Charged-black-hole electrostatics*

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We study electrostatics in the background gravitational and electric field of a charged massive body or black hole that is described by the Reissner-Nordström metric and its associated electric field. That is, we neglect the gravitational field associated with the perturbing charges and electric fields. Specifically, we consider the special case in which the charge e and mass M of the Reissner-Nordström metric are related by $e^2 = \kappa M^2$, with κ the gravitational constant, and obtain the multipole solutions of Maxwell's equations for small electrostatic perturbations. As one example we give the field of a point test charge in remarkably simple closed algebraic form and study its behavior as the charge approaches the black-hole surface: Loosely speaking, the test charge is rapidly "swallowed" by the black hole. Lastly, we show that our results are readily generalized to cover both electrically and magnetically charged test particles.

I. INTRODUCTION

We wish to study the perturbations in the electric field of a charged (e) and massive (M) body or black hole described by the Reissner-Nordström (RN) metric[1,2]; we will neglect the effect of the perturbing charges and electric fields on the gravitational field, i.e., the background metric. Thus we must solve Maxwell's equations in the RN background space. This is most easily done if we choose the charge-to-mass ratio of the body to be $e^2 = \kappa M^2$, where κ is the gravitational constant. The metric is then particularly simple. This charge-to-mass ratio is of independent theoretical interest as we will discuss further in Sec. II and Sec. III.[3] It is remarkably easy to obtain a complete set of multipole solutions as we shall do in Sec. II. In Sec. III we will apply the multipole solution to calculate the field of a point test charge, which may be expressed in simple closed algebraic form. Owing to this simple form, the behavior of the field of the test charge may be readily studied as the test charge approaches the black-hole radius.[4,5] Lastly, in Sec. IV we show that the field of a magnetically charged test particle is easily obtained without further calculation.[6-8]

II. MULTIPOLE SOLUTION OF MAXWELL'S EQUATIONS

The classic Reissner-Nordström (RN) solution represents a charged and massive spherically symmetric body with a line element,[1,2]

$$ds^2 = g_{00}c^2dt^2 - g_{00}^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\ , \quad (2.1)$$

$$g_{00} \equiv 1 - \frac{2m}{r} + \frac{\Lambda}{r^2}\ , \quad m \equiv \frac{\kappa M}{c^2}\ , \quad \Lambda \equiv \frac{\kappa e^2}{c^4}\ ,$$

and a Minkowski tensor with one nonzero component,

$$F^{01} = e/r^2\ . \quad (2.2)$$

(We use in general the notation and conventions of Ref. 2, but with cgs electromagnetic units instead of those of Heavyside and Lorentz.)

We will be interested in the special case $\Lambda = m^2$, or $e^2 = \kappa M^2$, in which case g_{00} takes the particularly simple form $(1 - m/r)^2$. This case is of special theoretical interest since a group of point particles with this charge-to-mass ratio can remain at relative rest, as shown by Papapetrou, Majumdar, Synge, Israel and Wilson, and Perjes[3]; this is the general-relativistic generalization of the balance between gravitational attraction and electric repulsion that obtains classically for this charge-to-mass ratio. For this reason we refer to this special case of the RN solution as optimally charged.[8] (In Sec. III we will consider the exact solutions mentioned above in more detail.)

We wish to obtain solutions for electrostatic perturbations to the optimally charged RN field; by this we mean we will neglect the gravitational field associated with the perturbing charges and electric fields and solve Maxwell's equations in the RN background metric space. For convenience we limit ourselves to cylindrically symmetric configurations and thus write the nonvanishing components of the Minkowski tensor as

$$F^{01} = f(r, \theta)\ , \quad F^{02} = h(r, \theta)\ . \quad (2.3)$$

Maxwell's equations may be written as

$$[(|g|)^{1/2}F^{\alpha\beta}]_{|\beta} = 0\ , \quad F_{\alpha\beta} = A_{\beta|\alpha} - A_{\alpha|\beta}\ , \quad (2.4)$$

where a vertical bar denotes an ordinary derivative.[2] With $A_0 = \phi$ these lead to

$$g_{00}(r^2\phi_{|r})_{|r} + \frac{1}{\sin\theta}(\sin\theta\phi_{|\theta})_{|\theta} = 0\ ,$$
$$f = -\phi_{|r}\ , \quad hr^2g_{00} = -\phi_{|\theta}\ . \quad (2.5)$$

This is a modified Laplace equation, and in the limit $g_{00} \to 1$ we retrieve flat-space electrostatics.

To solve (2.5) we use a product solution, where the angle-dependent factor is a Legendre polynomial as in elementary theory. Then

$$\phi(r,\theta)=\phi_l(r)P_l(\cos\theta)\,,\qquad \left(1-\frac{m}{r}\right)^2(r^2\phi_l')'-l(l+1)\phi_l=0\,, \tag{2.6}$$

where a prime denotes a derivative with respect to r. The solutions to this are simple products of powers of r and $r-m$, and the multipole solutions of (2.5) are accordingly

$$\phi=\sum_{l=0}^{\infty}\left[\frac{a_l}{(r-m)^l r}+\frac{b_l(r-m)^{l+1}}{r}\right]P_l(\cos\theta)\,, \tag{2.7}$$

where, as in elementary theory, the coefficients must be chosen to satisfy specific physical demands or boundary conditions. We will give one example of the choice of coefficients in the next section.

