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Self-consistent quark bag in three space dimensions*

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A schematic model of a quark bag with scalar gluons is considered quantitatively in three space dimensions. Stable spherically symmetric solutions of c-number self-consistent bag equations are obtained numerically. The structure constants of hadrons are calculated as functions of the coupling constant and the gluon mass. Good agreement of absolute values of the proton magnetic moment and the bag mass with experiment is found.

The forces between the quarks are not understood. Bardeen *et al.* have suggested[1] that the structure of hadrons may be described in a model theory in which the forces between quarks are mediated by scalar mesons (SLAC quark bag). In our model we adopt the simplest choice for the Lagrangian $\mathcal{L}$ of the interacting quark-gluon system and investigate the properties of solutions in the large coupling constant, $g^2 \gg 10$, limit:

$$\mathcal{L} = \overline{\Psi}(\gamma \cdot p - m_q)\Psi - \tfrac{1}{2}(\partial\Phi \cdot \partial\Phi + \mu^2\Phi^2) + g\Phi\overline{\Psi}\Psi \,, \quad (1)$$

where Ψ is the quark field, all quarks have mass m_q, and μ is the mass of the scalar gluon field.

Our investigations differ in two essential points from those of Refs. 1–3: (1) We discuss the quark bag quantitatively and obtain actual solutions by extensive numerical studies in the physical space rather than in a one-dimensional space; (2) we choose the gluon potential $U(\Phi^2) = -\frac{1}{2}\mu^2\Phi^2$ and show that in the physical space the existence of the quark bag does not critically depend on an inherent nonlinearity of the gluon field but is rather a consequence of the quark-gluon interaction.

Following Ref. 1 we reduce the q-number theory to the associated c-number bag model by considering an expectation value of the Hamiltonian belonging to $\mathcal{L}$. Our trial state $|t\rangle$ is formed in the direct-product space of mesons and fermions from a localized, coherent meson state[1,4] $|c\rangle$ and a quasifermion state[5,1] $|f\rangle$. We work in the zero-average-momentum frame of our trial state[1] and obtain for the dominant term

$$M_{(N)}^{(g^2)} \equiv \langle t|H|t\rangle$$
$$= \int d^3x \left(\sum_i (-)^{p_i} \{\psi_i^\dagger [\vec{\alpha}\cdot\vec{p} + \beta(m_q - g\phi_c)]\psi_i\} + \tfrac{1}{2}(\vec{\nabla}\phi_c)^2 + \tfrac{1}{2}\mu^2\phi_c^2 \right). \quad (2)$$

We understand the resulting c-number function M as the quark-bag mass. The subscript N stresses its dependence on the number of quarks present in the bag. M is a functional of the mean meson field $\phi_c = \langle t|\Phi|t\rangle$ and of the quark wave functions $\psi_i = \langle t|b_i^\dagger\psi|t\rangle$.

We note that M, Eq. (2), is a bounded-below functional of $\{\psi_i, \phi_c\}$ since ψ_i are eigenmodes of the Dirac field. The negative-frequency modes become positive-energy antiparticle states in consequence of the normal ordering of anticommuting fermion operators. This gives rise to the factor $(-)^{p_i}$; $p=0$ for positive-frequency modes and $p=1$ for negative-frequency modes. We consider Eq. (2) as the basis of our investigations.

The model defined by Eq. (2) is not to be taken seriously as an ultimate phenomenological description of the hadrons, which would perhaps include other mesons than the scalar gluon and a detailed treatment of the internal quark variables. However, it appears to provide a useful test case for the investigation of the consequences of quark bags in three-space dimensions, in particular, their hadronic structure.

The normalization of the quark wave function ψ_i is

$$\int \sum_{i=1}^{N} \psi_i^\dagger \psi_i d^3x = N \,. \quad (3)$$

N is 3 for the (qqq) baryons, and 2 for the $(\overline{q}q)$ mesons. Each ψ_i is normalized to unity. We solve the self-consistent set of Dirac and Klein-Gordon equations following from variation of M, Eq. (2), subject to the normalization (3) for the lowest-positive-energy state. With (dropping henceforth the indices i, c)

$$(4\pi)^{1/2} r\psi^\dagger(\vec{r}) = (u(r), 0, iv(r)\cos\gamma, -iv(r)\sin\gamma e^{-i\xi}) \quad (4)$$

for the spin-up state and a similar wave function for the spin-down state we find

$$u' - r^{-1}u - (m_q - g\phi + E_q)v = 0 \,, \quad (5a)$$

$$-v' - r^{-1}v + (m_q - g\phi - E_q)u = 0 \,, \quad (5b)$$

$$-r^{-1}(rg\phi)'' + \mu^2 g\phi = Ng^2(u^2 - v^2)r^{-2} \,. \quad (5c)$$

All radial quark densities are equal, even if different spin states are occupied. The $(-)^{p_i}$ sign and the sign of the negative-frequency scalar density cancel each other. Therefore all sums over the quark wave functions reduce to N times a single term in Eq. (5c) for an arbitrary number of quarks and antiquarks in the bag. For large r Eqs. (5) decouple; the spectrum of the Dirac equation is of the usual type with bound states for $m_q > E_q > -m_q$ and the continuum solutions elsewhere.

