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# Massive gauge fields in the SU(4) $\sigma$ model*

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We investigate the effects of adding a set of $1^{\pm}$ gauge fields to a chiral SU(4) $\times$ SU(4) $\sigma$ model of the $0^{\pm}$ fields. Special attention is given to calculating the entire mass spectrum with the choice of symmetry-breaking terms which transform as the smaller-dimensional representations of SU(4). Hadronic currents and some vector-meson decays are also discussed.

## I. INTRODUCTION

The discovery[1] of narrow $1^-$ resonances in the 3–4 GeV range has increased the speculation that SU(4) rather than SU(3) might be the symmetry structure of the hadrons. To further investigate this possibility we wish to consider adding a set of $1^{\pm}$ gauge fields to a previously discussed[2] SU(4) $\sigma$ model.[3]

The extension of the SU(4) $\sigma$ model in this manner[4] allows for a large number of possible symmetry-breaking terms which can then lead to an extremely complicated Lagrangian. In our case we have assumed that all symmetry breaking occurs in the $0^{\pm}$ potential terms, and that the possible symmetry-breaking terms themselves transform as the smaller-dimensional representations of SU(4). Thus, all the terms involving the gauge mesons are chiral-invariant. We then calculate the vector and axial-vector mass spectrum and fit the $\eta, \eta', \eta''$ masses. Assuming renormalizability of the potential we then calculate the pseudoscalar- and scalar-meson masses. As a check of the model, we also calculate various two-body vector-meson decays. Hadronic currents which can be used in weak decay models are also calculated. The $f_K/f_\pi$ ratio is compared to the experimental results using the standard Cabibbo scheme.

## II. MASS MATRIX

The model of Ref. 2 is described by

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{Tr}\,(\partial_\mu M \partial_\mu M^\dagger) - V\,, \tag{2.1}$$

where

$$\begin{aligned} &V = V_0 + V_{\rm SB}\,,\\ &M_a^b = S_a^b + i\,\phi_a^b\,,\\ &(M^\dagger)_a^b = S_a^b - i\,\phi_a^b\,, \end{aligned} \tag{2.2}$$

and $S_a^b$ is the 16-plet of scalar fields and $\phi_a^b$ is the pseudoscalar 16-plet. $V_0$ is the most general potential without derivatives of the following chiral SU(4)$\times$SU(4) invariants:

$$\begin{aligned} &I_1 = \mathrm{Tr} MM^\dagger\,,\\ &I_2 = \mathrm{Tr} MM^\dagger MM^\dagger\,,\\ &I_3 = \mathrm{Tr} MM^\dagger MM^\dagger MM^\dagger\,,\\ &I_4 = \mathrm{Tr} MM^\dagger MM^\dagger MM^\dagger MM^\dagger\,,\\ &I_5 = \det M + \det M^\dagger\,. \end{aligned} \tag{2.3}$$

In analogy to the basic SU(3) $\sigma$ model, $V_{\rm SB}$ is taken to be a simple linear and bilinear combination of $M_a^b$ and $(M^\dagger)_a^b$. Since $M_a^b$ transforms as $(4,4^*)$ and $(M^\dagger)_a^b$ as $(4^*,4)$, the linear term is proportional to $(M^\dagger)_a^a + M_a^a$. The coefficients of these terms can be considered as analogous to "quark-mass terms."

The bilinear terms are of the form $\sum_b M_a^b (M^\dagger)_b^a$ +H.c. and $\sum_b M_a^b M_b^a$ +H.c. The first of these transforms as $(1,15)\oplus(15,1)$ and it has been discussed in Ref. 2. The second term transforms as $(10,10^*)$ $\oplus(10^*,10)\oplus(6,6)$ (see the Appendix for further details). This is the bilinear term to be used in this paper since its irreducible representations are of lower dimension than those of the $M_a^b(M^\dagger)_b^a$ term.

