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Ultimate sensitivity limit of a resonant gravitational wave antenna using a linear motion detector*

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The sensitivity of a resonant-mass gravitational radiation antenna coupled to a motion detector with given noise properties is calculated in detail. It is shown that the quantum-mechanical limit of *linear* amplifier performance implies an important restriction on the sensitivity for any system using a *linear* motion detector. For a signal frequency ω_a, this fundamental limit requires that the gravitational radiation pulse be capable of driving the antenna from rest to an energy level exceeding $2\hbar\omega_a$.

I. INTRODUCTION

In recent attempts to detect gravitational radiation, several workers have failed to observe statistically significant cross correlations between the outputs of separated pairs of antennas.[1-5] These and other measurements[6,7] have placed upper limits on the flux of radiation incident in the form of brief pulses to which the antennas would respond. These limits contradict the results of other experiments.[8,9] In view of such results, it is desirable to consider in detail how the sensitivity of the antennas can be improved, and make a comparison with the available predictions of the ambient radiation flux.

A resonant-mass gravitational radiation antenna takes the form of a damped harmonic oscillator exhibiting a vibrational mode with nonvanishing mass quadrupole moment.[10-12] Coupling to the radiation field results in a driving force proportional to the oscillating components of the Riemann tensor. The eigenfrequency of the mechanical mode, typically in the approximate range 10^2 to 10^3 Hz, is chosen to lie within the estimated spectrum of the radiation.[11] The interaction between a gravitational radiation flux and a massive body has been discussed in detail.[10-14] A resonant antenna is most useful for detecting radiated energy arriving in a time τ_p of order 1 msec, which is very much shorter than τ_a, the damping time of the mechanical oscillations. If an antenna of given mechanical properties is exposed to an incoming pulse of radiation described by an energy spectral density $F(\omega)$ per unit area, a quantity U_s may be defined by the equation

$$U_s = \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(\omega)F(\omega)d\omega \ , \tag{1}$$

where $\chi(\omega)$ is an anisotropic polarization-dependent function determined by the mechanical properties and size of the antenna. The quantity U_s is equal to the energy which would be deposited by the pulse in an antenna previously at rest. In order to detect the mechanical oscillations resulting from signals, the antenna is coupled to a motion detector whose sensitivity can most generally be represented by a noise power per unit bandwidth, W. If the antenna is used to detect the presence of a signal pulse during a time interval τ_s, the required bandwidth is approximately equal to $(2\tau_s)^{-1}$, and the noise power is $W(2\tau_s)^{-1}$. The signal power is equal to the energy U_s divided by the duration of the experiment τ_s, and the power available for detection is known from the maximum power theorem to be one-fourth of this, or $U_s(4\tau_s)^{-1}$. In order for a signal to be detected the available signal power must exceed the noise power, resulting in the condition $U_s > 2W$. Heffner has shown, starting from the uncertainty principle, that all *linear* amplifiers are unavoidably noisy.[15] Application of the same argument to a *linear* motion detector shows that its mechanical noise power per unit bandwidth has an absolute minimum value $W = \hbar\omega_a$, where ω_a is the signal frequency. Therefore the fundamental sensitivity limit for a resonant-mass antenna using such a motion detector is given by the bandwidth-independent condition

$$U_s > 2\hbar\omega_a . \tag{2}$$

This condition, which appears not to have been recognized in the literature, has important consequences for refrigerated antennas at present under design and construction.

In order to achieve the noise level defined by the relation $U_s > 2W$, the signal power $U_s(\tau_s)^{-1}$ must exceed the noise power $k_B T_a(\tau_a/2)^{-1}$ associated with thermal fluctuations in the mechanical dissipation mechanism of the oscillator, assumed at temperature T_a. This requirement is not met in the room temperature antennas used hitherto, and a combination of thermal noise and detector noise controls the sensitivity. Optimization of the signal-to-noise ratio under these conditions has been considered by Gibbons and Hawking[11] and others,[16-18] and the performance of these detectors is well understood. It is reported that the sensitivity of room temperature detectors can eventually be imporved by between one and two orders of magnitude.[19] The corresponding noise level will still exceed the limit defined by Eq. (2) by about six orders of magnitude.

In order to improve the sensitivity further, the antenna may be cooled,[20] and materials with low mechanical losses employed,[21] thus reducing the thermal noise. At or below the temperature of liquid helium it is possible to use superconducting transducers and amplifiers to minimize motion detector noise.[18,20] In calculating the limit of the sensitivity which can be attained with any given motion detector, it was first recognized by Braginskii that the effect of detector noise on the motion of the antenna must be included.[21] Neglecting this effect leads to the physically unreasonable conclusion that sensitivity can be improved without limit by increasing the mechanical Q of the antenna and simultaneously reducing the measurement bandwidth, contrary to Eq. (2).