The coupled nonlinear Eqs. (5) have been solved numerically for various values of g^2, μ. Assuming a form for $(g\phi)_0$, we obtained u_0 and v_0 from Eqs. (5a) and (5b). Then $(g\phi)$ was calculated by solving Eq. (5c). The above steps were repeated until a self-consistent solution emerged. A solution is characterized uniquely by the choice of N, μ/m, and g^2, and can be obtained from different starting functions $(g\phi)_0$. To prove stability of our solutions it would be necessary to show that M is increased in second order by arbitrary variations of ϕ, with u and v chosen to give the least positive $E_q[\phi]$. The necessary constrained variation calculation requires a major computational effort and has not been carried out in that form. However, a larger number of $\delta\phi$ were chosen at random and the change in M was positive in every case. A separate investigation of nonradial solutions has shown that the spherical solutions are stable against deformations. In passing we note that our self-consistent solutions are stable solitons of relativistically invariant field theory in three space dimensions.[6]

In Fig. 1 we show some of the typical solutions. The self-consistent quark mass $m^* = m_q - g\phi$ is graphed for several values of the gluon mass and coupling constant. For $\mu = 0.04m_q$ and $\mu = 0.02m_q$ we also show the radial vector $[\rho_V = \psi^\dagger\psi/(4\pi)]$ and scalar $[\rho_S = \bar{\psi}\psi/(4\pi)]$ densities (in Fig. 1 ρ_S is enhanced by a factor of 10 in relation to ρ_V). For μ of the order of the bare quark mass, $m^*(r)$ has a pronounced minimum. For coupling constants smaller than those considered here, i.e., for $g^2 \lesssim 10$, we have also found solutions that were evenly distributed over the volume, in contrast to a conjecture made in Ref. 1.

In Fig. 2 the masses of the bound states $M_{(N)}$ are given as functions of g^2 for the gluon masses of $0.4m_q$, $0.1m_q$, $0.02m_q$. We see that smaller gluon masses give considerably smaller bag masses. As g^2 rises we find a point g_c^2 such that

$$2M_{(2)} < \mu, \quad g^2 > g_c^2(\mu). \tag{6}$$

Beyond this point quark bags are stable against annihilation into a gluon. We also found that $M_{(N)}$ can be made arbitrarily small (in units of m_q), as g^2 is increased, for fixed gluon mass μ.

Having calculated the wave functions we can proceed to evaluate the hadronic structure. We will use as an example the SU(6) classification and an extra internal quantum number — color — in which the baryonic wave functions are antisym-