Thus $V_{\rm SB}$ is then

$$V_{\rm SB} = -\sum_a A_a[(M)_a^a + (M^\dagger)_a^a] + \tfrac{1}{2}\sum_{a,b} B_a[(M^\dagger)_a^b (M^\dagger)_b^a + (M)_a^b (M)_b^a]\,, \tag{2.4}$$

with

$$\sum_a B_a = 0\,. \tag{2.5}$$

As in Ref. 2, we define the "ground state" as

$$\langle S_a^b\rangle_0 = \alpha_a\,\delta_a^b\,, \tag{2.6}$$

where $\langle\ \rangle_0$ indicates that the enclosed object should be evaluated at its classical equilibrium point. Assuming isotopic spin invariance for the entire Lagrangian, we have

$$\alpha_1 = \alpha_2\,,\quad A_1 = A_2\,,\quad B_1 = B_2\,. \tag{2.7}$$

It is also convenient to define

$$\alpha = \alpha_1 = \alpha_2, \quad w = \frac{\alpha_3}{\alpha}, \quad w' = \frac{\alpha_4}{\alpha}. \tag{2.8}$$

We then add the gauge fields[3] to Eq. (2.1) in the usual manner. We replace $\partial_\mu$ by its covariant derivative $\mathfrak{D}_\mu$ and we add a spin-1 kinetic Yang-Mills[5] term $F_{\mu\nu}$. We also add a chiral-invariant term which, after a spontaneous breakdwon of symmetry, will yield masses for the gauge particles. The Lagrangian (2.1) then becomes

$$\mathfrak{L} = -\tfrac{1}{2}\,\mathrm{Tr}(\mathfrak{D}_\mu M \mathfrak{D}_\mu M^\dagger) - \tfrac{1}{2}\,\mathrm{Tr}(F^l_{\mu\nu} F^l_{\mu\nu} + F^r_{\mu\nu} F^r_{\mu\nu}) - V_0 - V_{\mathrm{SB}} + \mathfrak{L}_I, \tag{2.9}$$

where

$$\begin{aligned}
\mathfrak{D}_\mu M &= \partial_\mu M - igl_\mu M + igMr_\mu, \\
\mathfrak{D}_\mu M^\dagger &= \partial_\mu M^\dagger + igM^\dagger l_\mu - igr_\mu M^\dagger, \\
l_\mu &= \tfrac{1}{2}(V_\mu + A_\mu), \\
r_\mu &= \tfrac{1}{2}(V_\mu - A_\mu), \\
F^l_{\mu\nu} &= \partial_\mu l_\nu - \partial_\nu l_\mu - ig[l_\mu, l_\nu]_-, \\
F^r_{\mu\nu} &= \partial_\mu r_\nu - \partial_\nu r_\mu - ig[r_\mu, r_\nu]_-, \\
\mathfrak{L}_I &= -C\,\mathrm{Tr}(l_\mu M r_\mu M^\dagger) - \frac{D}{2}\,\mathrm{Tr}(l_\mu l_\mu M M^\dagger + r_\mu r_\mu M^\dagger M).
\end{aligned} \tag{2.10}$$

Here $V_\mu$ and $A_\mu$ are respectively the vector and axial-vector gauge fields. The constants $A_a$, $B_a$, $C$, $D$, and $g$ are to be determined later.

As in Ref. 2, we expand (2.9) in normal coordinates where

$$S = \hat{S} + \langle S \rangle_0 \tag{2.11}$$

and

$$\phi = \hat{\phi}.$$

We then identify the coefficients of second-order terms as the masses of the mesons. After the substitution of (2.11) into (2.9) we find that it is necessary to redefine the vector and axial-vector gauge fields in order to eliminate terms proportional to $V_\mu \partial_\mu \hat{S}$ and $A_\mu \partial_\mu \hat{\phi}$. After this is done it is then necessary to "renormalize" the scalar and pseudoscalar fields so that the coefficients of the kinetic terms are all $-\frac{1}{2}$. The physical fields, denoted by the tilde, are then given by