In this paper the sensitivity of an antenna coupled to an ideal linear motion detector with given noise properties is calculated, and Eq. (2) is derived in detail. It is shown that the classical treatment employed is adequate in practice, and the ultimate sensitivity of the measuring process to the amplitude and phase of the antenna oscillation is shown to be close to the quantum-mechanical limit expressed by the uncertainty principle. Finally, the possibility of achieving the ultimate limit in experiments is discussed.

II. MECHANICAL MODEL OF ANTENNA AND MOTION DETECTOR

A selected mode of a mechanical oscillator driving a motion detector can be represented[12] by the model shown in Fig. 1. A convenient normalization identifies the displacement $z(t)$ in the model with $\xi(\vec{r}_d,t)$ the instantaneous change in length of the motion detector D due to the chosen mode of oscillation of the antenna. The value of M, the model mass, is normalized so that the kinetic energy of the model corresponds to the integrated kinetic energy of the antenna in the appropriate mode, that is,

$$\tfrac{1}{2}M\dot{\xi}^2(\vec{r}_d,t)=\tfrac{1}{2}\int\rho(\vec{r})\dot{\xi}^2(\vec{r},t)d^3r\ , \tag{3}$$

where $\xi(\vec{r},t)$ is the displacement at position $\vec{r}$ in the antenna and $\rho(\vec{r})$ is the density. For example, if a detector senses the displacement of one end face of a cylinder relative to the center of mass, the appropriate value of M is approximately one-half of the cylinder mass. When the model mass is correctly normalized, the values of compliance K^{-1} and damping μ are fixed by the requirement that the resonant frequency and the amplitude decay time of the model should be identical with the values observed with the antenna coupled to the motion detector. The mechanical input impedance of the motion detector is assumed to be in the form of a simple reactance and incorporated in the model parameters. The interaction of the antenna with the radiation field is represented in the model by the suitably normalized force $p(t)$.

In order to calculate the sensitivity of the antenna, it is necessary to model all sources of noise. Thermal noise associated with the total mechanical dissipation in the antenna and motion detector is represented by the random force $h(t)$ in Fig. 1. Remaining noise due to the motion detector can be modeled completely by two effective noise sources as shown. The series component of the detector noise is equivalent to a random error in the detected velocity, and is modeled by the fluctuating velocity error $\epsilon(t)$. The shunt component is a noise force applied to the antenna by the motion detector, and is modeled by the random force $f(t)$. These sources are the mechanical duals of the voltage and current generators which can always be used to represent the total noise of an electrical two-port. In an ideal detector, the effective sources are stationary and uncorrelated. The equation of motion of the model may now be written

$$M\ddot{z}(t)+\mu\dot{z}(t)+Kz(t)+h(t)+f(t)+p(t)=0. \tag{4}$$

Equation (4) represents a harmonic oscillator of resonant frequency $\omega_a=(K/M)^{1/2}$ with an amplitude decay time τ_a given by

$$\tau_a=2M/\mu\ . \tag{5}$$

In the absence of the noise represented by $h(t)$ and $f(t)$, the response of the oscillator to a harmonic

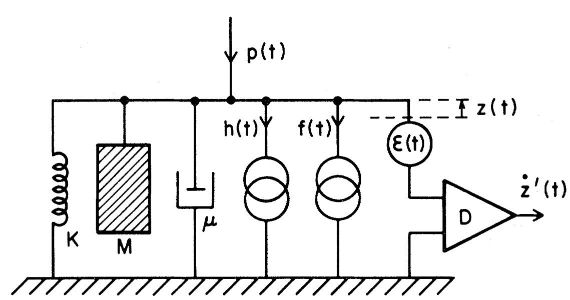

FIG. 1. Mechanical model of a single mode of a resonant-mass gravitational radiation antenna. The driving force $p(t)$ results in a displacement $z(t)$ which is sensed by the motion detector D. The noise sources $h(t)$, $f(t)$, and $\epsilon(t)$ are discussed in the text.

