Representations of the Pomeron propagator*

William R. Frazer and Hanna Hoffman
Department of Physics, University of California, San Diego, La Jolla, California 92093

José R. Fulco and Robert L. Sugar
Department of Physics, University of California, Santa Barbara, California 93106
(Received 19 July 1976)

A general method for constructing explicit representations of the Pomeron propagator in the presence of several dimensionless parameters is developed. Three cases are presented: the introduction in the bare Pomeron of a cutoff in k^2, the calculation of the angular distribution for elastic scattering, and the investigation of the behavior of the total cross section when the intercept of the renormalized Pomeron is less than unity. It is found that in the one-loop approximation a perturbation expansion of the Pomeron propagator in powers of the bare triple-Pomeron coupling constant is valid at intermediate energies, provided that the intercept shift is evaluated nonperturbatively. It is also found that the first-order correction to the asymptotic behavior of the angular distribution for elastic scattering is fairly small.

I. INTRODUCTION

After the work of Abarbanel and Bronzan[1] and of Migdal, Polyakov, and Ter-Martirosyan[2] demonstrated the applicability of the renormalization-group approach to the problem of finding the asymptotic behavior of Green's functions in the Gribov Reggeon calculus, the problem remained of finding explicit representations of these Green's functions and thereby calculating quantities of physical interest. For example, one would like to be able to calculate not only the asymptotic power behavior of hadronic cross sections, but also terms which describe the rate of approach to this limit. One would like to be able to exhibit not only scaling laws, but also explicit forms for angular distributions.

A method for finding an integral representation of the Pomeron propagator was introduced by Sugar and White,[3] and developed and generalized by Frazer and Moshe[4] and by Abarbanel, Bronzan, Bartels, and Sidhu.[5] The method consists of deriving differential equations for Green's functions, in which the differentiation is with respect to the dimensionless parameters of the theory. These equations can be integrated to yield integral representations for the Green's functions.

In this paper we confine our attention to the Pomeron propagator. The Pomeron propagator was analyzed in Refs. 4 and 5, as a function of both $Y=\ln s$ and of $k^2=-t$. We extend previous work in the following directions: (i) We avoid the $\epsilon=4-D$ expansion, which was used in all the papers referred to above. By remaining in the physical number of dimensions, $D=2$, we avoid the ambiguities and gross approximations of the ϵ expansion. However, we do not go beyond the one-loop approximation, which is the most serious limitation of the present work. (ii) We are able to discuss not only the asymptotic angular distribution, as in Ref. 5, but also the finite-energy corrections, as in Ref. 4. (iii) Working at $D=2$ we are able to investigate more fully the nature of the infrared singularity at $J=1$ and to determine the limitations imposed by this singularity on the validity of the perturbation expansion. (iv) We develop a technique for investigating the dependence on additional variables, working always at $D=2$ (in Refs. 4 and 5 additional variables were investigated, but in the ϵ expansion). Three examples are discussed: the dependence of the total cross section on a cutoff in k^2, the angular dependence of the elastic cross section, and the dependence of the total cross section on the renormalized Pomeron intercept.

The organization of this paper is as follows: We begin in Sec. II with a discussion of the Pomeron propagator at $k^2=0$, in the absence of a cutoff or any other additional parameters. Although this section is primarily a review, it serves to introduce our notation and a few new technical features. For example, the discussion is simplified by recognizing that it is unnecessary to renormalize the slope parameter α_0' when one is working only at $k^2=0$. At the end of Sec. II we discuss the perturbation expansion of the representation developed for the Pomeron propagator. In agreement with Sugar and White[3] we find that the intercept shift (difference between bare and renormalized Pomeron intercept) cannot be evaluated perturbatively. More positively, we find that one *can* use perturbation theory to evaluate the Pomeron propagator, provided that the intercept shift is evaluated nonperturbatively and provided that $|1-J|>E_0$, a quantity which determines the basic energy scale of the theory. Translating this into statements about the total cross section as a function of Y

$=\ln s$, it means that at energies $Y \lesssim E_0^{-1}$ perturbation theory should be valid, whereas for energies $Y \gg E_0^{-1}$ asymptotic formulas based on the renormalization-group analysis should be valid. Our estimate of the energy scale agrees with that of Ref. 5, $E_0^{-1} \approx 5-10$, which places the transition region (between perturbation theory and the asymptotic region) at Fermilab and CERN ISR energies.

In Sec. III we discuss the effect of a cutoff in k^2 on the Pomeron propagator at $k^2=0$. Such a cutoff is physically resonable and is necessary if perturbation theory is to be well defined at $D=2$. We use this simple case to describe the method for introducing new variables in the problem. We find that in our approximation the critical exponents depend on the new variable, a dependence which can be shown to be absent in the exact exponents. We then use a simple approximation which eliminates this problem.

In Sec. IV our method for incorporating additional variables is used to investigate the k^2 dependence of the Pomeron propagator. Here we find new complexities in the integral representation for the Pomeron propagator. These can be reduced to tractable form by a proper choice of variables. We then are able to study the k^2 dependence of the Pomeron propagator (a quantity related to the angular distribution in elastic scattering) in a manner which combines the desirable features of the calculations given in Refs. 4 and 5. The asymptotic form agrees with Ref. 5, if the critical exponents are evaluated in the ϵ expansion. In addition we find the leading finite-energy correction. In Sec. V the method is applied to the problem of the noncritical Pomeron with intercept $\alpha_P(0)<1$. Since this problem is formally almost identical to the problem of the k^2 dependence, the results of Sec. IV can be used with very little modification. One finds that for $\Delta=\alpha_P(0)-1$ small and negative, the behavior of the total cross section as a function of Y is modified by the introduction of a new energy scale $|\Delta|^{-1}$. For $Y<E_0^{-1}$ perturbation theory is applicable, as discussed above. For $E_0^{-1}<Y<|\Delta|^{-1}$ the usual renormalization-group "asymptotic" result of a total cross section rising as a small power of Y is applicable, whereas for $Y>|\Delta|^{-1}$ a simple renormalized Pomeron pole dominates. Although present data can probably tolerate such a situation, the intercept gap Δ would have to be quite small, $|\Delta| \lesssim 0.01$.

II. THE SINGLE-POMERON PROPAGATOR AT $D=2$ AND $k^2=0$

We begin with the simplest problem, the calculation of the single-Pomeron Green's function $G^{1,1}(E,k^2)$ in the absence of a cutoff or any other additional parameters. As usual[1] we write

$$i\Gamma^{1,1}(E,k^2)=i[G^{1,1}(E,k^2)]^{-1} = E-\alpha_0' k^2 - \Sigma(E,k^2)+\delta\Delta\,. \tag{2.1}$$

and we impose the condition that the Pomeron intercept lies at $J=1$,

$$\Gamma^{1,1}(0,0)=0\,. \tag{2.2}$$

Under these conditions one finds that the intercept renormalization $\delta\Delta$ is not well defined at $D=2$. Both ultraviolet and infrared divergences are present. The ultraviolet divergences are not physically interesting, since they are a result of oversimplification of the k and E dependence of the theory. Although this problem can be removed by the introduction of a physically reasonable cutoff, such a procedure introduces the complexity of an additional parameter. Deferring discussion of the cutoff theory to Sec. III, let us first consider the usual procedure of dimensional regularization, working at arbitrary space dimension D and setting $D=2$ at the end of the calculation. However, we shall not, expand in powers of $\epsilon=4-D$.

At $D=2$ no divergences remain after the intercept renormalization indicated in Eq. (2.1). If we introduce, as usual, the quantity

$$Z_3^{-1}(E)\equiv \frac{d}{dE}\, i\Gamma^{1,1}(E,0)\,, \tag{2.3}$$

it follows that $Z_3(E)$ is well defined even at $D=2$. It is therefore advantageous to work with this dimensionless quantity, developing representations for it. Finally, when we integrate $Z_3(E)$ to find $i\Gamma^{1,1}(E,0)$ we shall be able to study the nature of the singularities which develop.

Now we follow the procedure of Sugar and White[3] to find a representation for $Z_3(E)$. First one calculates the lowest-order contribution to $\Sigma(E,0)$ which arises from the single-bubble diagram. From this, one finds at $D=2$

$$Z_3^{-1}=1-\frac{g_0^2}{16\pi}\,, \tag{2.4}$$

where

$$g_0^2(E)\equiv -\frac{r_0^2}{\alpha_0' E}\,. \tag{2.5}$$

This expression for the dimensionless coupling constant $g_0^2(E)$ shows that an expansion in powers of this coupling constant is unlikely to be valid in the neighborhood of $E=0$. Later in this section we shall find the region of validity of the perturbation expansion in powers of g_0. In order to discuss the neighborhood of $E=0$ we follow what is called the renormalization-group method, in which one seeks a new expansion parameter $g(E)$ which remains small as $E \to 0$. Following Abarbanel and Bronzan we define

$$r(E) \equiv (2\pi)^{3/2} Z_3^{3/2} \Gamma^{1,2}(E, E/2, E/2; 0, 0, 0) \tag{2.6}$$

and

$$g^2(E) \equiv -\frac{r^2(E)}{\alpha_0' E}. \tag{2.7}$$

It is unnecessary at this point, since we are not concerning ourselves with the k^2 dependence, to define a renormalized slope α'.