$$\begin{aligned}
\hat{\phi}^b_a &= \frac{1}{Z_{ab}}\,\tilde{\phi}^b_a, \quad \hat{S}^b_a = \frac{1}{X_{ab}}\,\tilde{S}^b_a, \\
(V_\mu)^b_a &= (\tilde{V}_\mu)^b_a + \frac{\Lambda_{ab}}{X_{ab}}\,\partial_\mu(\tilde{S})^b_a, \\
(A_\mu)^b_a &= (\tilde{A}_\mu)^b_a + \frac{\Gamma_{ab}}{Z_{ab}}\,\partial_\mu(\tilde{\phi})^b_a,
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
(Z_{ab})^2 &= 1 - \frac{g^2(\alpha_a + \alpha_b)^2}{4m^2(A^b_a)}, \\
(X_{ab})^2 &= 1 - \frac{g^2(\alpha_a - \alpha_b)^2}{4m^2(V^b_a)}, \\
\Lambda_{ab} &= \frac{ig(\alpha_a - \alpha_b)}{2m(V^b_a)}, \\
\Gamma_{ab} &= \frac{g(\alpha_a + \alpha_b)}{2m(A^b_a)}, \\
m^2(V^b_a) &= \frac{1}{4}\left[(\alpha_a - \alpha_b)^2\left(\frac{2g^2 + D - C}{2}\right) + (\alpha_a + \alpha_b)^2\left(\frac{D + C}{2}\right)\right], \\
m^2(A^b_a) &= \frac{1}{4}\left[(\alpha_a + \alpha_b)^2\left(\frac{2g^2 + D - C}{2}\right) + (\alpha_a - \alpha_b)^2\left(\frac{D + C}{2}\right)\right].
\end{aligned} \tag{2.13}$$

Here $m(V^b_a)$ and $m(A^b_a)$ are the masses of the appropriate vector and axial-vector mesons.

The mass terms for the gauge mesons are usually added by a term

$$m_0^{\,2}\,\mathrm{Tr}(V_\mu V_\mu + A_\mu A_\mu), \tag{2.14}$$

but $\mathfrak{L}_I$ yields a more realistic mass spectrum[4] than does (2.14). In fact, from (2.13) we see that if we identify $(1/\sqrt{2})[(\tilde{V}_\mu)^1_1 - (\tilde{V}_\mu)^2_2] = \rho(0.780)$, $(\tilde{V}_\mu)^3_3 = \Phi(1.020)$, $(\tilde{V}_\mu)^4_4 = \Psi(3.095)$, and $(1/\sqrt{2})[(\tilde{A}_\mu)^1_1 - (\tilde{A}_\mu)^2_2] \equiv \rho_A = A_1(1.04)$, the rest of the spin-1 masses can be calculated. They[6] are listed in Table I.

The $\rho$ and $\omega$ are degenerate as are the $\rho_A$ and $\omega_A$.

The chiral invariance of $V_0$ gives Ward-type identities among some spin zero masses. These can be obtained by referring to Eqs. (2.7)–(2.11) of

TABLE I. Input values (Ref. 9) and mass predictions for the spin-1 mesons. All masses are given in GeV with experimental values in parentheses.

| Quantity | Prediction | Quantity | Prediction |
|---|---|---|---|
| $m_\rho(0.780)$ | Input | $m_{\phi_A}(1.42?)$ | 1.359 |
| $m_\phi(1.020)$ | Input | $m_{D^*}$ | 2.477 |
| $m_\psi(3.095)$ | Input | $m_{F^*}$ | 2.479 |
| $m_{\rho_A}(1.04)$ | Input | $m_{D^*_A}$ | 2.831 |
| $m_{K^*}(0.892)$ | 0.914 | $m_{F^*_A}$ | 2.933 |
| $m_{K^*_A}(1.24?)$ | 1.206 | $m_{\psi_A}$ | 4.127 |