force of the form $p(t)=-p_0\exp(-i\omega t)$ may be written in terms of the resulting velocity, $\dot{z}(t)=v_0\exp(-i\omega t)$, where

$$v_0=(-i\omega p_0/M)(\omega_a^2-\omega^2-2i\omega/\tau_a)^{-1}\,. \tag{6}$$

In practice the resonance is very narrow and for frequencies close to ω_a one may write Eq. (6) in the approximate form

$$v_0=(p_0\tau_a/2M)[1-i(\omega-\omega_a)\tau_a]^{-1}\,. \tag{7}$$

When the Q of the oscillator is very high it is convenient to express the instantaneous velocity, which is a narrowband process with zero mean, in terms of a pair of quadrature components $x(t)$ and $y(t)$, where

$$\dot{z}(t)=x(t)\cos\omega_a t-y(t)\sin\omega_a t\,. \tag{8}$$

The amplitude and phase of the oscillation are now represented by a slowly changing vector $\vec{V}(t)$ which has components $x(t)$ and $y(t)$ in the x-y plane. The total kinetic and elastic energy $E(t)$ and phase $\phi(t)$ of the oscillation are given by

$$E(t)=\tfrac{1}{2}M[x^2(t)+y^2(t)] \tag{9a}$$

and

$$\phi(t)=\tan^{-1}[x(t)/y(t)]\,. \tag{9b}$$

The response of the antenna to a brief pulse has been considered in Eq. (1). Since the function $\chi(\omega)$ is only nonzero close to the mechanical eigenfrequencies of the antenna, and $F(\omega)$ must be comparatively wide for a brief pulse, the equation can be simplified by considering only a single mode at ω_a, so that

$$U_s=\sigma_a F(\omega_a)\,. \tag{10}$$

The anisotropic parameter σ_a, which may be referred to as the cross section for the ath mode for the wave type considered, depends on the mass and dimensions of the antenna and the velocity of longitudinal sound in it. When the energy of the oscillation is not zero at the time of arrival of the pulse, the quantity U_s given by Eq. (10) no longer corresponds to the energy deposited, but is related to resulting changes in the amplitude and phase of the existing oscillation. The corresponding changes in the model variables can conveniently be expressed in terms of the vector $\vec{V}(t)$ defined above. It can be shown[12] that

$$(\Delta V_s)^2=2U_s/M\,, \tag{11}$$

where ΔV_s is the magnitude of the vector change in the model variable $\vec{V}(t)$ corresponding to the change in the state of the antenna caused by the arrival of the pulse described by $F(\omega)$. The quantity U_s can conveniently be used to describe the effect of the pulse on a given antenna and will be referred to as the energy equivalent of the pulse. The orientation of the vector change in the x-y plane cannot be specified because the energy spectrum $F(\omega)$ contains no phase information, and the signals resulting from incoherent signal pulses will be randomly directed.

In practice, the sudden change in $\vec{V}(t)$ caused by a signal pulse will be accompanied by random noise from the antenna and motion detector. The spectrum of random antenna motion can be calculated in terms of the spectral densities $S_h(\omega)$ and $S_f(\omega)$ corresponding to the noise sources $h(t)$ and $f(t)$. If the resulting spectral density of antenna velocity noise is $S_a(\omega)$, Eqs. (5) and (7) show that

$$S_a(\omega)=\left(\frac{\tau_a}{2M}\right)^2\frac{[S_h(\omega)+S_f(\omega)]}{[1+(\omega\pm\omega_a)^2\tau_a^2]}\,, \tag{12}$$

since $h(t)$ and $f(t)$ are uncorrelated.

The output of the motion detector will in general be an amplified electrical signal. In the case to be considered below, the output signal $\dot{z}'(t)$ of the linear detector is written in a form proportional to the instantaneous velocity at the detector input, which is the sum of $\dot{z}(t)$ and the series noise error $\epsilon(t)$. It is convenient to normalize the output signal to the power level at the output, so that the motion detector power gain need not be considered. Thus

$$\dot{z}'(t)=\dot{z}(t)+\epsilon(t)\,. \tag{13}$$

In order to minimize the effect of the series noise, the detector output may be processed by a band-limiting filter with a response chosen to optimize the overall signal-to-noise ratio. If it is assumed for convenience that the filter has a Lorentzian power response centered at ω_a and characterized by a time constant $\tau_f\ll\tau_a$, the spectrum $S_n(\omega)$ of noise of the detector output will be given close to ω_a by

$$S_n(\omega)\approx\left(\frac{\tau_a}{2M}\right)^2\frac{[S_h(\omega)+S_f(\omega)]}{[1+(\omega\pm\omega_a)^2\tau_a^2]}+\frac{S_\epsilon(\omega)}{[1+(\omega\pm\omega_a)^2\tau_f^2]}\,, \tag{14}$$

where $S_\epsilon(\omega)$ is the spectral density of the velocity error $\epsilon(t)$. Equation (14) shows that the detector shunt noise is indistinguishable from thermal noise in the antenna, but that the series noise component appears independently. The conditions for minimizing the overall noise level will be considered in Sec. III.