In order to investigate the dependence of g and Z_3 on E we introduce the usual renormalization-group functions,

$$\beta \equiv E\,\partial g/\partial E\big|_{r_0, \alpha_0'}, \tag{2.8a}$$

$$\gamma \equiv E\,\partial \ln Z_3/\partial E\big|_{r_0, \alpha_0'}. \tag{2.8b}$$

A perturbation calculation in the one-loop approximation yields for β the result

$$\beta(g) = -\tfrac{1}{2} g\left(1 - \frac{g^2}{g_1^2}\right), \tag{2.9}$$

where at $D = 2$

$$g_1^2/16\pi = (16 \ln 2 - 3)^{-1}. \tag{2.10}$$

Now the function $\beta(g)$ defined by Eq. (2.8a) can also be written in the form

$$\beta(g) = -\tfrac{1}{2} g_0 \frac{dg}{dg_0}. \tag{2.11}$$

This form shows clearly the significance of the function $\beta(g)$ in defining the mapping from g_0 to g, as well as the significance of g_1 as a critical point of that mapping. Substituting the one-loop expression (2.9) into Eq. (2.11) one can solve for g, with the boundary condition $g/g_0 \to 1$ as $g_0 \to 0$, to obtain the result

$$g = g_0(1 + g_0^2/g_1^2)^{-1/2}. \tag{2.12}$$

One then solves for Z_3 by recognizing that since this dimensionless quantity can be expressed as a function of g, Eq. (2.8b) can be expressed in the form

$$\gamma(g) = \beta(g) \frac{d}{dg} \ln Z_3(g). \tag{2.13}$$

Using Eq. (2.9), as well as the fact that in the one-loop approximation $\gamma(g) = -g^2/16\pi$, one can solve this differential equation with the boundary condition $Z_3 = 1$ at $g = 0$ to obtain

$$Z_3 = (1 + g_0^2/g_1^2)^{-C_3}, \tag{2.14a}$$

where

$$C_3 = \gamma(g_1^2)/\beta'(g_1) \tag{2.14b}$$

$$= (3 - 16 \ln 2)^{-1}$$

$$= -0.124 \tag{2.14c}$$

and $\beta'(g_1) = 1$ in the one-loop approximation.

The above numerical result, $C_3 \approx -\frac{1}{8}$, differs from the result of Abarbanel and Bronzan, $C_3 = -\frac{1}{6}$, because different approximations have been used: We have evaluated C_3 at $D = 2$ instead of employing an ϵ expansion, and we have defined the renormalized coupling constant g in terms of α_0' instead of α'. Uncertainties such as these are inherent in the crude one-loop approximation which we are using. A reliable evaluation of C_3 and other critical exponents is one of the central problems which must be solved before the Reggeon calculus can provide a quantitative theory of high-energy scattering. In this paper we regard C_3 as a parameter, calculations of which have yielded values in the range from about $-\frac{1}{10}$ to $-\frac{1}{2}$.[6] The best present estimate[7] is $C_3 \approx -0.22$.

Integrating Eq. (2.3) and using Eq. (2.14) we find for the Pomeron propagator the result

$$i\Gamma^{1,1}(E, 0) = \int_0^E dE' \left(1 - \frac{E_0}{E'}\right)^{C_3}, \tag{2.15}$$

where

$$E_0 = \frac{r_0^2}{\alpha_0' g_1^2} = \frac{r_0^2}{-16\pi \alpha_0' C_3}. \tag{2.16}$$

The quantity E_0 sets the scale for approximations to $i\Gamma^{1,1}$. For $E \ll E_0$, or $Y = \ln s \gg E_0^{-1}$, the asymptotic form

$$i\Gamma^{1,1}(E) \sim E^{1-C_3} \tag{2.17}$$

is valid. For $Y \ll E_0^{-1}$ it is commonly assumed that ordinary perturbation theory in powers of g_0 is valid. We shall examine later the extent to which such an assumption is true.

To evaluate E_0 we require knowledge of the bare triple-Pomeron coupling r_0. Interpreting high-mass diffraction dissociation experiments at Fermilab in terms of a bare triple-Pomeron vertex, the Fermilab Single-Arm Spectrometer Group[8] has found the value $r_0\sqrt{2} = 0.8 \pm 0.03$.[9] Using the value of g_1 given by Eq. (2.10) and using $\alpha_0' \approx 0.3$, one finds that $E_0^{-1} \approx 6$, indicating that the transition region between perturbation theory and the asymptotic form of the Pomeron propagator may be occurring at ISR energies. Corrections, such as inclusion of lower trajectories in analyzing the Fermilab data, will probably raise the value of E_0^{-1}. In Fig. 1 we compare the "exact" propagator (calculated in the next section, using a cutoff) with the asymptotic form, using a more conservative value of $E_0^{-1} = 8.5$.

Now we turn to the question of the perturbation expansion of the Pomeron propagator given by Eq. (2.15) in powers of E_0 (powers of r_0^2). We see immediately that, since the integration extends to $E=0$, no expansion of the full propagator in terms of this parameter is possible, as was pointed out by Sugar and White.[3] After we separate out the intercept-renormalization term $\delta\Delta$, which is the cause of the difficulty, we shall find that a perturbation expansion is indeed possible for $E > E_0$. In order to identify $\delta\Delta$ we note that as $E \to -\infty$ all diagrams with one or more loops will vanish provided that the integrals $\int d^2k$ are cutoff (as we have remarked, such a cutoff is physically reasonable). In particular, $\Sigma(-\infty, k^2) = 0$ and according to Eq. (2.1) it follows that

$$\delta\Delta = \lim_{E\to-\infty} [i\Gamma^1(E,0) - E] = \lim_{\Lambda\to\infty} \int_0^\Lambda dx \left[1 - \left(1 + \frac{E_0}{x}\right)^{C_3}\right]. \tag{2.18}$$

As long as $C_3 < 0$ one finds that $\delta\Delta > 0$ and the bare Pomeron has an intercept at $J > 1$. However, as written, the integral above diverges. Again this divergence can be removed by a cutoff in k^2. The result is that E_0 is replaced by a function of E which vanishes as $1/E$ for large E. Assuming that a cutoff has been used to define the integral, we can write for the propagator

$$i\Gamma^{1,1}(E,0) - E - \delta\Delta = -\int_E^\infty dE' \left[\left(1 - \frac{E_0}{E'}\right)^{C_3} - 1\right]. \tag{2.19}$$

This integral can now be expanded in powers of E_0 as long as $E > E_0$. However, note that no perturbation expansion is possible for $\delta\Delta$. We find then, for $E > E_0$, that

$$i\Gamma^{1,1}(E,0) = E + \delta\Delta + O(r_0^2). \tag{2.20}$$

We shall discuss this expansion in more detail in Sec. III, after introduction of a cutoff.

We have shown that one can use perturbation theory for $E > E_0$ provided that $\delta\Delta$ is evaluated nonperturbatively. However, it should be emphasized that the perturbation series is an asymptotic expansion. As one would expect in a nonlinear field theory it does not converge.[10] The corresponding situation in the $Y = \ln s$ plane is shown in Fig. 1, where the lowest-order bare Pomeron approximation is seen to be valid for $Y \ll E_0^{-1}$.

III. POMERON PROPAGATOR WITH CUTOFF

As we saw in Sec. II the calculation of the Pomeron propagator at $D=2$ is complicated by both

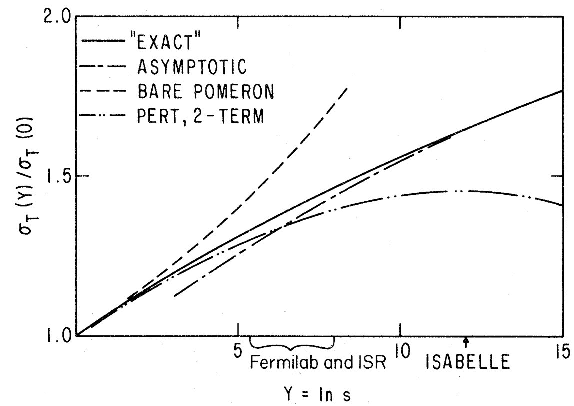

FIG. 1. Comparison between the total cross section obtained by using the first two terms of the perturbation expansion, the Sommerfeld-Watson transform of the representation (3.24), and the asymptotic form of that representation. See Table I for values of parameters.

infrared and ultraviolet divergences. The infrared complication is essential, since it reflects the nature of the singularities we are investigating at $E=0$ ($J=1$). However the ultraviolet divergences are inessential complications which can be removed by the introduction of a cutoff. We choose an exponential cutoff in k^2, a form which is both convenient and in accord with experiment.[11] That is, we replace the bare Pomeron propagator as follows:

$$\frac{1}{E - \alpha_0' k^2 - \Delta_0} \to \frac{e^{-bk^2}}{E - \alpha_0' k^2 - \Delta_0}. \tag{3.1}$$

We shall consider here only the case $k^2 = 0$, and write

$$i\Gamma^{1,1}(E,b) = E - \Sigma(E,b) + \delta\Delta. \tag{3.2}$$

The condition that the Pomeron intercept lie at $J=1$ now reads

$$\Gamma^{1,1}(0,b) = 0. \tag{3.3}$$

In the one-loop approximation one finds at $D=2$ that

$$\Sigma(E,b) = \frac{r_0^2}{16\pi\alpha_0'} e^{\omega} E_1(\omega), \tag{3.4}$$

where the dimensionless variable ω is defined as

$$\omega = -bE/\alpha_0', \tag{3.5}$$

and where the function $E_1(x)$ is the usual exponential integral. Again one finds that the intercept renormalization $\delta\Delta$ is undefined in perturbation theory, as a result of the infrared divergence. As in Sec. II, we deal with this problem by analyzing the derivative

$$Z_3^{-1}(E,b) \equiv \frac{\partial}{\partial E} i\Gamma^{1,1}(E,b). \tag{3.6}$$

In the one-loop approximation at $D=2$ one finds from Eq. (3.4) that

$$Z_3^{-1}=1+\frac{g_0^2}{16\pi}[\omega e^{\omega}E_1(\omega)-1], \tag{3.7}$$

where, as in Sec. II,

$$g_0^2(E)=-r_0^2/\alpha_0'E. \tag{3.8}$$

We also retain the same definition of g given by Eq. (2.7), but g is now a function of both E and b.