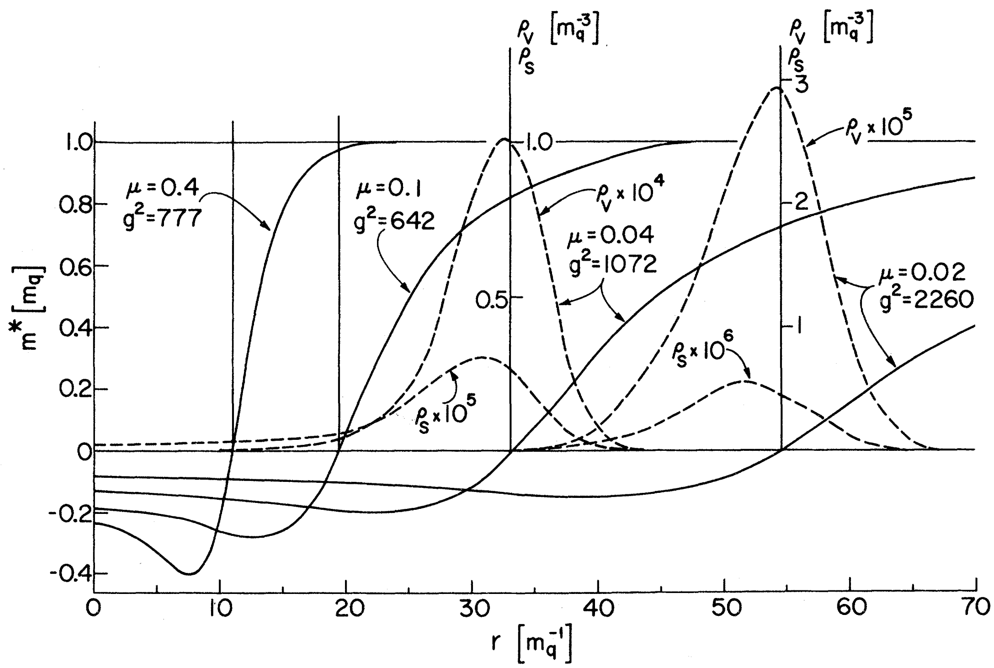

FIG. 1. The bound-quark mass $m^* = m_q - g\phi$ (in units of m_q) as a function of r (in units of m_q^{-1}) for $\mu = 0.02m_q$, $0.1m_q$, $0.4m_q$ and $g^2 = 2260$, 1072, 642, 777 correspondingly. The dashed curves are the vector and scalar densities for $\mu = 0.04m_q$, $g^2 = 642$ and $\mu = 0.02m_q$, $g^2 = 2260$. The vector density is normalized to unity.

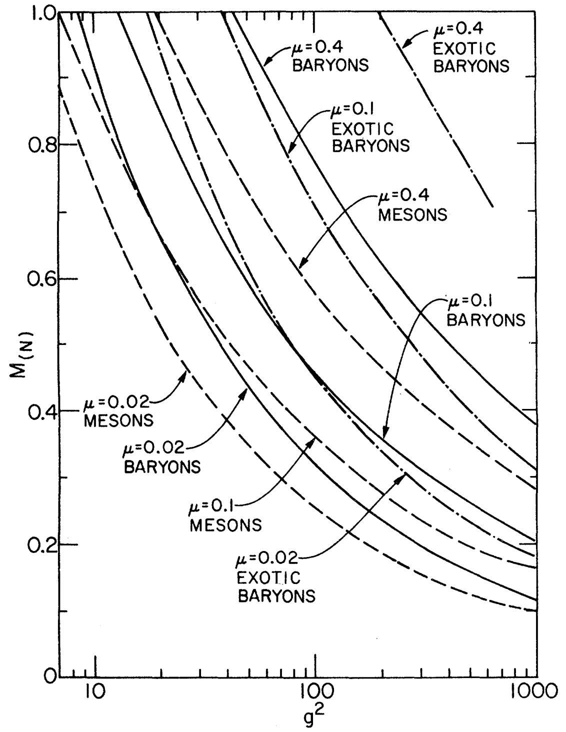

FIG. 2. The mass $M_{(N)}$ of the bag in units of the quark mass for several quark states [baryons ($N=3$), mesons ($N=2$), and exotic baryons ($N=6$)] as a function of the coupling constant g^2 for values of the gluon mass $\mu=0.02m_q$, $0.1m_q$, $0.4m_q$.

metric.[1] Introduction of color excludes all colored states from the spectrum.[1] As was shown in Fig. 2, we have found weakly bound exotic colorless states. When more complicated interactions are included in our model, those may easily become unbound.

Since we do not know the absolute value of the bare quark mass m_q, which fixes the scale in our model, we may consider only quantities that are scale independent. The most prominent ones are the products of the rms radius of the baryons and mesons with their masses [Figs. 3(a) and 3(b)]. The experimental number to compare with for baryons is, most likely, the product of the proton charge radius, 0.8 fm, with the average mass of the $\underline{56}$ multiplet, $M_{\underline{56}}=1280$ MeV, which is 5.2 [$\hbar c$]. We see that this lies well within values spanned by our calculations.

We can also calculate the absolute value of the magnetic moment of the proton, using as the basic unit $e\hbar/2M_{\underline{56}}c$. Scaling up the experimental value to account for the larger multiplet mass than that of the proton, we obtain for comparison with Fig. 3(c) a value $\mu_p^{\exp}=2.79(1280/938)=3.8$. μ_p was calculated from[1]

$$\mu_p=-\tfrac{4}{3}\int_0^\infty vur\,dr\,M_{(3)}\,. \tag{7}$$

We consider also the axial-vector coupling constant g_A of the neutron decay process[1]