III. SIGNAL DETECTION AND SENSITIVITY

A convenient signal-processing technique for detecting sudden excitations of a resonant antenna in the presence of noise is well known.[6] The prefiltered motion detector output signal is fed to two phase-sensitive detectors at frequency ω_a to recover the quadrature components of the antenna velocity. These are then sampled simultaneously at times separated by a chosen measuring interval τ_s. In order to minimize the effects of noise while preserving the independence of successive samples, the prefilter time constant τ_f should be approximately equal to τ_s. A difference vector $\Delta\vec{V}'$ is calculated from successive samples of the envelope vector $\vec{V}'(t)$ whose components $x'(t)$ and $y'(t)$ are the quadrature components of the antenna motion contaminated by detector series noise. The magnitude $\Delta V_j'$ of the jth vector difference is

$$\Delta V_j' = [(x_j' - x_{j-1}')^2 + (y_j' - y_{j-1}')^2]^{1/2}, \tag{15}$$

where x_j' denotes the jth sample of $x'(t)$, etc.

A brief signal pulse, causing a certain vector change in the state of the antenna oscillation with the magnitude given by Eq. (11), will be detected as as an equal change in the vector $\vec{V}'(t)$. If the duration of the signal τ_p is shorter than τ_s, the magnitude of $\Delta V_s'$ of the vector difference which spans the interval is thus given by $\Delta V_s' = (2U_s/M)^{1/2}$, where U_s is the energy equivalent given by Eq. (10). More sophisticated algorithms, employing optimum filtering techniques, can be used to give an output which is independent of the arrival time of the signal.[6,22,23]

The statistical properties of the noise which accompanies the signal can easily be calculated from the noise spectrum given in Eq. (14). It is shown in the first part of the Appendix that, in the absence of signals, the mean square detected vector difference Δ^2, where $\Delta^2 = \langle(\Delta V_j')^2\rangle$, is given by

$$\Delta^2 = \tau_s M^{-2} S_h(\omega_a) + \tau_s M^{-2} S_f(\omega_a) + 4\tau_s^{-1} S_\epsilon(\omega_a), \tag{16}$$

where the condition $\tau_s = \tau_f$ has been assumed. The first term on the right of Eq. (16) represents the effect of thermal noise in the antenna and transducer. The spectral density of the thermal noise is given by the Nyquist relation in terms of T_a, the temperature of the mechanical dissipation mechanisms. Typical antennas will satisfy the condition $k_B T_a \gg \hbar\omega_a$, and the high-temperature limit, $S_h(\omega_a) = 2K_B T_a \mu$, will be valid. If the antenna and detector are in internal and mutual thermal equilibrium, T_a will be the antenna temperature. The second and third terms of Eq. (16) represent the consequences of motion detector noise. The relationship between the two terms can be clarified by expressing the detector noise in terms of two new parameters W and Z given by

$$W = [S_f(\omega_a) S_\epsilon(\omega_a)]^{1/2}, \tag{17a}$$

$$Z = [S_f(\omega_a)/S_\epsilon(\omega_a)]^{1/2}. \tag{17b}$$

The total noise power per unit bandwidth of the detector is thus equal to W, and Z is a characteristic impedance which can in principle be selected at will. The overall noise level is conveniently expressed by the energy equivalent U_n of the signal pulse which would cause a vector difference equal in magnitude to the root mean square noise increment $(\Delta^2)^{1/2}$. Using Eq. (11) and the redefined noise parameters, Eq. (16) gives

$$U_n = 2k_B T_a(\tau_s/\tau_a) + W(Z\tau_s/2M + 2M/Z\tau_s). \tag{18}$$

In principle, brief signal pulses can be detected with a signal-to-noise ratio of unity if their energy equivalent is equal to U_n. For a chosen value of τ_s the detector noise, represented by the second and third terms on the right of Eq. (18), is minimized when the noise properties of the detector satisfy the unique impedance-matching condition

$$Z = 2M/\tau_s. \tag{19}$$

The minimum detectable signal pulse is then characterized by an optimum energy equivalent U_{n0}, where

$$U_{n0} = 2k_B T_a(\tau_s/\tau_a) + 2W. \tag{20}$$

IV. MINIMUM ATTAINABLE NOISE LEVEL

Equation (20) gives U_{n0}, the energy equivalent of the signal pulse which generates a vector difference output equal to the root mean square noise. The sensitivity of an antenna is thus optimized by reducing as far as possible the terms on the right-hand side. The first term represents the noise due to thermodynamic fluctuations in the dissipative mechanisms of the antenna and motion detector. In principle this noise can be reduced indefinitely by using better materials to increase τ_a and by reducing the temperature; therefore the sensitivity must eventually be limited by the detector noise represented by the second term.