The dependence on an additional dimensionless parameter ω causes very little additional complication if one generalizes the method of Sec. II by taking partial derivatives at fixed ω. For example, β is now defined as

$$\beta(g,\omega)=E\left.\frac{\partial g}{\partial E}\right|_{r_0,\alpha_0',\omega}$$

$$=-\tfrac{1}{2}g_0\left.\frac{\partial g}{\partial g_0}\right|_{\omega}, \tag{3.9}$$

from which one obtains in the one-loop approximation the usual result,

$$\beta=-\tfrac{1}{2}g(1-g^2/g_1^2(\omega)). \tag{3.10}$$

The additional complexity enters only in the fact that g_1 is now a function of ω,

$$16\pi/g_1^2=3\omega e^{\omega}E_1(\omega)-3+16e^{(3/4)\omega}E_1(\tfrac{3}{4}\omega)$$

$$-16e^{(3/2)\omega}E_1(\tfrac{3}{2}\omega). \tag{3.11}$$

One can integrate Eq. (3.9) at fixed ω to find that Eq. (2.12) again holds. To find $Z_3(g,\omega)$ we observe that

$$\frac{\partial \ln Z_3(g,\omega)}{\partial g}=\left.\frac{\partial E}{\partial g}\right|_{\omega}\left.\frac{\partial \ln Z_3}{\partial E}\right|_{\omega} \tag{3.12a}$$

$$=\frac{\gamma(g,\omega)}{\beta(g,\omega)}, \tag{3.12b}$$

where

$$\gamma(g,\omega)=E\left.\frac{\partial \ln Z_3}{\partial E}\right|_{r_0,\alpha_0',\omega}$$

$$=-\tfrac{1}{2}g_0\left.\frac{\partial \ln Z_3}{\partial g_0}\right|_{\omega}. \tag{3.13}$$

In the one-loop approximation one sees from Eq. (3.7) that

$$\gamma(g,\omega)=-\frac{g^2}{16\pi}[1-\omega e^{\omega}E_1(\omega)]. \tag{3.14}$$

We can now proceed to integrate Eq. (3.12) at fixed ω, since the boundary condition

$$Z_3(0,\omega)=1 \tag{3.15}$$

is independent of ω. The result is

$$Z_3(g,\omega)=\left[1+\frac{g_0^2}{g_1^2(\omega)}\right]^{-C_3(\omega)}, \tag{3.16a}$$

where

$$C_3(\omega)=-\frac{g_1^2(\omega)}{16\pi}[1-\omega e^{\omega}E_1(\omega)]. \tag{3.16b}$$

In the asymptotic region $E\to 0$, the parameter ω also goes to zero. Hence the above results reduce to the same asymptotic expression we found in Sec. II, as one would expect.[12]

The function $C_3(\omega)$ is a very slowly varying function of ω, going from a value of -0.12 at $\omega=0$ to a value of -0.07 at $\omega=\infty$. However, it has been shown in Ref. 5 that the critical exponent C_3 should in principle not depend on ω at all. The argument is as follows: From Eq. (3.16a) one finds that

$$\left.\frac{\partial \ln Z_3}{\partial\omega}\right|_{g}=\frac{dC_3}{d\omega}\ln(g_1^2-g^2)+\frac{C_3}{g_1^2-g^2}\frac{dg_1^2}{d\omega}+\cdots, \tag{3.17}$$

where the remaining terms are nonsingular at $g=g_1$. However, it is possible to derive a differential equation analogous to Eq. (3.12) for $\partial \ln Z_3/\partial\omega$. When the right-hand side of that equation is evaluated in the one-loop approximation, no logarithmic singularity at $g=g_1$ appears. Consistency with Eq. (3.17) occurs only if the coefficient of the logarithmic singularity vanishes, or

$$dC_3/d\omega=0. \tag{3.18}$$

The fact that our result, Eq. (3.16b), does not satisfy this condition is an inadequacy of the one-loop approximation. In other words, C_3 would not be independent of ω if one works to finite order in perturbation theory. The approximate constancy of $C_3(\omega)$ as given by Eq. (3.16b) is as good as we have any right to expect in the one-loop approximation. It therefore seems quite within the spirit of that approximation to simply set $C_3(\omega)$ equal to a constant value,

$$C_3(\omega)\approx c_3\equiv C_3(0). \tag{3.19}$$

Looking back at Eq. (3.16b) one sees that this approximation implies another, namely,

$$g_1^2(\omega)\approx -16\pi c_3[1-\omega e^{\omega}E_1(\omega)]^{-1}. \tag{3.20}$$

Finally, we obtain for Z_3

$$Z_3^{-1}=\left[1-\frac{E_0(\omega)}{E}\right]^{c_3}, \tag{3.21}$$

where

$$E_0(\omega)=\overline{E}_0[1-\omega e^{\omega}E_1(\omega)], \tag{3.22}$$

and where $\overline{E}_0$, which is equal to the E_0 defined in Sec. II, is given by

$$\overline{E}_0 = -\frac{r_0^2}{16\pi\alpha_0' c_3} = \frac{r_0^2}{\alpha_0' g_1^2(0)} . \qquad (3.23)$$

The method we have used to obtain Eq. (3.21) can be applied quite generally to the calculation of the Pomeron propagator in the presence of any additional dimensionless parameters. In the one-loop approximation Eq. (3.21) always results; only the form of the dependence of g_1 and E_0 on the parameters will vary. In Secs. IV and V we shall discuss two more examples.

Using Eq. (3.21) in Eq. (3.6) and integrating we finally find for the Pomeron propagator

$$i\Gamma^{1,1}(E,b) = \int_0^E dE' \left[1 - \frac{E_0(\omega')}{E'}\right]^{c_3}, \qquad (3.24a)$$

where

$$\omega' = -bE'/\alpha_0' . \qquad (3.24b)$$

This is the representation promised at the end of Sec. II. Since as $E \to -\infty$ all diagrams with one or more loops vanish, it follows that

$$\Sigma(-\infty, b) = 0 , \qquad (3.25)$$

and therefore from Eq. (3.2)

$$\delta\Delta = \lim_{E\to -\infty} [i\Gamma^{1,1}(E,b) - E] \qquad (3.26a)$$

$$= \int_0^{-\infty} dE' \left\{\left[1 - \frac{E_0(\omega')}{E'}\right]^{c_3} - 1\right\}. \qquad (3.26b)$$

Subtracting this expression from Eq. (3.24) and deforming the contour one can write finally

$$i\Gamma^{1,1}(E,b) - E - \delta\Delta = \Sigma(E,b) = -\int_E^{\infty} dE' \left\{\left[1 - \frac{E_0(\omega')}{E'}\right]^{c_3} - 1\right\}. \qquad (3.27)$$

As we discussed at the end of Sec. II, one can see from these representations that although $\delta\Delta$ cannot be expanded in powers of r_0^2, the quantity $\Sigma(E,b)$ does have a convergent perturbation expansion for $E > E_0$. In fact, the lowest-order term in the expansion of the right-hand side Eq. (3.27) reproduces Eq. (3.4) correctly. A numerical integration of the Sommerfeld-Watson transform of Eq. (3.27) is shown in Fig. 1 in comparison with the first two terms of the perturbation expansion, which is a good approximation up to moderate energies, and which begins to break down only at the highest energies presently available. The values of $\delta\Delta$ [the bare Pomeron intercept lies at $\alpha_0(0) = 1 + \delta\Delta$] calculated according to Eq. (3.26) are given in Table I for various choices of the parameters. Note that our values are several times larger than the value $\delta\Delta \approx 0.01$ found by Capella and Kaplan, using a perturbative expression.[13]

One can also make an asymptotic expansion of Eq. (3.27), the first two terms of which yield the result

$$\sigma_T \propto (\overline{E}_0 Y)^{-c_3} \{1 + (\overline{E}_0 Y)^{-\lambda}[a_0 + a_1 \ln(\overline{E}_0 Y)]\}, \qquad (3.28a)$$

where

$$a_0 = \frac{c_3^2(1-c_3)}{2-c_3}\left[1 + \overline{\omega}\gamma - \frac{\overline{\omega}}{2-c_3} + \overline{\omega}\pi\cot\pi c_3 + \overline{\omega}\ln\overline{\omega} + \overline{\omega}\psi(1-c_3)\right], \qquad (3.28b)$$

$$a_1 = -\frac{c_3^2(1-c_3)}{2-c_3}\,\overline{\omega} , \qquad (3.28c)$$

$$\overline{\omega} = b\overline{E}_0/\alpha_0' . \qquad (3.28d)$$

γ is the Euler-Mascheroni constant and ψ is the logarithmic derivative of the gamma function.