Ref. 2 and noting that we are now dealing with the "renormalized" fields $\tilde{S}$ and $\tilde{\phi}$. Thus

$$m^2(\tilde{\phi}_a^b)=\left\langle\frac{\partial^2 V}{\partial\tilde{\phi}_a^b\,\partial\tilde{\phi}_b^a}\right\rangle_0 =\frac{1}{(Z_{ab})^2}\,\frac{1}{\alpha_a+\alpha_b}\times[2(A_a+A_b)+(\alpha_a-\alpha_b)(B_a-B_b)]\quad(a\neq b)$$

and (2.15)

$$m^2(\tilde{S}_a^b)=\left\langle\frac{\partial^2 V}{\partial\tilde{S}_a^b\,\partial\tilde{S}_b^a}\right\rangle_0 =\frac{1}{(X_{ab})^2}\,\frac{1}{\alpha_a-\alpha_b}\times[2(A_a-A_b)-(\alpha_a+\alpha_b)(B_a-B_b)]\quad(a\neq b).$$

By referring to Eq. (2.13), we see that

$$Z_\pi\equiv Z_{11}=Z_{12}=Z_{22}=Z_{33}=Z_{44}. \tag{2.16}$$

The mass-squared matrix for the four neutral pseudoscalars $(\pi^0,\eta,\eta',\eta'')$ is just

$$\left\langle\frac{\partial^2 V}{\partial\tilde{\phi}_a^a\,\partial\tilde{\phi}_b^b}\right\rangle_0=\frac{2}{(Z_\pi)^2}\begin{bmatrix}\frac{A_1}{\alpha_1}-U\frac{\alpha_2\alpha_3\alpha_4}{\alpha_1} & -U\alpha_3\alpha_4 & -U\alpha_2\alpha_4 & -U\alpha_2\alpha_3\\ -U\alpha_3\alpha_4 & \frac{A_2}{\alpha_2}-U\frac{\alpha_1\alpha_3\alpha_4}{\alpha_2} & -U\alpha_1\alpha_4 & -U\alpha_1\alpha_3\\ -U\alpha_2\alpha_4 & -U\alpha_1\alpha_4 & \frac{A_3}{\alpha_3}-U\frac{\alpha_1\alpha_2\alpha_4}{\alpha_3} & -U\alpha_1\alpha_2\\ -U\alpha_2\alpha_3 & -U\alpha_1\alpha_3 & -U\alpha_1\alpha_2 & \frac{A_4}{\alpha_4}-U\frac{\alpha_1\alpha_2\alpha_3}{\alpha_4}\end{bmatrix}, \tag{2.17}$$

where

$$U\equiv\left\langle\frac{\partial V_0}{\partial I_5}\right\rangle_0. \tag{2.18}$$

This is just Eq. (3.3) of Ref. 2 with the substitution of $2/(Z_\pi)^2$ for the factor 2 multiplying the matrix. By Eq. (2.15) we can identify

$$2\frac{A_1}{\alpha}=(Z_\pi)^2 m^2(\pi^0)$$

and

$$2\frac{A_3}{\alpha}=(Z_\pi^2)\left[(1+w)m^2(K)\left(\frac{Z_K}{Z_\pi}\right)^2-m^2(\pi^0)-\frac{w-1}{(Z_\pi)^2}(B_3-B_1)\right]. \tag{2.19}$$

From (2.13) we see that

$$w=\frac{m(\phi)}{m(\rho)}\simeq1.307$$

and (2.20)

$$w'=\frac{m(\psi)}{m(\rho)}\simeq3.968\,.$$

We can then use the method described in the Appendix of Ref. 2 to calculate the mass of the $\eta''$ in terms of $B_3-B_1$ and $(Z_K/Z_\pi)$, using as additional input,

$$\begin{aligned}(\pi^0)^2&=0.0182\ \text{GeV}^2,\\ (K^0)^2&=13.60(\pi^0)^2,\\ (\eta)^2&=16.54(\pi^0)^2,\\ (\eta')^2&=50.35(\pi^0)^2,\end{aligned} \tag{2.21}$$