Heffner[15] has used an argument based on the uncertainty principle to show that no linear amplifier can have a noise temperature lower than the value

$$T_{n0}|_{\min} = \hbar\omega/k_B \ln 2. \tag{21}$$

A *linear* amplifier is defined to be one in which the amplitude and phase of the output signal are linearly related to the input amplitude and phase. The arguments which lead to Eq. (21) are equally applicable to a device whose input is a mechanical signal, and they are thus appropriate to the case of any motion detector which is linear. All motion detectors used at present or proposed for gravity wave detectors appear to fall within the definition of linear devices, with the possible exception of the "quantum nonperturbing" detector discussed by Braginskii *et al.*[24,25]

It is shown in the second part of the Appendix that a motion detector operating at a frequency ω, with a noise level characterized by the parameter W defined in Eq. (17a), has an optimum noise temperature T_{n0} satisfying the equation

$$W=\hbar\omega/[\exp(\hbar\omega/k_BT_{n0})-1].$$

Since this value of T_{n0} must be greater than the minimum allowed by Eq. (21), the condition

$$W\geq\hbar\omega \tag{22}$$

is obtained. A motion detector with a smaller value of W would potentially be capable of making a measurement to an accuracy which is not allowed by the uncertainty principle, and therefore such a device cannot be realized.

Since the ultimate lower limit to W is of quantum-mechanical origin, it is necessary to check the validity of the classical calculation used to obtain Eq. (20). Equations (9a) and (11) show that E_{n0}, the mean change in resonator energy corresponding to a signal equal to the minimum noise level U_{n0}, is given by

$$E_{n0}=2(\overline{E}U_{n0})^{1/2}, \tag{23}$$

where $\overline{E}$ is the average value of the resonator energy $E(t)$. If $\overline{E}$ takes the thermal equilibrium value K_BT_a, and U_{n0} takes the minimum possible value $2\hbar\omega_a$, the value of E_{n0} is given by

$$E_{n0}/\hbar\omega_a=(8k_BT_a/\hbar\omega_a)^{1/2}. \tag{24}$$

Using optimistic values of the parameters, $T_a=10^{-3}$ K, $\omega_a=2\pi\times10^3$ sec^{-1}, the value of $(8k_BT_a/\hbar\omega_a)^{1/2}$ is found to exceed 10^2. Thus the energy changes corresponding to detectable signals are considerably greater than $\hbar\omega_a$, and the classical derivation of Eq. (20) is justified for practical antennas.

It is interesting to determine how closely the performance of the signal-processing technique described in Sec. III approaches the limit of the uncertainty principle. The quantity Δ^2, whose value is given in Eq. (16), represents the mean square uncertainty involved in determining a change in the state of the antenna oscillation by means of two successive measurements. Because the autocorrelation functions of the quadrature components $x'(t)$ and $y'(t)$ are identical, the probability distribution associated with a single measurement of the state of the antenna in x-y space may be expected to be isotropic with a mean square radius of $\frac{1}{2}\Delta^2$.

The state of the oscillation can be given in terms of the phase ϕ and quantum number n of the oscillator using Eqs. (9a) and (9b), in which case the length of the envelope vector $\vec{V}(t)$ in x-y space is given by $|\vec{V}(t)|=[2\hbar\omega_a(n+\frac{1}{2})/M]^{1/2}$. If the quantities n and ϕ are measured by the technique described in Sec. III, and the resulting root mean square uncertainties are Δn and $\Delta\phi$, respectively, the condition that the overall uncertainty be isotropic in x-y space with variance $\frac{1}{2}\Delta^2$ requires that

$$\Delta n=\Delta\phi(M/\hbar\omega_a)\langle V^2\rangle, \tag{25}$$

and

$$(\Delta n)^2=\Delta^2(M/\hbar\omega_a)^2\langle V^2\rangle, \tag{26}$$

where $\langle V^2\rangle$ is the mean square of $|\vec{V}(t)|$. Simple manipulation of these equations shows that

$$\Delta n\Delta\phi=\Delta^2(M/2\hbar\omega_a). \tag{27}$$

For the minimum value of W allowed by Eq. (22), Δ^2 cannot be smaller than $4\hbar\omega_a/M$, giving $\Delta n\Delta\phi\geq2$. The uncertainty principle applied to a harmonic oscillator defines the minimum value $\Delta n\Delta\phi=\frac{1}{2}$, and thus the signal processing technique discussed in Sec. III appears to be slightly less than optimal. Moreover, it follows that the sensitivity of an antenna with a linear detector cannot be substantially improved beyond the limit $U_{n0}=2\hbar\omega_a$ by using more sophisticated signal-processing techniques. More sensitivity may be achieved if a nonlinear motion detection system with greater sensitivity to energy changes at the expense of phase resolution, or vice versa, can be devised. This problem has been considered by Braginskii *et al.*[24,25]