In the one-loop approximation the critical index λ is found to have the value $\lambda = 1$. The above result differs slightly from that found by Frazer and Moshe[4] in the ϵ expansion without a cutoff, in that a term emerges of the form $\ln(\overline{E}_0 Y)/\overline{E}_0 Y$, which is multiplied by a coefficient proportional to the cutoff parameter b. In other words, the introduction of a cutoff influences the rate of approach to the scaling limit, but not the form of that limit itself.

The asymptotic expansion is also shown in Fig. 1. The two-term expansion is a good approximation down to energies approaching the highest presently available. Using perturbation theory at the lower energies and the asymptotic expansion at the higher energies enables one to approximate

TABLE I. The value of $\delta\Delta$ [the bare Pomeron intercept is $\alpha_0(0) = 1 + \delta\Delta$] for various values of parameters c_3 and $\overline{E}_0$. In case (a), which corresponds to Fig. 1, the parameters have been chosen independently, whereas in cases (b)–(d) they have been chosen to satisfy Eq. (3.23) with $r_0^2 = 0.32$ (see Refs. 8 and 9) and $\alpha_0' = 0.3$. In all cases we take $b = 1$, as found in Ref. 8.

	c_3	$\overline{E}_0^{-1}$	$\delta\Delta$
(a)	−0.32	8.5	0.069
(b)	−0.22	10.4	0.043
(c)	−0.124	5.8	0.038
(d)	−0.32	15.1	0.046

the Pomeron propagator of Eq. (3.27) quite well over almost the entire energy spectrum.

IV. k^2 DEPENDENCE OF THE POMERON PROPERAGATOR AT $D=2$

A. Representation for Z_3

The k^2 dependence of the Pomeron propagator has been investigated previously by Frazer and Moshe[4] and by Abarbanel, Bartels, Bronzan, and Sidhu.[5] The present work avoids the ϵ expansion, working at the physical value $D=2$, and evaluates corrections to the asymptotic limit. As in Sec. II, we analyze the derivatives of the Pomeron propagator in order to avoid considering the singular quantity $\delta\Delta$. According to convention,[1] we define

$$Z_3^{-1}(E,k^2)=\frac{\partial}{\partial E}\,i\Gamma^{1,1}(E,k^2)\,, \tag{4.1}$$

$$-\alpha'(E,k^2)Z_3^{-1}(E,k^2)=\frac{\partial}{\partial k^2}\,i\Gamma^{1,1}(E,k^2)\,. \tag{4.2}$$

We find it convenient to define the renormalized coupling constant r at a more general point,

$$r(E,k^2)\equiv(2\pi)^{3/2}Z_3^{3/2}\Gamma^{1,2}(E,E/2,E/2;\vec{k},\vec{k}/2,\vec{k}/2)\,, \tag{4.3}$$

and to define

$$g^2(E,k^2)\equiv-\frac{r^2(E,k^2)}{\alpha'(E,k^2)E}\,. \tag{4.4}$$

The second dimensionless parameter is taken to be

$$h\equiv-\alpha'k^2/E\,. \tag{4.5}$$

The reader may wonder at this point why we have introduced the renormalized intercept α' instead of using α_0' as in the preceding sections. The reason is that, in contrast to the cutoff problem, the introduction of nonzero k^2 does not leave the asymptotic solution unmodified. We shall find that it is not possible to derive a representation for $i\Gamma^{1,1}(E,k^2)$ by simply integrating Eq. (4.1) at fixed k^2, but it will be convenient to integrate at fixed h. Were we to define h in terms of α_0' instead of α', we would encounter difficulties in the integration, as we shall point out at the appropriate point.

We can now proceed to derive a representation for $Z_3(g,h)$ exactly as in Sec. III, by taking partial derivatives at fixed h. The dependence of α' on E, which we have not yet evaluated, causes no additional complexity at the one-loop level, since $\alpha'=\alpha_0'+O(g^2)$. One finds that

$$\gamma(g,h)=-\frac{g^2}{16\pi}\,(1+\tfrac{1}{2}h)^{-1} \tag{4.6}$$

and

$$\beta(g,h)=-\frac{g}{2}\left[1-\frac{g^2}{g_1^2(h)}\right], \tag{4.7}$$

where

$$32\pi/g_1^2=\frac{16}{\sqrt{q}}\ln\frac{(\frac{1}{2}+\frac{1}{2}h+\sqrt{q})(\frac{1}{2}+\frac{1}{4}h-\sqrt{q})}{(\frac{1}{2}+\frac{1}{2}h-\sqrt{q})(\frac{1}{2}+\frac{1}{4}h+\sqrt{q})}-5(1+\tfrac{1}{2}h)^{-1}\,, \tag{4.8a}$$

with

$$q=\tfrac{1}{4}(\tfrac{5}{4}h^2+3h+1)\,. \tag{4.8b}$$

For $Z_3(g,h)$ one finds that

$$Z_3(g,h)=\left[1+\frac{g_0^2}{g_1^2(h)}\right]^{-C_3(h)}, \tag{4.9}$$

where

$$C_3(h)=-\frac{g_1^2(h)}{16\pi}\,(1+\tfrac{1}{2}h)^{-1}\,. \tag{4.10}$$

Again we observe that $C_3(h)$ is a slowly varying function of h, although in principle should not depend on h at all. As in Sec. III we make the approximation

$$C_3(h)\approx c_3\equiv C_3(0)\,, \tag{4.11}$$

which implies that

$$g_1^2(h)\approx-16\pi c_3(1+\tfrac{1}{2}h)\,. \tag{4.12}$$

Finally, in this approximation we can write for Z_3

$$Z_3^{-1}=\left(1-\frac{E_0(h)}{E}\right)^{c_3}, \tag{4.13}$$

where

$$E_0(h)=\overline{E}_0(1+\tfrac{1}{2}h)^{-1}\,, \tag{4.14}$$

and where $\overline{E}_0$, which is approximately equal to the E_0 defined in Sec. II, is given by

$$\overline{E}_0=-\frac{r_0^2}{16\pi\,\alpha_0'c_3}\,. \tag{4.15}$$

B. Integration to find $\Gamma^{1,1}$

From Eq. (4.1) one sees that one can obtain $\Gamma^{1,1}$ from Z_3^{-1} by integration with respect to E at fixed k^2. However, this is not suitable with respect to the approximations we have made. If we hold k^2 fixed and integrate up from $E=0$ we force h to range to infinity. From Eq. (4.12) we see that at $h=\infty$, $g_1^2=\infty$ also, and our approximation scheme breaks down. This is not unexpected; at fixed k^2 the Pomeron trajectory is not at $J=1$ and there is no buildup of singularities.

However, one can circumvent this difficulty by integrating at fixed h,

$$i\Gamma^{1,1}(E,h)=\int_0^E dE'\left.\frac{\partial i\Gamma}{\partial E'}\right|_h . \quad (4.16)$$

Note that this choice of path enforces the boundary condition (2.2). Changing variables one obtains

$$\left.\frac{\partial i\Gamma}{\partial E}\right|_h=\left.\frac{\partial i\Gamma}{\partial E}\right|_{k^2}+\left.\frac{\partial i\Gamma}{\partial k^2}\right|_E\left.\frac{\partial k^2}{\partial E}\right|_h \quad (4.17a)$$

$$=Z_3^{-1}(E,h)\left(1-\alpha'\left.\frac{\partial k^2}{\partial E}\right|_h\right) \quad (4.17b)$$

and finally

$$i\Gamma^{1,1}(E,h)=\int_0^E dE' Z_3^{-1}(E',h)\times\left\{1+h\left[1-\left.\frac{\partial\ln\alpha'}{\partial\ln(-E)}\right|_h\right]\right\}. \quad (4.18)$$

To evaluate this expression we require knowledge of $\alpha'=Z_\alpha\alpha_0'$. Analogously to the procedure above, we find that

$$Z_\alpha(E,h)=\left(1-\frac{E_0(h)}{E}\right)^{-c_\alpha}, \quad (4.19)$$

where in the one-loop approximation

$$c_\alpha=\tfrac{1}{2}c_3 . \quad (4.20)$$

Putting the above results together one finds that

$$i\Gamma^{1,1}(E,h)=-E_0(h)\int_0^{-E/E_0(h)} dx\, x^{-c_3}(1+x)^{c_3}\times\left[1+h\left(1-\frac{c_\alpha}{1+x}\right)\right], \quad (4.21)$$

where h is related to E and k^2 through the implicit relation

$$h=-\frac{\alpha' k^2}{E}=-\frac{\alpha_0' k^2}{E}\left[1-\frac{E_0(h)}{E}\right]^{-c_\alpha} . \quad (4.22)$$

Note that the two terms inside the square brackets in Eq. (4.21) give rise to the same asymptotic power behavior as $E\to 0$. These two terms arise from the two terms on the right-hand side of Eq. (4.17a). If one adopts the apparently simpler definition $h_0=\alpha_0'k^2/E$, instead of defining h in terms of the renormalized α', then at this point one encounters the difficulty that the power behavior of the two terms in the representation equivalent to Eq. (4.21) no longer matches and one does not see that scaling behavior found in Refs. 4 and 5. Moreover, performing the integration at fixed h_0 instead of h one finds that the representation will give information on the scaling functions only at $h=0$.