where for brevity, the particle mass has been replaced by its symbol. Equation (2.15) tells us that with the exception of $m^2(\tilde{S}_1^3)\equiv m^2(\kappa)$, this is all we can calculate. From (2.17) we see that the $\eta$-$\eta'$-$\eta''$ mass matrix does not depend on the value of $B_4$, and from (2.15) chiral invariance gives no information on the masses of the $I=1$ scalar mesons. Thus, as an additional condition we shall assume that the potential $V$ is renormalizable. With the notation that

$$V_i\equiv\left\langle\frac{\partial V_0}{\partial I_i}\right\rangle_0,\quad V_{11}=\left\langle\frac{\partial^2 V_0}{\partial I_1\partial I_1}\right\rangle_0, \tag{2.22}$$

we note that the renormalizable $V_0$ is just

$$V_0=[V_1-\langle I_1\rangle_0 V_{11}]I_1+\tfrac{1}{2}V_{11}(I_1)^2+V_2 I_2+UI_5\,. \tag{2.23}$$

The additional information comes from the stability equations

$$\left\langle\frac{\partial V}{\partial S_a^a}\right\rangle_0=\left\langle\frac{\partial V_0}{\partial S_a^a}\right\rangle_0+\left\langle\frac{\partial V_{SB}}{\partial S_a^a}\right\rangle_0 =0, \tag{2.24}$$

which take on the explicit form

$$\alpha_1(V_1+2V_2\alpha_1^2)+\alpha_2\alpha_3\alpha_4U=A_1-B_1\alpha_1\,, \qquad (2.25a)$$

$$\alpha_3(V_1+2V_2\alpha_3^2)+\alpha_1\alpha_2\alpha_4U=A_3-B_3\alpha_3\,, \qquad (2.25b)$$

$$\alpha_4(V_1+2V_2\alpha_4^2)+\alpha_1\alpha_2\alpha_3U=A_4-B_4\alpha_4\,. \qquad (2.25c)$$

From Eqs. (2.16) and (2.25) we now see that

$$m^2(\tilde{S}_1^2)=\frac{2}{(X_{12})^2}\left[(V_1+B_1)+6V_2\alpha^2-ww'U\alpha^2\right], \qquad (2.26)$$

where

$$V_1+B_1=\frac{\frac{A_1}{\alpha}w^2-\frac{A_3}{\alpha w}+U\alpha^2w'\left(\frac{1}{w}-w^3\right)+(B_3-B_1)}{w^2-1}, \qquad (2.27)$$

$$V_2\alpha^2=\frac{\frac{A_1}{\alpha}-\frac{A_3}{\alpha w}-(B_3-B_1)-U\alpha^2w'\left(w-\frac{1}{w}\right)}{2(1-w^2)}$$

and from the trace of $\langle\partial^2V/\partial\tilde{S}_a^a\partial\tilde{S}_b^b\rangle_0$,

$$U^2=-\frac{1}{2}\,\frac{(Z_\pi)^2(\eta''^2+\eta'^2+\eta^2-\pi^2)-2\frac{A_3}{\alpha w}-2\frac{A_4}{\alpha w'}}{2ww'+\frac{w'}{w}+\frac{w}{w'}}. \qquad (2.28)$$

Thus if we identify the $I=1$ scalar mesons with the $\delta(0.965)$, we can fit most the scalar and all the pseudoscalar masses since $(B_3-B_1)$ and $(Z_\pi)^2$ are now determined by the masses of the $\delta$ and the $\eta''$. The quantity $B_4$ can be calculated through Eqs. (2.5) and (2.25). Along with the masses of the mesons, we can also calculate the $\eta$-$\eta'$-$\eta''$ mixing angles. Again as in Ref. 2, we let

$$\begin{pmatrix}\eta\\ \eta'\\ \eta''\end{pmatrix}=\begin{pmatrix}\cos y&0&-\sin y\\ 0&1&0\\ \sin y&0&\cos y\end{pmatrix}\begin{pmatrix}1&0&0\\ 0&\cos z&-\sin z\\ 0&\sin z&\cos z\end{pmatrix}\begin{pmatrix}\cos x&-\sin x&0\\ \sin x&\cos x&0\\ 0&0&1\end{pmatrix}\begin{pmatrix}\eta_{NM}\\ \eta'_{NM}\\ \eta''_{NM}\end{pmatrix}, \qquad (2.29)$$

