, these waves cannot be treated in the same way as can those of a linear theory such as Maxwell's electrodynamics. The principle of superposition, and with it the use of such techniques as Fourier analysis, can no longer be used. One can, of course, restrict attention to the linear approximation, but this loses many features of the theory.

It is of interest to examine the possibility of combining exact wavelike solutions of the Einstein equations. Solutions representing, in some sense, combinations of gravitational waves provide classical analogs for processes of graviton-graviton scattering, including possible absorption and production of gravitons. I have suggested previously that such processes may have been important in the very early universe.[1] Exact solutions representing colliding gravitational waves have been given by Szekeres.[2]

The present paper deals with combinations of gravitational plane waves moving in the same direction, with particular attention being given to the frequencies and amplitudes of the waves, rather than to geometric considerations. While the equations involved are nonlinear, the nonlinearity is of a quite simple type. Judicious choices of the wave form reduce the problem to one involving standard linear differential equations, and allow easy contact with the results of the linearized theory.

PLANE GRAVITATIONAL WAVES

A metric for plane waves traveling in the positive z direction can be written

$$ds^2 = L^2[\exp(2\beta)dx^2 + \exp(-2\beta)dy^2] + dz^2 - dt^2 \tag{1}$$

in the notation of Misner, Thorne, and Wheeler,[3] to be followed here. L and β are functions of $u=t-z$, restricted by the only nontrivial Einstein equation

$$L'' + (\beta')^2 L = 0\,, \tag{2}$$

a prime indicating differentiation with respect to u. Such metrics, and especially the geometric properties of plane-wave space-times, have been studied extensively.[4] Here I want to discuss the consequences of particular choices for the function $\beta(u)$, in which the nonlinearity of Eq. (2) primarily resides. If (β_1, L_1) and (β_2, L_2) are pairs of functions satisfying Eq. (2), $(\beta_1+\beta_2, L_1+L_2)$ will, in general, *not* be a solution. But once β is chosen, we need only solve a linear equation for L.

For $\beta' \ll 1$, Eq. (2) becomes just $L''=0$, and we take $L=1$ so that (1) will become the Minkowski metric when $\beta=0$ and space-time if flat. The metric is then

$$ds^2 \approx (1+2\beta)dx^2 + (1-2\beta)dy^2 + dz^2 - dt^2\,. \tag{3}$$

The deviation from flat space-time is thus described by a transverse, traceless tensor $h_{\mu\nu}$ with $h_{11}=2\beta$, $h_{22}=-2\beta$, and all other components zero. The fact that β is a function only of u (and not of $v=t+z$) means that it automatically satisfies the wave equation $\partial^2\beta/\partial u \partial v = 0$ in flat space-time.

Any function β of u which is not too pathological will describe linearized gravitational plane waves. Functions of the form $\cos\omega u$ and $\sin\omega u$ provide a complete set of solutions, and much of the treatment of waves in linear theories has been in terms of these functions. It is natural to proceed in the same way, as far as possible, in dealing with the exact equation (2).

MONOCHROMATIC WAVES

Solutions of (2) with β having the form $\beta = F\cos\omega u$, where F and ω are constants, will be called *monochromatic* waves. (This terminology differs from that of Avez.[5]) If we set $\phi \equiv \omega u$ and $\psi(\phi) \equiv L(u)$, our basic equation (2) becomes

$$d^2\psi/d\phi^2 + F^2(1-\cos^2\phi)\psi = 0\,, \tag{4}$$

which is the special case of the Mathieu equation,

$d^2\psi/d\phi^2 + (b - h^2\cos^2\phi)\psi = 0$, when $b = h^2 = F^2$.[6] We may study the form of solutions of (4) for different values of F, corresponding to monochromatic superpositions of waves in the linearized theory. It would be possible to consider more general superpositions, with β a Fourier series in $\cos\omega u$ and $\sin\omega u$. This would lead to a linear differential equation for L of the general type of Hill's equation.[7] However, I restrict myself to monochromatic waves.