C. Asymptotic expansion and Sommerfeld-Watson transform

Although the expression for $i\Gamma^{1,1}(E,h)$ given in Eq. (4.21) is the representation of the Pomeron propagator which we have been seeking, it is not yet in a form which is tractable for explicit calculation. However, if we seek only an asymptotic expansion (small E) of Eq. (4.21) we can do the integration analytically, leaving only the Sommerfeld-Watson transform for numerical integration. Expanding the integrand and performing the integration at fixed h one finds for the first two terms that

$$-i\Gamma^{1,1}/\overline{E}_0\approx\left(-\frac{E}{\overline{E}_0}\right)^{1-c_3}\left\{[1+h(1-c_\alpha)]/(1-c_3)-\frac{E}{\overline{E}_0}(1+\tfrac{1}{2}h)[c_3+hc_\alpha+c_3h(1-c_\alpha)]/(2-c_3)\right\}. \quad (4.23)$$

It is reassuring that, despite the seemingly different approximations made, the leading term in the above equation agrees with the result found by Abarbanel, Bronzan, Bartels, and Sidhu [Ref. 5, Eq. (75)] provided that c_3, c_α, and g_1 are set equal to their lowest-order ϵ-expansion values.

In order to calculate the contribution of the Pomeron propagator to the angular distribution we must take the Sommerfeld-Watson transform of the propagator at fixed $k^2=-t$; that is, we wish to calculate

$$F(Y,k^2)=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[i\Gamma^{1,1}(E,k^2)]^{-1}e^{-EY}dE . \quad (4.24)$$

In order to perform this integration at fixed k^2 we must take into account the fact that h is defined as an implicit function of E and k^2 through Eq. (4.22). Again, this can be made more tractable by an expansion at small E. Since it will prove convenient to change the variable of integration in Eq. (4.24) from E to h, we solve Eq. (4.22) by iteration to obtain the form

$$-E/\overline{E}_0=\rho\varphi_1(h)+\rho^2\varphi_2(h)+\cdots , \quad (4.25a)$$

where

$$\rho=(\alpha_0'k^2/\overline{E}_0)^{1/(1-c_\alpha)} , \quad (4.25b)$$

$$\varphi_1(h)=h^{1/(c_\alpha-1)}(1+\tfrac{1}{2}h)^{-c_\alpha/(c_\alpha-1)} , \quad (4.25c)$$

$$\varphi_2(h)=\varphi_1^2(h)(1+\tfrac{1}{2}h)c_\alpha/(c_\alpha-1) . \quad (4.25d)$$

Note that this expansion is in effect a small-ρ expansion, and therefore limits us to small values of k^2.

Let us first discuss the Sommerfeld-Watson transform of the leading term of Eq. (4.23), keeping only the leading term in Eq. (4.25a),

$$-E/\overline{E}_0 \approx \rho h^{1/(c_\alpha-1)}(1+\tfrac{1}{2}h)^{-c_\alpha/(c_\alpha-1)}. \quad (4.26)$$

One easily sees, as pointed out in Ref. 5, that the apparent singularity at $E=0$ disappears when one takes into account the fact that $h\sim E^{-1}$ as $E\to 0$. Moreover, the singularity at $h=-2$ is mapped to $E=\infty$. The leading singularity in the E plane for $k^2>0$ is a moving (as a function of k^2) pair of complex conjugate branch points which arise from the singularities of $h(E)$. Although it is not in general possible to invert the mapping specified by Eq. (4.26) to display the singularity explicitly, we can locate the singularity by calculating the derivative dh/dE. One finds that this derivative fails to exist at the critical point

$$h=h_c=\frac{2}{c_\alpha-1}. \quad (4.27)$$

The resulting branch points in the E plane are shown in Fig. 2. The lowest-order amplitude given by the first term of Eq. (4.23) also has a pole, located at

$$h=h_0=\frac{1}{c_\alpha-1}. \quad (4.28)$$

We find, using Eqs. (4.26)–(4.28), that the leading branch point and the pole do not satisfy the familiar relation

$$\alpha_c(t)=2\alpha_P(t/4)-1 \quad (4.29)$$

unless $c_\alpha=0$. (This conclusion differs from that of Ref. 5.) However, this should not cause concern; Eq. (4.29) is the position of the two-Pomeron cut, and is recovered if one makes a perturbation expansion of Eq. (4.21). The full amplitude is doing its best to go beyond perturbation theory, and the cuts we find should be regarded as singularities which are trying to approximate the effect of the sum of all higher-order multi-Pomeron cuts.

The contour shown in Fig. 2 is inconvenient for numerical integration. Fortunately, it maps in the h plane into the relatively simple contour shown in Fig. 3, which enables one to write

$$F(Y,k^2)\approx\frac{(1-c_3)(\overline{E}_0Y)^{-c_3}}{\Gamma(1-c_3)}\,F_1(x), \quad (4.30)$$

where the scaling variable x is defined (following Ref. 5) as

$$x=(\rho\overline{E}_0Y)^{1-c_\alpha}, \quad (4.31)$$

and where $F_1(x)$, normalized such that $F_1(0)=1$, is given by

$$F_1(x)=\frac{\Gamma(1-c_3)h_0}{c_\alpha-1}\left[\frac{1}{\pi}\,\mathrm{P}\int_{-2}^{0}dh\,\frac{\mathrm{Im}\,f_0(h,x)}{h-h_0}-\mathrm{Re}\,f_0(h_0+i\epsilon,x)\right], \quad (4.32)$$

where

$$f_0(h,x)=\frac{1}{h}\,[\xi\varphi_1(h)]^{c_3}(1+\tfrac{1}{2}h)^{c_3}\varphi_3(h)\exp[\xi\varphi_1(h)], \quad (4.33a)$$

$$\xi=x^{1/(1-c_\alpha)}, \quad (4.33b)$$

$$\varphi_3(h)=1-c_\alpha\tfrac{1}{2}h(1+\tfrac{1}{2}h)^{-1}, \quad (4.33c)$$

and where $\varphi_1(h)$ and $\varphi_2(h)$ are given by Eq. (4.25). Again, this result agrees with Ref. 5, Eq. (81), provided that the c's are given their lowest-order ϵ-expansion values. The function $F_1(x)$ is shown in Fig. 4(a).

The second term in the asymptotic expansion of $F(Y,k^2)$ can be found similarly. The result is of the form

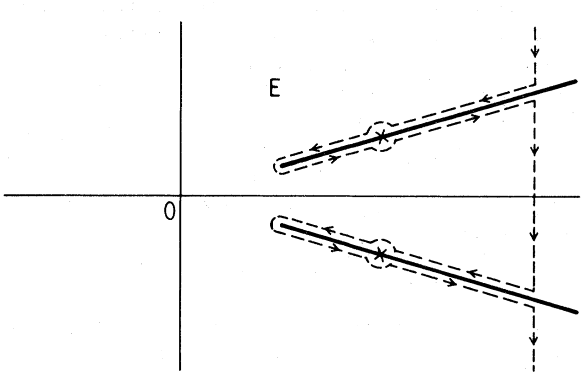

FIG. 2. Singularities of the asymptotic Pomeron propagator [first term of Eq. (4.23)] in the complex E plane for $k^2>0$, for the case $c_\alpha=-0.1$. Crosses show location of poles.

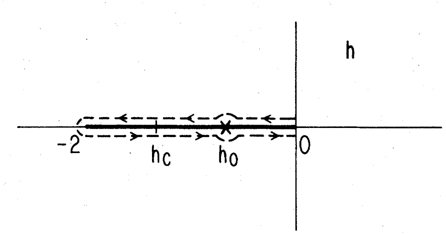

FIG. 3. Singularities of the asymptotic Pomeron propagator in the complex h plane. The cross at h_0 shows the location of the pole, and the point h_c denotes the image of the branch points in the E plane.