where

$$\begin{bmatrix}\pi^0\\ \eta_{NM}\\ \eta'_{NM}\\ \eta''_{NM}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}&0&0\\ \frac{1}{\sqrt6}&\frac{1}{\sqrt6}&-\frac{2}{\sqrt6}&0\\ \frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&0\\ 0&0&0&1\end{bmatrix}\begin{bmatrix}\tilde{\phi}_1^1\\ \tilde{\phi}_2^2\\ \tilde{\phi}_3^3\\ \tilde{\phi}_4^4\end{bmatrix} \qquad (2.30)$$

Values of meson masses and the $\eta$-$\eta'$-$\eta''$ mixing angles for some specific choices of $\eta''$ are listed in Table II.

For each choice of $\eta''$,

$$(Z_\pi)^2=0.390 \qquad (2.31)$$

and $m(\kappa)=1.11$ GeV, both independent of the specific choice of $\eta''$. Furthermore, (2.13) and (2.20) tell us that

$$(Z_K/Z_\pi)^2=1.015,\quad (Z_D/Z_\pi)^2=1.262,$$
$$(Z_F/Z_\pi)^2=1.196\,,\quad (X_{13})^2=0.981, \qquad (2.32)$$
$$(X_{14})^2=0.763,\quad (X_{34})^2=0.810\,.$$

Again, we can understand this by referring to the analysis in the Appendix of Ref. 2. It tells us that the mass of a large $\eta''$ is *extremely* sensitive to the value of $A_3/\alpha$, so that if we set $(Z_\pi)^2$ by the $\delta(0.965)$, then by (2.19) a slight change in $(B_3-B_1)$ will drastically change the value of $\eta''$. This is also shown in Table II. By (2.15) a slight change in $(B_3-B_1)$ will not appreciably effect the mass of the $\kappa$.

We now turn our attention to the $I=0$ chargeless scalar mesons, the $\sigma$-$\sigma'$-$\sigma''$. These are the scalar analogs of the $\eta$-$\eta'$-$\eta''$. Explicit differentiation of the renormalizable potential $V$ shows us that

$$\left\langle\frac{\partial^2V}{\partial\tilde{S}_a^a\partial\tilde{S}_b^b}\right\rangle_0=2\left(2V_{11}\alpha_a\alpha_b+U\frac{\alpha_1\alpha_2\alpha_3\alpha_4}{\alpha_a\alpha_b}\right)+2\left[V_1+6V_2\alpha_a^{\,2}+B_a-U\frac{\alpha_1\alpha_2\alpha_3\alpha_4}{(\alpha_a)^2}\right]\delta_a^b. \qquad (2.33)$$

Reference to (2.7) and (2.13) tells us that

$$X_{11}=X_{12}=X_{22}=X_{33}=X_{44}=1. \qquad (2.34)$$

All the parameters above except $V_{11}$ are known for various choices of $\eta''$. For completeness we shall identify the $S^*(.993)$ with the $\sigma'$. Then using the same formalism as was used to calculate the $\eta$-$\eta'$-$\eta''$ masses, we fit the value of the $\sigma''$ to various choices of the $\eta''$. In all cases, the mass of the $\sigma$ was about 0.728 GeV. Again, the $\sigma$ mass

TABLE II. Masses, mixing angles, and some system parameters for various choices of $\eta''$. In all cases $m(\delta)=0.965$ GeV, $m(\kappa)=1.11$ GeV, and $(Z_\pi)^2=0.390$.