V. EXPERIMENTAL SENSITIVITY

The energy equivalent of a pulse which is detectable at unity signal-to-noise ratio is given by Eq. (18). However, in an experiment it is necessary to consider the expected frequency of the signals and the statistics of the noise. The noise output of the detection algorithm described in Sec. III is a normal distribution of vector differences, with the mean square value U_n, referred to energy, given by Eq. (18). The probability $P(U)$ of the square of a single vector difference exceeding a threshold corresponding to an input pulse of energy equivalent U is therefore

$$P(U)=\exp(-U/U_n). \tag{28}$$

Thus the threshold U_n' which is exceeded by the noise output at a repetition rate R is given by $P(U_n')\tau_s^{-1}=R$, so that

$$U_n'=U_n\ln(R\tau_s)^{-1}. \tag{29}$$

Since the detection efficiency for signals will only approach unity when the energy equivalent is equal to or greater than U_n', the value necessary for detection of signals occurring at rate R is greater than U_n by a factor of approximately $\ln(\eta)$, where $\eta=(R\tau_s)^{-1}$. The minimum signal energy equivalent U'_{s0} detectable at this accidentals rate with optimum noise matching is thus given by

$$U'_{s0}=\ln(\eta)[2k_BT_a(\tau_s/\tau_a)+2W]. \tag{30}$$

Equation (30) gives the best sensitivity which can be obtained with given values of the parameters T_a, τ_a, τ_s, and W. Using Eq. (10) this may be rewritten in terms of the spectral density $F(\omega_a)$ required for detection of signal pulses at an accidentals rate of R. The condition for detection is

$$F(\omega_a)\geq\sigma_a^{-1}\ln(\eta)[2k_BT_a(\tau_s/\tau_a)+2W]. \tag{31}$$

In practice the sensitivity of a detector will depend on how far the quantities on the right-hand side of Eq. (31) can be reduced. It is always useful to increase the cross section σ_a as far as possible. Since the factor $\ln(\eta)$ may be expected to vary only slowly with typical values of R and τ_s, the most significant remaining factor is the quantity U_{n0}, with the dimensions of energy, between brackets. The first term represents the noise due to thermodynamic fluctuations in the dissipative mechanism of the antenna. This is traditionally considered to be under the control of the experimenter since it can be reduced by the technical expedient of using better materials to lengthen the decay time and by reducing the temperature. It is also advantageous to use the shortest possible sampling time permitted by the inequality $\tau_s\geq\tau_p$, and by any constraints that may be imposed by the matching condition. It is clear that the sensitivity of the antenna cannot be increased indefinitely by increasing τ_a or reducing T_a since the detector noise will eventually become dominant. It has been shown above that, in order to obtain the minimum detector noise $2W$, the detector noise impedance Z must satisfy the condition $Z=2M/\tau_s$, and that any departure from ideal matching will result in poorer sensitivity. In practice, matching is achieved by correct design of the motion detector and by choosing the correct point of attachment to the antenna, which varies the model mass M. The use of a value of τ_s longer than τ_p in order to achieve matching is undesirable since it will significantly increase the contribution of resonator noise unless the condition

$$k_BT_a(\tau_s/\tau_a)\ll W \tag{32}$$

can still be satisfied. This equation therefore gives the condition for thermal noise not to degrade sensitivity for given values of the parameters τ_s and W. If the matching condition cannot be satisfied at a value of τ_s which gives negligible thermal noise, the value giving optimum sensitivity must be found by differentiating Eq. (18).

The ultimate sensitivity of a practical antenna using a linear detector is obtained when W takes the lowest theoretically possible value $\hbar\omega_a$. If thermal noise can be made unimportant, and the detector is perfectly matched to the antenna, the condition for detection is

$$F(\omega_a)\geq 2(\hbar\omega_a/\sigma_a)\ln(\eta). \tag{33}$$

It is important to notice that, as a result of the use of a linear detection system, neither the amplitude of the detected signal nor the detector noise level depends on $E(t)$, the level of oscillation. The ultimate sensitivity given by Eq. (33) would consequently be unaffected by coherent excitation of the antenna. Braginskii and Vorontsov[25] have calculated a much smaller limiting noise level than Eq. (33) assuming that the last two terms of Eq. (16) may be independently minimized. The discussion in Sec. IV above shows that the necessary motion detector must sacrifice sensitivity to the phase of the oscillation in order to be more sensitive to energy. It is not yet clear that this can be achieved with the proposed motion detector, which appears capable of linear amplification.