$$F(Y,k^2)\approx\frac{(1-c_3)}{\Gamma(1-c_3)}(\bar{E}_0Y)^{-c_3}[F_1(x)+F_2(x)(\bar{E}_0Y)^{-\lambda}c_3{}^2(1-c_3)/(2-c_3)]\,, \tag{4.34}$$

where $F_2(0)=1$. The integral expression for $F_2(x)$ is shown below and the numerical results are displayed in Fig. 4(b):

$$F_2(x)=h_0{}^2\Gamma(1-c_3)\frac{(2-c_3)}{c_3{}^2(1-c_3)}I(x)\,, \tag{4.35}$$

$$I(x)=\frac{1}{\pi}\int_{-2}^{0}dh\,\mathrm{Im}\left[\frac{f_2(h)-f_2(h_0)}{h-h_0}+\varphi_4(h_0)\frac{f_0(h)-f_0(h_0)-f_0'(h_0)(h-h_0)}{(h-h_0)^2}\right]$$

$$-\mathrm{Re}[\varphi_4(h_0)f_0'(h_0)+f_2(h_0)]+\frac{1}{\pi}\mathrm{Im}[\varphi_4(h_0)f_0'(h_0)+f_2(h_0)]\ln\left|\frac{h_0}{2+h_0}\right|+\frac{1}{\pi}\mathrm{Im}\left[2f_0(h_0)\frac{\varphi_4(h_0)}{h_0(2+h_0)}\right], \tag{4.36}$$

where

$$f_2(h,x)=f_1(h,x)+f_0(h,x)\varphi_5(h)\,, \tag{4.37a}$$

$$f_1(h,x)=\xi f_0(h,x)\left[\xi\varphi_2(h)+c_3\frac{\varphi_2(h)}{\varphi_1(h)}+\frac{c_\alpha}{2}h\frac{\varphi_1(h)}{\varphi_3(h)}+\frac{c_\alpha}{c_\alpha-1}(1+\tfrac{1}{2}h)\varphi_1(h)\right], \tag{4.37b}$$

$$\varphi_4(h)=\frac{c_3-1}{(1-c_\alpha)(2-c_3)}\xi\varphi_1(h)(1+\tfrac{1}{2}h)[c_3+c_3h(1-c_\alpha)+hc_\alpha]\,, \tag{4.37c}$$

$$\varphi_5(h)=\frac{\varphi_4(h)-\varphi_4(h_0)}{h-h_0}\,, \tag{4.37d}$$

and where $f_0(h,x)$, $\varphi_1(h)$, $\varphi_2(h)$, and $\varphi_3(h)$ are given by Eqs. (4.25) and (4.33).

The contribution of the Pomeron propagator, Eq. (4.34), to the angular distribution of a $2\to2$ process is given by

$$\frac{d\sigma}{dt}=\frac{[\beta_1(t)\beta_2(t)]^2}{16\pi}F^2(Y,-t)\,, \tag{4.38}$$

where the Pomeron residue functions $\beta_i(t)$ are arbitrary functions of t. In order to see how much of the observed angular dependence in high-energy p-p scattering might be attributed to the Pomeron propagator, we have set the residue functions equal to constants and have adjusted the parameters of the theory ($\bar{E}_0$, c_3, α_0') within reasonable limits imposed by theoretical and experimental uncertainties. Comparison with experimental data is shown in Figs. 5(a)–5(c).

Although the qualitative resemblance between the data and our calculated Pomeron-propagator contribution is remarkable, we have neglected many effects which would have to be considered in a careful phenomenological treatment. In Eq. (4.35) only the imaginary part of the amplitude has been included. A rough estimate of the real part shows this to be negligible near $t=0$.[14] Nonenhanced graphs must be considered, as well as the possibility of important four- (or more-) Pomeron couplings. Finally, we have no justification for believing in the validity of our approximations out to the large values of t in Fig. 5, where the parameter ρ of Eq. (3.40b) is no longer small.

One new feature has resulted from the evaluation of the correction term $F_2(x)$: Note that $F_2(x)$ drops very steeply to zero at $x\approx0.07$. This results in a sharpening of the angular distribution at small t, qualitatively similar to the behavior of the data. However, the coefficient of $F_2(x)$ is too small by an order of magnitude to account completely for this effect with the values of the parameters we have chosen.

V. POMERON INTERCEPT $\alpha_P(0)<1$

The method developed in Sec. IV for investigating the k^2 dependence of the Pomeron propagator can be applied with very little modification to the formally similar problem of the subcritical Pomeron, that is, the Pomeron with intercept $\alpha_P(0)<1$. The Pomeron propagator at $k^2=0$ is modified from the usual form given in Eq. (2.1) to the form

$$i\Gamma^{1,1}(E,\delta_0)=E-\Sigma(E,\delta_0)-\delta_0+\delta\Delta\,, \tag{5.1}$$

with the condition that the Pomeron intercept lie at $J=1$ when $\delta_0=0$,

$$i\Gamma^{1,1}(0,0)=0\,. \tag{5.2}$$

That is, the new parameter δ_0 measures the displacement of the bare Pomeron intercept from the critical value which results in a renormalized intercept $\alpha_P(0)=1$. Again we define

$$Z_3{}^{-1}(E,\delta_0)=\frac{\partial}{\partial E}i\Gamma^{1,1}(E,\delta_0)\,, \tag{5.3}$$

and, in parallel to the definition of α' in Eq. (4.2), we define a renormalized intercept parameter δ,

$$-\delta(E,\delta_0)Z_3^{-1}(E,\delta_0)=\delta_0\frac{\partial}{\partial\delta_0}i\Gamma^{1,1}(E,\delta_0). \quad (5.4)$$

The renormalized coupling constant r is defined at the point

$$r(E,\delta)=(2\pi)^{3/2}Z_3^{3/2}\Gamma^{1,2}(E,E/2,E/2;\vec{k}=0,\delta_0), \quad (5.5)$$

with the dimensionless coupling g given by

$$g^2(E,\delta)=-\frac{r^2(E,\delta)}{\alpha_0'E}. \quad (5.6)$$

We shall be concerned with calculating Z_3 and Z_6 where

$$\delta=\delta_0 Z_6. \quad (5.7)$$

The dimensionless Z's are now functions of the two dimensionless parameters g and p, where

$$p=-4\delta/E. \quad (5.8)$$

Again proceeding just as in Secs. III and IV we write

$$\left.\frac{\partial \ln Z_3}{\partial g}\right|_p=\frac{\gamma(g,p)}{\beta(g,p)}. \quad (5.9)$$

In the one-loop approximation in perturbation theory one finds that

$$\gamma=-\frac{g^2}{16\pi}(1+\tfrac{1}{2}p)^{-1} \quad (5.10)$$

and, as usual,

$$\beta=-\frac{g}{2}\left(1-\frac{g^2}{g_1^2}\right), \quad (5.11)$$

where

$$16\pi/g_1^2(p)=16\ln\left(\frac{2+p}{1+p}\right)-\frac{6}{2+p}. \quad (5.12)$$

Proceeding as above one solves Eq. (5.10) to find

$$Z_3(g,p)=\left[1+\frac{g_0^2}{g_1^2(p)}\right]^{-C_3(p)}, \quad (5.13)$$

where

$$C_3(p)=-\frac{g_1^2(p)}{16\pi}(1+\tfrac{1}{2}p)^{-1} \quad (5.14)$$

Again it is possible to show that $C_3(p)$ should be independent of p. The expression above is in fact a slowly varying function of p, varying from $C_3(0)=-0.124$ to $C_3(\infty)=-0.20$. We again make the approximation of setting $C_3(p)=C_3(0)\equiv c_3$. Then Eq. (5.14) implies the approximation

$$g_1^2(p)\approx-16\pi c_3(1+\tfrac{1}{2}p). \quad (5.15)$$

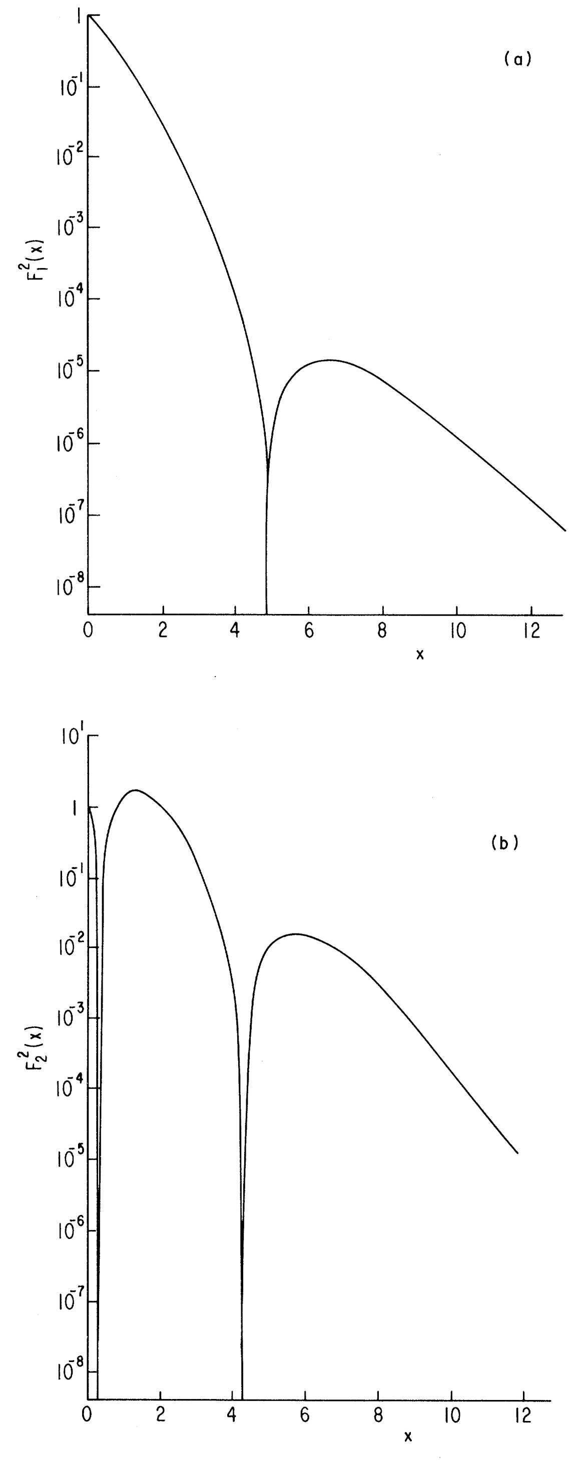

FIG. 4. (a) The square of the asymptotic scaling function $F_1^2(x)$ [Eq. (4.32)]. The values of the parameters are $c_3=-0.25$, $E_0=0.2$, and $\alpha_0'=0.45$. (b) The square of the first correction term $F_2^2(x)$ [Eq. (4.35)] to the asymptotic behavior of the Sommerfeld-Watson transform of the Pomeron propagator. The values of the parameters are the same as in (a).