| $m(\eta'')$ | $m(D)$ | $m(F)$ (GeV) | $m(D_S)$ | $m(F_S)$ | $x$ | $y$ (deg.) | $z$ | $\frac{B_3-B_1}{(Z_\pi)^2}$ [units of $m^2(\pi^0)$] |
|---|---|---|---|---|---|---|---|---|
| 2.80 | 1.90 | 1.97 | 2.79 | 2.85 | −11.9 | 0.28 | 0.90 | −8.49510 |
| 3.05 | 2.20 | 2.26 | 2.86 | 2.91 | −11.9 | 0.24 | 0.75 | −8.49800 |
| 3.40 | 2.61 | 2.64 | 2.96 | 3.00 | −11.9 | 0.19 | 0.59 | −8.50043 |
| 3.53 | 2.76 | 2.78 | 3.00 | 3.04 | −11.9 | 0.17 | 0.54 | −8.50104 |

is relatively invariant under changes in the $\eta''$ mass since it does not depend on $B_4$. The mixing angles $x_S$, $y_S$, and $z_S$ are the exact scalar analogs of the $x$, $y$, and $z$ mixing angles. The results are listed in Table III. We note that if $m(\eta'')>3.40$ GeV, then $m(\sigma'')<m(\eta'')$, contrary to the usual expectation.

## III. CURRENTS AND VECTOR-MESON DECAYS

In addition to calculating a mass matrix, we can also use the basic Lagrangian to calculate hadronic Noether currents. These currents can then be used to test various models of current-current hadronic and semileptonic weak interactions. In the case of the Lagrangian in Eq. (2.9) the vector current $(\mathcal{V}_\mu)_a^b$ and the axial-vector current $(\mathcal{A}_\mu)_a^b$ are

$$(\mathcal{V}_\mu)_a^b=-i[\mathfrak{D}_\mu\phi,\phi]_a^b-i[\mathfrak{D}_\mu S,S]_a^b -i[A_{\mu\nu},A_\nu]_a^b-i[V_{\mu\nu},V_\nu]_a^b$$

and (3.1)

$$(\mathcal{A}_\mu)_a^b=\{\mathfrak{D}_\mu\phi,S\}_a^b-\{\mathfrak{D}_\mu S,\phi\}_a^b -i[A_{\mu\nu},V_\nu]_a^b-i[V_{\mu\nu},A_\nu]_a^b\ ,$$

where

$$(V_{\mu\nu})_a^b=(F^l_{\mu\nu}+F^r_{\mu\nu})_a^b,\qquad (A_{\mu\nu})_a^b=(F^l_{\mu\nu}-F^r_{\mu\nu})_a^b. \tag{3.2}$$

After substituting in the physical fields of equation (2.12), the currents become

$$(\mathcal{V}_\mu)_a^b=-i(\alpha_b-\alpha_a)X_{ab}\partial_\mu\tilde{S}_a^b-\frac{g}{2}(\alpha_a-\alpha_b)^2(\tilde{V}_\mu)_a^b$$
$$+\sum_c\left(\frac{i}{X_{ac}X_{cb}}\left\{\tilde{S}_a^c\partial_\mu\tilde{S}_c^b\left[X_{cb}{}^2-i\frac{g}{2}\Lambda_{cb}(\alpha_b-\alpha_a)\right]-\partial_\mu\tilde{S}_a^c\tilde{S}_c^b\left[X_{ac}{}^2-i\frac{g}{2}\Lambda_{ac}(\alpha_b-\alpha_a)\right]\right\}\right.$$
$$\left.+\frac{i}{Z_{ac}Z_{cb}}\left\{\tilde{\phi}_a^c\partial_\mu\tilde{\phi}_c^b\left[Z_{cb}{}^2-\frac{g}{2}\Gamma_{cb}(\alpha_b-\alpha_a)\right]-\partial_\mu\tilde{\phi}_a^c\tilde{\phi}_c^b\left[Z_{ac}{}^2+\frac{g}{2}\Gamma_{ac}(\alpha_b-\alpha_a)\right]\right\}\right)$$
$$+\cdots$$

and (3.3)