III. FIELD OF A POINT CHARGE

In the absence of the central body, m is equal to zero and we have ordinary flat-space electrostatics. Then the field of a point test charge e_0 at $r=a$ and $\theta=0$ is described by a potential ϕ_0 given by

$$\phi_0=e_0(r^2+a^2-2ar\cos\theta)^{-1/2}=e_0\sum_{l=0}^{\infty}\frac{a^l}{r^{l+1}}P_l(\cos\theta)\,. \tag{3.1}$$

It is clear that to describe the analogous situation in the presence of the central body we must choose $b_l=0$ in (2.7) so that ϕ has the correct behavior for large r. The a_l must be chosen so that a singularity of the appropriate nature occurs in ϕ at $r=a$, $\theta=0$. To do this we demand that, for $\theta\to 0$ and $r\to a$, ϕ behave like ϕ_0, since sufficiently close to the test charge the potential should be independent of the background field. The limiting behavior of the series (2.7) should thus be forced to be the same as the singular limiting behavior of (3.1), which gives

$$\phi\to\sum_{l=0}^{\infty}\frac{a_l}{a(a-m)^l}\,,\quad \phi_0\to e_0\sum_{l=0}^{\infty}\frac{1}{a}\,,\qquad a_l=e_0(a-m)^l\,. \tag{3.2}$$

We thus obtain a simple expression for ϕ, which, moreover, may be summed in closed form:

$$\begin{aligned}\phi&=e_0\sum_{l=0}^{\infty}\frac{(a-m)^l}{r(r-m)^l}P_l(\cos\theta)\\&=e_0\left(1-\frac{m}{r}\right)[(r-m)^2+(a-m)^2-2(r-m)(a-m)\cos\theta]^{-1/2}\\&\equiv\left(1-\frac{m}{r}\right)\frac{e_0}{\tilde{r}}\,.\end{aligned} \tag{3.3}$$

In Fig. 1 we have shown the equipotential lines of this remarkably simple expression. Note that the black-hole surface is an equipotential.

The result (3.3) may be compared with the exact solution which we may obtain from Refs. 3. For two point particles with charges e and e_0 and masses M and M_0 (obeying the optimal charge condition $e^2=\kappa M^2$ and $e_0{}^2=\kappa M_0{}^2$) the exact solution in isotropic coordinates is given by[3]

$$ds^2=\psi^{-2}c^2dt^2-\psi^2[d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\phi^2)]\,,$$

$$\psi=1+\frac{m}{\rho}+\frac{m_0}{\rho_0}\,, \tag{3.4a}$$

$$A_0=\frac{c^2}{\sqrt{\kappa}}\left(1-\frac{1}{\psi}\right)=\frac{e/\rho+e_0/\rho_0}{1+m/\rho+m_0/\rho_0}\,, \tag{3.4b}$$

where ρ_0 is the distance between the field point and the charge e_0, located on the z axis, and $m=\kappa M/c^2$, $m_0=\kappa M_0/c^2$. Our perturbation solution is in ordinary Schwarzschild coordinates. With the transformation $r=\rho+m$ we may put the RN solution into isotropic coordinates corresponding to (3.4). Under this transformation the potential A_0 behaves like a scalar. Thus the *total* potential of our perturbative solution in isotropic coordinates is

$$A_0=\frac{e}{r}+\left(1-\frac{m}{r}\right)\frac{e_0}{\tilde{r}}=\left(1+\frac{m}{\rho}\right)^{-1}\left(\frac{e}{\rho}+\frac{e_0}{\rho_0}\right)\,, \tag{3.5}$$

$$\tilde{r}=[\rho^2+(a-m)^2-2\rho(a-m)\cos\theta]^{-1/3}\,.$$

Identifying the radial position of the test charge in isotropic coordinates as $a-m$ we have $\tilde{r}=\rho_0$, and we see that the exact solution (3.4b) and our approximate solution (3.5) agree: The gravitational effect of the test charge is contained in the small third term of the denominator of (3.4b).