VI. DISCUSSION AND CONCLUSIONS

The use of a linearly responding motion detector is shown by Eq. (20) to imply a bandwidth-independent minimum noise level in the absence of thermal fluctuations. When the fundamental limit of linear detector sensitivity is considered, it is found that the ultimate lower limit of noise in an antenna system at frequency ω_a is given by $U_{n0}>2\hbar\omega_a$, where U_{n0} is the energy equivalent of the noise. The existence of this absolute noise limit is not apparent from earlier sensitivity calculations[11,16,17] which made use of various approximations, valid at room temperature, which fail in the general case of a refrigerated antenna.

In order to approach the ultimate noise limit with practical antennas, several important problems have to be solved. A motion detector with sensitivity close to the quantum-limiting value, with a characteristic impedance matched to the antenna, must be devised, and all other forms of noise must be reduced to insignificant levels. The most formidable problem appears to be the construction of the detector. The most sensitive amplifiers available at present are traveling wave masers. These have a noise level which is already within an order of magnitude of the quantum limit at their operating frequencies.[26] In order to use such an amplifier, the signal at the antenna frequency ω_a must be up-converted to the microwave input frequency ω_e either by means of a motion transducer followed by a parametric converter, or by means of a parametric transducer. Ideally, such a device need add no noise, and the relationship between the amplifier noise level and the quantum limit at ω_e can be reproduced at the input frequency ω_a.[15] A second essential function of the transducer or converter is to provide the correct noise match between the amplifier and the antenna. Transducers for this application pose unique problems, and existing designs rely on the properties of superconductors to achieve low-noise and parametric action.[27-29]

The remaining problem of thermal noise is perhaps less severe. In the most demanding case where the detector is quantum limited, the inequality $(k_B T_a/\tau_a) < (\hbar\omega_a/\tau_s)$ must be satisfied to achieve full sensitivity. Reasonably optimistic parameter values of $Q = 3.10^6$ and $\tau_s = 10^{-3}$ sec require a temperature $T_a < 0.07$ K, which is accessible by continuous refrigeration. The problem would become worse if the oscillator Q were lower, or if matching could only be achieved for longer values of τ_s. It is important to notice that once the condition of Eq. (32) has been met, no further increase of the ratio (Q/T_a) is useful.

With a given noise level the spectral intensity of a detectable pulse is given by Eq. (31). It is clear that the greatest sensitivity results from using the maximum possible value of the cross section σ_a. The largest antenna reportedly under construction[18,20] will have a cross section of 8×10^{-25} m^2, assuming a favorably polarized source in a suitable direction. The ultimate sensitivity obtainable with this detector is shown by Eq. (33) to correspond to an energy spectral density of about 3.10^{-5} J m^{-2} Hz^{-1} at an accidentals rate of 10 per year. This intensity corresponds to the isotropic conversion into a bandwidth of 10^3 Hz of 2.10^{-7} solar masses at the distance to the center of the galaxy.

It is clear that existing antennas are far from any fundamental restrictions on performance. However, the existence of a fundamental noise limit is very significant since proposals for more sensitive experiments are now being made.[25,30]

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APPENDIX

1. Noise level for vector difference detection

In Sec. II it is shown that the motion detector output $\dot{z}'(t)$ of the antenna system modeled in Fig. 1 has a narrowband noise spectrum $S_n(\omega)$ which is given at frequencies close to the antenna resonance ω_a by

$$S_n(\omega) = \frac{(\tau_a/2M)^2[S_h(\omega_a)+S_f(\omega_a)]}{[1+(\omega\pm\omega_a)^2\tau_a{}^2]} + \frac{S_\epsilon(\omega_a)}{[1+(\omega\pm\omega_a)^2\tau_f{}^2]}\,. \tag{A1}$$

The quadrature components $x'(t)$ and $y'(t)$ of the detector output are defined by

$$\dot{z}'(t) = x'(t)\cos\omega_a t - y'(t)\sin\omega_a t. \tag{A2}$$

Since the spectrum $S_n(\omega)$ is symmetric about ω_a, the quadrature components are uncorrelated and have identical spectra equal to $S_q(\omega)$, such that[31]

$$S_n(\omega) = \tfrac{1}{2}S_q(\omega-\omega_a) + \tfrac{1}{2}S_q(\omega+\omega_a). \tag{A3}$$

The spectrum $S_q(\omega)$ is thus given by

$$S_q(\omega) = \frac{2(\tau_a/2M)^2[S_h(\omega)+S_f(\omega)]}{1+\omega^2\tau_a{}^2} + \frac{2S_\epsilon(\omega)}{1+\omega^2\tau_f{}^2}\,. \tag{A4}$$

The autocorrelation function $R_q(\tau)$ for each quadrature noise signal is therefore equal to

$$R_q(\tau) = \tau_a(2M)^{-2}[S_h(\omega_a)+S_f(\omega_a)]e^{-\tau/\tau_a} + (\tau_f)^{-1}S_\epsilon(\omega_a)e^{-\tau/\tau_f}. \tag{A5}$$