$$g_A=\tfrac{5}{3}\int_0^\infty (u^2-\tfrac{1}{3}v^2)\,dr\,. \tag{8}$$

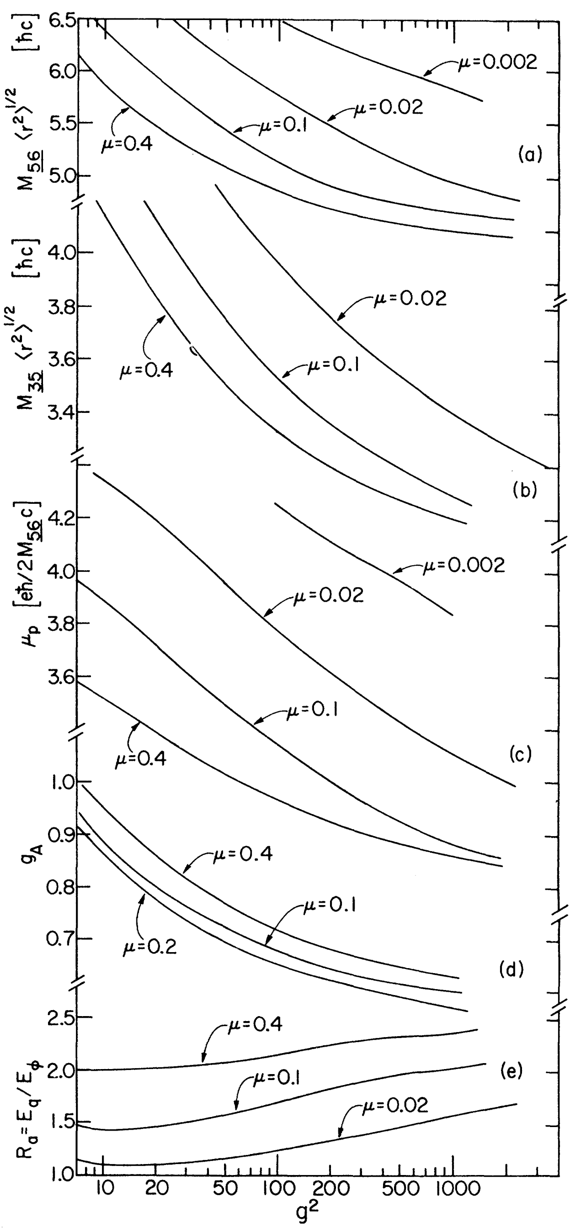

FIG. 3. The structure constants of SU(6) hadrons as a function of the coupling constant g^2 for values of the gluon mass $\mu=0.02m_q$, $0.1m_q$, $0.4m_q$. (a) The product of the mass of the baryon with its size ($\langle r^2\rangle^{1/2}$). (b) The same as in (a) for mesons. (c) The magnetic moment of protons. (d) The axial-vector coupling constant g_A. (e) The ratio R_a of the energy (momentum) carried by the quarks to that carried by the neutral glue.

Here we do not expect a good agreement with the experimental value, $g_A^{exp} = 1.25$, since the axial-vector current is nonconserved in all models with massive bare quarks. The results are shown in Fig. 3(d).

Another interesting function is the ratio R_a of the mass of the baryon carried by the charged quarks to that carried by the neutral glue. From deep-inelastic experiments it is believed that this number is of the order of unity. In our model we have

$$M_{(N)} = NE_q(1 + R_a^{-1}), \tag{9a}$$

and using Eqs. (5) and (2)

$$R_a = \frac{2E_q}{\int_0^\infty g\phi(u^2 - v^2)dr}. \tag{9b}$$

In Fig. 3(e) our results are shown: $1 \lesssim R_a \lesssim 2$. For the SLAC bag[1] $R_a = 2$ and for the MIT bag[2] $R_a = 3$.

The strongly bound solutions show very small expectation values of the quark mass $m^* = m_q - g\phi$ in units of $M_{(3)}$. We find, for example, for the bag with $\mu = 0.02\ m_q$ shown in Fig. 1 that the effective quark mass $m_{q\,\mathrm{eff}} = \int m^* \rho_V d^3r$ to be $0.013 M_{(3)} = 17$ MeV. Empirically one needs a number around this magnitude to explain the mass splitting in the $\underline{56}$ multiplet.[7]

Finally we mention that the hadronic structure constants are not independent of each other. We can express the matrix element for μ_p with the help of Eq. (5) in terms of g_A,

$$\mu_p = (0.5 + 0.3 g_A) N(1 + R_a^{-1}). \tag{10}$$

Furthermore, the deeply bound solutions satisfy $E_q \cong \langle r^2 \rangle^{-1/2}$. Therefore

$$\langle r^2 \rangle^{1/2} M_{(N)} \approx N(1 + R_a^{-1}) \tag{11}$$

which implies for $\underline{56}$ baryons that $R_a \approx 1.4$, if we want a bag of the right mass and size. From Eqs. (11) and (10) we find for protons

$$g_A = 3.3\,[\mu_p / (\langle r^2 \rangle^{1/2} M_{\underline{56}}) - 0.5], \tag{12}$$

which suggests that if we want the right value of the magnetic moment, then g_A will be ~ 0.7.

The above qualitative discussion shows the intercorrelation of the hadronic structure. There is *a priori* no reason that our self-consistent solutions yield the correct hadronic structure. A surprising characteristic of our model is that our solutions have the desired properties. We have studied several other self-consistent bag models and found significantly worse phenomenological behavior.

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