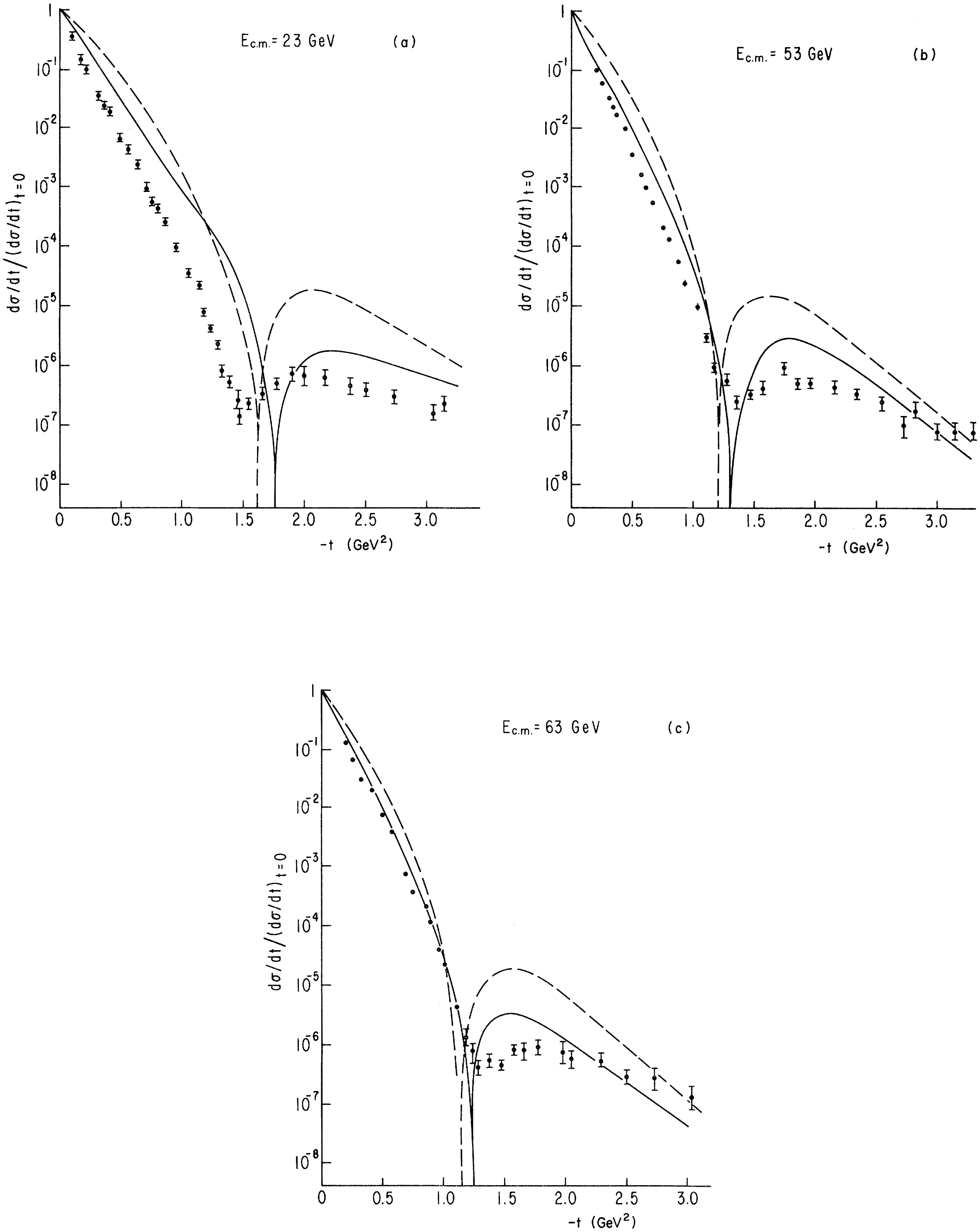

FIG. 5. Comparison between the ISR data, the differential cross section obtained by using the asymptotic form $d\sigma/dt \propto F^2(Y,k^2)$ [Eq. (4.30)] (dashed curve), and the two-term asymptotic expansion $d\sigma/dt \propto F^2(Y,k^2)$ [Eq. (4.34)] (solid curve) for (a) $\sqrt{s}=23$ GeV, (b) $\sqrt{s}=53$ GeV, and (c) $\sqrt{s}=$ 63 GeV. The values of the parameters are the same as in Fig. 4(a).

Finally, in this approximation one has for Z_3

$$Z_3^{-1} = \left[1 - \frac{E_0(p)}{E}\right]^{c_3}, \tag{5.16}$$

where

$$E_0(p) = \overline{E}_0(1 + \tfrac{1}{2}p)^{-1}, \tag{5.17}$$

$$\overline{E}_0 = -r_0^2/16\pi\alpha_0' c_3. \tag{5.18}$$

Comparing these results with Eqs. (4.12)–(4.15) one sees that they are identical if one identifies h with p. We can therefore apply the results of Sec. IV, suitably reinterpreted, to the solution of the problem of the noncritical Pomeron. However, there is one point at which a difference occurs. If we proceed as in Sec. IV B to integrate at fixed p to find $\Gamma^{1,1}(E,p)$ we find instead of Eq. (4.21) the result

$$i\Gamma^{1,1}(E,p) = -E_0(p)\int_0^{-E/E_0(p)} dx\, x^{-c_3}(1+x)^{c_3} \times\left[1 + \frac{p}{4}\left(1 - \frac{c_\delta}{1+x}\right)\right]. \tag{5.19}$$

Note in the final brackets the factor $p/4$, instead of the factor h which appears in Eq. (4.21). Note also that in this and all other formulas the critical exponent c_α is replaced by the exponent c_δ. In the one-loop approximation one finds that

$$c_\delta = -c_3, \tag{5.20}$$

in contrast to Eq. (4.20).

These small differences between the present problem and the problem of k^2 dependence treated in Sec. IV have a major effect on the structure of the J-plane singularities of the Pomeron propagator. We shall analyze in detail only the asymptotic form of the propagator and we shall assume that $\delta > 0$ [intercept $\alpha_P(0) < 1$]. The effect of the factor $p/4$ in Eq. (5.19) is to move the pole onto the positive real axis in the E plane. The branch point also moves onto the real x axis, and the singularities appear as shown in Fig. 6. This is in accord with one's expectations that a pole should be the leading singularity when the Pomeron intercept is less than unity.

Although the movement of the pole away from the branch cut simplifies the numerical evaluation, the expressions found in Sec. IV are formally almost unchanged. The asymptotic form of the cross section is given by

$$\sigma_T(Y,\delta_0) \propto \frac{(1-c_3)(\overline{E}_0 Y)^{-c_3}}{\Gamma(1-c_3)} F_\delta(x_\delta), \tag{5.21}$$

where

$$x_\delta = \frac{4\delta_0}{E_0}(\overline{E}_0 Y)^{1-c_\delta}, \tag{5.22}$$

and where $F_\delta(x) = F_1(x)$ is given by Eq. (4.32), except that $c_\alpha \to c_\delta$ and the pole position h_0 should be replaced by p_0, where

$$p_0 = \frac{4}{c_\delta - 1}. \tag{5.23}$$

For large Y the cross section falls off approximately exponentially with increasing rapidity Y,

$$\sigma_T(Y,\delta_0) \sim e^{-\Delta Y}, \tag{5.24a}$$

at a rate which can be found from the pole term to be

$$\Delta = \overline{E}_0(\delta_0/\overline{E}_0)^{1/(1-c_\delta)}(1-c_\delta)(1+c_\delta)^{c_\delta/(1-c_\delta)}. \tag{5.24b}$$

The results of this calculation of the Pomeron propagator for $\delta_0 > 0$ are the same as those found by Abarbanel *et al.*[15] There are three important rapidity domains. For $\overline{E}_0^{-1} > Y$ one can use perturbation theory in the sense described in Sec. II. For $\overline{E}_0^{-1} < Y < \Delta^{-1}$ one finds essentially the same results obtained in the case of the critical Pomeron. This is apparent from the "uncertainty principle" which indicates that at rapidity Y the resolution in the J plane is of order $\Delta J \approx 1/Y$. For $Y > \Delta^{-1}$ the renormalization-group analysis still holds but the renormalized pole dominates.