To solve (4), one looks for solutions having the form

$$\psi = e^{is\phi}\sum_{n=-\infty}^{\infty} a_n \exp(2in\phi)\,, \tag{5}$$

where s and the a_n's must be determined. In general, s will not be such as to make these solutions periodic in ϕ. The real and imaginary parts of (5) will then give the real solutions for the metric amplitude L. Such solutions will be stable—will not blow up exponentially for large values of $\pm\phi$—if s is real. Examination of the stability chart for the Mathieu equation[8] reveals that the solutions with $b = h^2$ are stable for sufficiently small values of b and h. However, as b and h increase we will eventually reach the value $s = 1$ and a periodic solution for ψ, an odd Mathieu function, will exist. Further increase of $b = h^2$ introduces an imaginary part of s, and consequent instability. (This shows, in passing, that flat space-time is stable with respect to small perturbations of the plane-wave type.)

Since a three-term recursion formula for s and the a_n's results when (5) is substituted into the differential equation (4), we cannot write down general solutions in a very neat form. The solutions are most surveyable if we now assume F to be small. In this case, a few terms in the general series (5) will give a fairly good picture of the solution. To lowest order, we find $s \approx 2^{-1/2}F$ and $a_1 = a_{-1} \approx -F^2 a_0/16$. The real solution of (4), with the omission of terms of order F^3 and higher, yields

$$L(u) \approx [A\cos(2^{-1/2}F\,\omega u) + B\sin(2^{-1/2}F\,\omega u)] \times [1 - (F^2/8)\cos 2\omega u]\,. \tag{6}$$

I shall set $B = 0$ for simplicity and $A = 1$, so that $L(u) \equiv 1$ when $F = 0$. One can easily work to higher accuracy in F. More harmonics of the basic frequency ω would have to be considered, and the formulas for s and the a_n's would change. However, the simple approximation (6) already presents some interesting features.

Since F is small, $L(u)$ is a long-period oscillation multiplied by a function which oscillates with small amplitude about unity with frequency 2ω. The long-period oscillation can be thought of as the average curvature of space-time produced by the effective energy-momentum tensor of the short-period oscillations.[3] The metric components g_{11}, g_{22} are

$$L^2\exp(\pm 2\beta) \approx [(1 + F^2) \pm 2F\cos\omega u + (3F^2/4)\cos 2\omega u] \times \cos^2(2^{-1/2}F\omega u)\,.$$

As expected from previous experience with nonlinear oscillations, the primary effect of nonlinearity is the production of harmonics (here with angular frequencies 0 and 2ω) of the fundamental frequency of the linearized theory. These harmonics will show themselves in the curvature tensor, which produces differential accelerations of test particles in the x-y plane, and in the formulas for energy transport.

The Landau-Lifshitz pseudotensor[9] gives, for the flux of power across a surface in the x-y plane,

$$(-g)t^{03}_{L\text{-}L} = L^4\beta'^2/4\pi \tag{7}$$

(in units with $c = G = 1$) for the general plane wave described by the metric (1). If we neglect the long-period variation of the metric, the power flux averaged over a linearized period $\Delta t = 2\pi/\omega$ is, to order F^4, consistent with our approximation,

$$\langle -gt^{03}_{L\text{-}L}\rangle = \frac{1}{4\pi}F^2\cos^4(2^{-1/2}F\,\omega u)[\omega^2/2 + F^2\omega^2/8]\,. \tag{8}$$

The first term in the square brackets is that for the oscillations with frequency ω, while the second term gives the power flux for the first overtone. In general, a combination of oscillations with frequencies ω_1 and ω_2 would give harmonics with frequencies $\omega_1 \pm \omega_2$. In our case, the $\omega_1 - \omega_2$ term gives merely a static field, which of course does not contribute to the energy flow.

Things will be much more complicated for more realistic wave solutions, such as those with curved wave fronts or those representing colliding waves. However, the ideas discussed here may be of some interest in dealing with such solutions, and in determining some features of the graviton-graviton interaction in the semiclassical regime.

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