To discuss the electric field we introduce a tetrad basis field $\tilde{e}^a$ defined as

$$e^a_\mu=(|g_{\mu\mu}|)^{1/2}\delta^a_\mu\,,\qquad ds^2=\eta_{cd}(e^c_\alpha dx^\alpha)(e^d_\beta dx^\beta)\,, \tag{3.6}$$

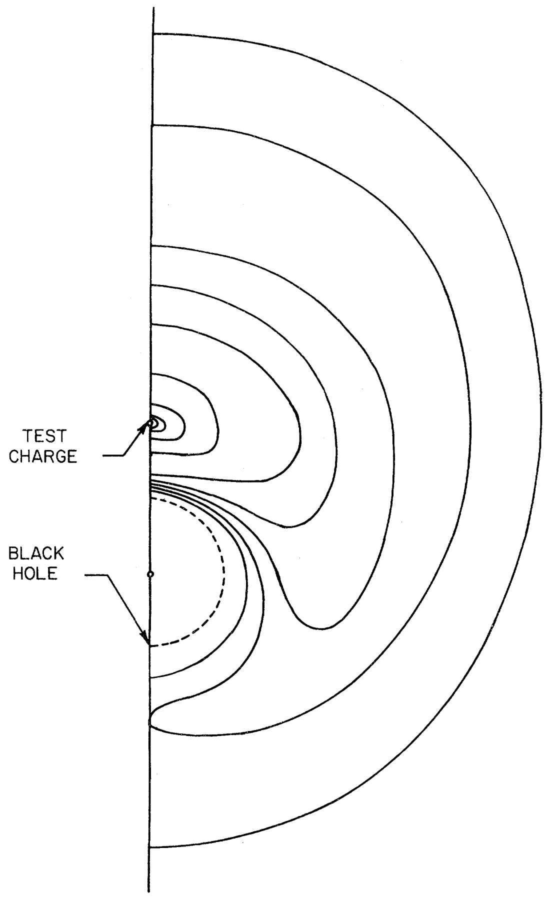

FIG. 1. Qualitative plot of the equipotential surfaces for a point test charge near an optimally charged black hole. We choose $a=2m$ and only plot the angular region $\theta=0$ to π.

where the Latin letters $a, b, \ldots$ signify tetrad indices and η_{cd} is the Lorentz metric. This basis provides a local tangent space in which the components of the Minkowski tensor have the usual flat-space interpretation. We denote the tetrad components of F_{ab} by E_r for 01 and E_θ for 02; they are related to the tensor components by

$$F_{ab}=e_a^\alpha e_b^\beta F_{\alpha\beta}\,,$$

$$E_r=-(|g^{00}g^{11}|)^{1/2}\phi_{|r}\,,\quad E_\theta=-(|g^{00}g^{22}|)^{1/2}\phi_{|\theta}\,, \tag{3.7}$$

$$E_r=-\frac{e_0}{r^2}\frac{m}{\tilde{r}}+\frac{e_0}{\tilde{r}^3}\left(1-\frac{m}{r}\right)[(r-m)-(a-m)\cos\theta]\,,$$

$$E_\theta=-\frac{e_0}{\tilde{r}^3 r}[(r-m)(a-m)\sin\theta]\,.$$

As measured with respect to the tetrad basis the electric field is radial at the black-hole radius, owing to the factor $r-m$ in E_θ.

In either of the forms for ϕ in (3.3) the behavior as $a\to m$ is transparent: Quite rapidly the higher multipoles vanish, leaving only the $l=0$ term e_0/r. The perturbing charge e_0 is merely added to the central charge e, or is "swallowed" by the black hole. Qualitatively similar behavior occurs in the Schwarzschild metric as shown in Refs. 4 and 5.

IV. MAGNETIC CHARGE

It is easy to generalize the preceding results to include magnetically as well as electrically charged particles.[6-8] The Minkowski tensor will contain both electric and magnetic cylindrically symmetric fields. By a duality rotation we may add to $F_{\alpha\beta}$ of the preceding section the correct magnetic terms; this is explicitly done by adding to it $^*F_{\alpha\beta}g_0/e_0$, where g_0 is the magnetic charge on the perturbing particle. Then $F_{\alpha\beta}$ will have two new nonzero magnetic components given by

$$F_{13}=(g_0/e_0)\sin\theta\,\phi_{|\theta}/(1-m/r)^2\,,$$
$$F_{23}=-(g_0/e_0)r^2\sin\theta\,\phi_{|r}\,. \tag{4.1}$$

The magnetic field of a point magnetic perturbing charge is one obvious example: We merely take the limit $e_0\to 0$.

Note that a duality rotation of the type we have used is not in general permissible owing to nonlinear effects. For example, the exact Majumdar-Papapetrou solution discussed in Sec. III cannot undergo a duality rotation to a solution for one electrically and one magnetically charged particle as we have done here; instead a more complicated procedure is necessary, and in fact this problem does not seem as yet to be completely solved.[9]

V. SUMMARY

For the particular case of an optimally charged Reissner-Nordström background field, defined by the relation $\kappa M^2=4\pi e^2$ between charge e and mass M, we have obtained the multipole solutions of Maxwell's equations in Eq. (2.3). The special case of a point test charge e_0 outside the black-hole surface at $r=\kappa M/c^2$ is given by (3.3). For magnetic charge the same basic equations hold, and the field of a magnetic test charge is also given by (3.3) with the electric charge e_0 replaced by a magnetic charge g_0.

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