VI. CONCLUSIONS

We have developed, in Secs. III and IV of this paper, a general method for constructing explicit representations of the Pomeron propagator in the presence of additional dimensionless parameters,

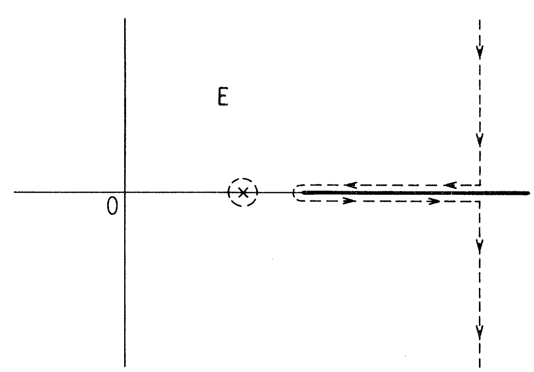

FIG. 6. Singularities of the asymptotic Pomeron propagator in the complex E plane for the case of a renormalized Pomeron intercept $\alpha_P(0) < 1$. The figure is drawn for the case $c_\delta = 0.2$.

and have applied the method to three examples: the introduction of a cutoff in k^2, the calculation of the angular distribution of the Pomeron-propagator contribution to elastic scattering, and the investigation of the behavior of the total cross section when the renormalized Pomeron intercept is slightly less than unity.

The representation with a cutoff in k^2 permits investigation of the regions of validity of the asymptotic and perturbation expansions. We find that in the one-loop approximation, a perturbation expansion of the Pomeron propagator in powers of the bare triple-Pomeron coupling r_0 is valid, provided that the intercept shift is evaluated nonperturbatively, and provided that $|1-J| > E_0$. In terms of rapidity $Y = \ln s$ this means that for $Y < E_0^{-1}$, the perturbation expansion is valid, whereas for $Y > E_0^{-1}$, one can use the asymptotic expansion. We expect the same results to hold in general, although perturbation theory will lead to an asymptotic (for small r_0) rather than a convergent series expansion of the scattering amplitude for $Y < E_0^{-1}$. The most recent measurement of r_0 combined with our one-loop calculation of the fixed point g_1^2 yields a value $E_0^{-1} = 6$, with uncertainties of the order of a factor 2. It seems that the transition region between perturbation theory and asymptotic expansions very likely lies in the Fermilab and ISR energy range. We emphasize that the existence of an energy region in which perturbation theory can be used—a conclusion which is nontrivial in the presence of the infrared singularities of the theory—is necessary if we are to be able to identify the parameters of the bare theory with physically measurable quantities.

The determination of the parameters of the theory by comparison with measurements at $Y < E_0^{-1}$ requires a formidable phenomenological effort, including many effects not considered in this paper, such as nonenhanced graphs, more general interactions, and multiloop diagrams. The importance of nonenhanced graphs and more general interactions is greatest at low energies, whereas the one-loop approximation is best at low energies. From Eq. (2.12) one sees that

$$\frac{g^2(E)}{g_1^2} = \frac{E_0}{E + E_0} \,. \tag{6.1}$$

At $E = E_0$, the effective coupling constant squared, $g^2(E_0)$, is only half as large as its asymptotic value $g^2(0) = g_1^2$. Quite possibly the one-loop approximation is useful at ISR energies, even though it has been found not to be quantitatively reliable at asymptotic energies.

The problems considered in Secs. IV and V, the k^2 dependence and the subcritical Pomeron [renormalized Pomeron intercept $\alpha_P(0) < 1$], are qualitatively different from the cutoff problem of Sec. III. The cutoff introduces a type of dimensionless parameter, $\omega = -bE/\alpha_0'$, which we might call an irrelevant parameter. That is, it has no effect on the asymptotic limit. As $E \to 0$, $\omega \to 0$, and all quantities return to the values taken in the absence of the cutoff. In other words, the solution is driven to the same stable fixed point, and the absence of dependence on the cutoff is an example of Wilson's universality.[12] On the other hand, the k^2 dependence and the subcritical Pomeron introduce "nontrivial" parameters. For example, the quantity δ introduced in Sec. V is a measure of how far the Pomeron intercept lies below unity. The corresponding dimensionless parameter, $p = -4\delta/E$, is nontrivial in that it does not leave the asymptotic behavior unaffected. The limit $E \to 0$ does not force quantities to their values in the critical theory with $\delta = 0$. In other words, the fixed point is unstable with respect to variations in the bare intercept.

Physically, the absence of a stable fixed point for the subcritical Pomeron is to be expected: At asymptotic energies the leading singularity is a simple renormalized Pomeron pole. Since there is no buildup of singularities, and no critical phenomenon, the asymptotic solution should not be characterized by an approach to a fixed point. It is nevertheless possible to extend the method developed in Sec. III for an irrelevant parameter to the case of nontrivial parameters. The trick is to form the integral representation of the propagator by integrating with the appropriate chosen dimensionless parameter held fixed (fixed h or p, not fixed k^2 or δ). The result for the subcritical Pomeron is that for sufficiently small δ_0, the displacement of the bare-Pomeron intercept below its critical value, there exists a large energy region $E_0^{-1} < Y < \Delta^{-1}$ in which the asymptotic solution found for the critical case is still valid to a good approximation. Only for $Y > \Delta^{-1}$ does the behavior change to that of a simple renormalized Pomeron pole. Since a value of Δ much larger than 0.01 would be difficult in fitting the rising cross section observed at the ISR, it is sufficient at today's energies to treat the Pomeron as critical, even if it should eventually turn out to be subscritical.

Our results on the k^2 dependence for the leading asymptotic behavior are in agreement with those of Abarbanel, Bronzan, Bartels, and Sidhu[5] provided that we set our critical exponents equal to their ϵ-expansion values. This is not too surprising since, although we work at $D = 2$, just as in their case we do not go beyond the one-loop approximation. In addition we have calculated the first correction term to the asymptotic behavior coming from the Pomeron propagator. The fact that this correction is small for reasonable values of the parameters is certainly encouraging. However, the remarkable similarity between theory and experiment should not be taken too seriously because there is not good justification for including only the Pomeron propagator at ISR energies. It should also be noticed that we have neglected the k^2 dependence of the particle-Pomeron couplings.

ACKNOWLEDGMENTS

We want to express our appreciation to John Bronzan for extensive and illuminating conversations during the initial period of this work. We are grateful to John Cardy, Moshe Moshe, and Frank Paige for numerous discussions and to Dan Birx for his help with the numerical computations. Finally, we would like to express our appreciation for the hospitality extended to us at the University of Washington where this work was begun.

---

*Work supported in part by the U. S. Energy Research and Development Administration and the National Science Foundation.

[1]H. D. I. Abarbanel and J. B. Bronzan, Phys. Lett. 48B, 345 (1974); Phys. Rev. D 9, 2397 (1974).

[2]A. A. Migdal, A. M. Polyakov, and K. A. Ter-Martirosyan, Phys. Lett. 48B, 239 (1974); Zh. Eksp. Teor. Fiz. 67, 84 (1974) [Sov. Phys.—JETP 40, 420 (1974)].

[3]R. L. Sugar and A. R. White, Phys. Rev. D 10, 4074 (1974).

[4]W. R. Frazer and M. Moshe, Phys. Rev. D 12, 2370 (1975).

[5]H. D. I. Abarbanel, J. Bartels, J. B. Bronzan, and D. Sidhu, Phys. Rev. D 12, 2799 (1975).

[6]M. Baker, Nucl. Phys. 80B, 62 (1974); J. B. Bronzan and J. W. Dash, Phys. Rev. D 10, 4208 (1974); 12, 1850 (1975); J. W. Dash and S. Harrington, Phys. Lett. 59B, 249 (1975); J. Ellis and R. Savit, Nucl. Phys. B94, 477 (1975). See also Refs. 1 and 2.

[7]J. B. Bronzan, J. A. Shapiro, and R. L. Sugar, Phys. Rev. D 14, 618 (1976).

[8]Fermilab Single-Arm Spectrometer Group, University of Utah Report No. UUCR-155 (unpublished). This experiment provides reliable normalization by presenting the ratio of the high-mass dissociation to the elastic scattering, measured in the same experiment. This method also has the advantage of practically eliminating dependence on the bare Pomeron intercept.

[9]The factor $\sqrt{2}$ is a result of confusion in the definition of the coupling constant. In terms of the Abarbanel-Bronzan Reggeon rules, which we use, Eq. (112) of Ref. 5 requires an additional factor $\sqrt{2}$ and Eq. (12) of Ref. 4 requires an additional factor $\pi\sqrt{2}$.

[10]P. Suranyi, Phys. Rev. D 14, 1149 (1976).

[11]J. Wallace, Ph.D. dissertation, UCSD, 1975 (unpublished).

[12]K. G. Wilson and J. B. Kogut, Phys. Rep. 12C, 75 (1974).

[13]A. Capella and J. Kaplan, Phys. Lett. 52B, 498 (1974).

[14]An approximate expression for the real part of the amplitude near $t=0$ may be obtained by differentiating the imaginary part with respect to $Y=\ln s$. That is,

$$\mathrm{Re}\left[\frac{1}{s}A(Y,-t)\right]=\frac{\pi}{2}\frac{d}{dY}\left[\frac{1}{s}\,\mathrm{Im}A(Y,-t)\right].$$

The function $F(Y,-t)$ in Eq. (4.38) is related to $A(Y,-t)$ by

$$F(Y,-t)=\frac{1}{s}\,\mathrm{Im}A(Y,-t).$$

[15]H. D. I. Abarbanel, J. B. Bronzan, A. Schwimmer, and R. L. Sugar, Phys. Rev. D 14, 632 (1976).