In the vector difference detection procedure described in Sec. III, the quadrature components $x'(t)$ and $y'(t)$ are sampled at intervals of τ_s. The mean square detected vector difference Δ^2 is given by

$$\Delta^2 = \langle\{[x'(t)-x'(t-\tau_s)]^2 + [y'(t)-y'(t-\tau_s)]^2\}\rangle. \tag{A6}$$

In the absence of signals, $x'(t)$ and $y'(t)$ have identical correlation functions $R_q(\tau)$ given by Eq. (A5), and Δ^2 is given by

$$\Delta^2 = 4[R_q(0) - R_q(\tau_s)]. \tag{A7}$$

In practice, the conditions $\tau_a \gg \tau_s$ and $\tau_f \approx \tau_s$ will obtain, and Δ^2 is given by

$$\Delta^2 \approx (\tau_s/M^2)[S_h(\omega_a)+S_f(\omega_a)] + 4(\tau_s)^{-1}S_\epsilon(\omega_a), \tag{A8}$$

where $\exp(-\tau_s/\tau_f)$ has been neglected in comparison with unity.

2. The noise temperature of a linear motion detector

The effective noise temperature of a linear mechanical amplifier or motion detector can be calculated by reference to the model shown in Fig. 2. The detector noise is represented by the two generators $f(t)$ and $\epsilon(t)$ corresponding respectively to a random force with spectral density $S_f(\omega)$ and a random velocity error with spectral intensity $S_\epsilon(\omega)$. The detector is connected to measure the velocity difference across a mechanical resistance γ whose noise is represented by a shunt random force $g(t)$. The corresponding spectral density $S_g(\omega)$ is given by the Nyquist relation in terms of the temperature T_γ,

$$S_g(\omega)=2\gamma\hbar\omega[\exp(\hbar\omega/k_BT_\gamma)-1]^{-1}. \tag{A9}$$

The equation of motion for the velocity difference $v(t)$ is

$$\gamma v(t)+g(t)+f(t)=0. \tag{A10}$$

It has been assumed that the mechanical input susceptance of the detector is negligible or has been tuned out by its complex conjugate at the frequency of interest. The detector output $v'(t)$, referred to as the power level at the input, is given by

$$v'(t)=[g(t)+f(t)]\gamma^{-1}+\epsilon(t). \tag{A11}$$

If the noise sources are assumed uncorrelated in an ideal detector, the spectrum $S_v(\omega)$ of $v'(t)$ is

$$S_v(\omega)=2\hbar\omega\gamma^{-1}[\exp(\hbar\omega/k_BT_\gamma)-1]^{-1}+\gamma^{-2}S_f(\omega)+S_\epsilon(\omega). \tag{A12}$$

When T_γ is equal to the effective noise temperature T_n, the first term in Eq. (A12), corresponding to the source thermal noise, is equal to the sum of the remaining two terms which represent the detector noise; thus

$$2\hbar\omega[\exp(\hbar\omega/k_BT_n)-1]^{-1}=\gamma S_\epsilon(\omega)+\gamma^{-1}S_f(\omega). \tag{A13}$$

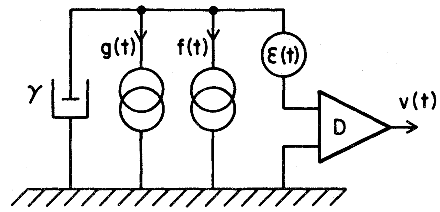

FIG. 2. Model of a motion detector D connected to a mechanical resistance γ. The noise sources $g(t)$, $f(t)$, and $\epsilon(t)$ are discussed in the text.

The detector noise is minimized when the source is matched to the detector so that $\gamma=[S_f(\omega)/S_\epsilon(\omega)]^{1/2}$, and the resulting optimum noise temperature T_{n0} is given by

$$\hbar\omega[\exp(\hbar\omega/k_BT_{n0})-1]^{-1}=W, \tag{A14}$$

where

$$W=[S_f(\omega)S_\epsilon(\omega)]^{1/2}.$$

Heffner has shown[15] that the minimum realizable noise temperature for a linear amplifier is

$$T_{n0}\big|_{\min}=\hbar\omega/(k_B\ln 2). \tag{A15}$$

The derivation of this relationship is equally applicable to a linear amplifier whose input signal is mechanical. Thus for any realizable, linear, motion detector, Eqs. (A14) and (A15) show that the value of W must satisfy the condition

$$W\geq\hbar\omega. \tag{A16}$$

Any detector in which the phase and amplitude of the output signal are not proportional to those at the input is not a linear amplifier, and thus Eq. (A16) need not apply.

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