$$(\mathcal{A}_\mu)_a^b=(\alpha_b+\alpha_a)Z_{ab}\partial_\mu\tilde{\phi}_a^b-\frac{g}{2}(\alpha_a+\alpha_b)^2(\tilde{A}_\mu)_a^b$$
$$+\sum_c\left(\frac{1}{X_{ac}Z_{cb}}\left\{\tilde{S}_a^c\partial_\mu\tilde{\phi}_c^b\left[Z_{cb}{}^2-\frac{g}{2}\Gamma_{cb}(\alpha_a+\alpha_b)\right]-\partial_\mu\tilde{S}_a^c\tilde{\phi}_c^b\left[X_{ac}{}^2+i\frac{g}{2}\Lambda_{ac}(\alpha_a+\alpha_b)\right]\right\}\right.$$
$$\left.-\frac{1}{Z_{ac}X_{cb}}\left\{\tilde{\phi}_a^c\partial_\mu\tilde{S}_c^b\left[X_{cb}{}^2-i\frac{g}{2}\Lambda_{cb}(\alpha_a+\alpha_b)\right]-\partial_\mu\tilde{\phi}_a^c\tilde{S}_c^b\left[Z_{ac}{}^2-\frac{g}{2}\Gamma_{ac}(\alpha_a+\alpha_b)\right]\right\}\right)$$
$$+\cdots\ .$$

TABLE III. Mass and mixing angles of the $\sigma''$ and system parameters for various choices $\eta''$. In all cases $m(\sigma')=0.993$ GeV and thus $m(\sigma)=0.728$ GeV.

| $m(\eta'')$ (GeV) | $m(\sigma'')$ | $x_S$ | $y_S$ (deg.) | $z_S$ | $\frac{V_{11}\alpha^2}{(Z_\pi)^2}$ [units of $m^2(\pi^0)$] |
|---|---|---|---|---|---|
| 2.80 | 3.13 | 13.6 | 3.30 | 7.37 | 11.65 |
| 3.05 | 3.17 | 13.6 | 3.16 | 7.02 | 11.40 |
| 3.40 | 3.23 | 13.6 | 2.97 | 6.59 | 11.15 |
| 3.53 | 3.25 | 13.6 | 2.90 | 6.42 | 11.05 |

From (3.3) we can identify the pseudoscalar decay constants as

$$\begin{aligned} f_\pi &= 2\alpha Z_\pi, \\ f_K &= \alpha(1+w)Z_K, \\ f_D &= \alpha(1+w')Z_D, \\ f_F &= \alpha(w+w')Z_F. \end{aligned} \quad (3.4)$$

Assuming the usual Cabibbo scheme[7] for weak decays and using the explicit currents of (3.3) we see that

$$\Gamma(K\to\mu\nu)=\frac{G^2}{8\pi}\sin^2(\theta_C)(f_K)^2 m^2(\mu)m(K) \times\{1-[m(\mu)/m(K)]^2\}^2$$

and

$$\Gamma(\pi\to\mu\nu)=\frac{G^2}{8\pi}\cos^2(\theta_C)(f\pi)^2 m^2(\mu)m(\pi) \times\{1-[m(\mu)/m(\pi)]^2\}^2,$$

where $\theta_C$ is the Cabibbo angle. If we let[8] $\sin\theta_C=0.230$, then[9] by (3.5)

$$f_K/f_\pi=1.163 \text{ and } f_\pi=0.99m(\pi^0). \quad (3.6)$$

Independently from (2.20), (2.30), and (3.4), we see that

$$\begin{aligned} f_K/f_\pi &= \tfrac{1}{2}(1+w)Z_K/Z_\pi \\ &= 1.162, \\ f_D/f_\pi &= \tfrac{1}{2}(1+w')Z_D/Z_\pi \\ &= 2.790, \\ f_F/f_\pi &= \tfrac{1}{2}(w+w')Z_F/Z_\pi \\ &= 2.884. \end{aligned} \quad (3.7)$$

The $f_K/f_\pi$ ratio comes out in remarkably good agreement with experiment. We also see that SU(4) is more seriously broken in the values for $f_D$ and $f_F$.[10]

In addition to the calculation of hadronic currents, we can also use the simple phenomenological Lagrangian (2.9) to calculate in the tree approximation some simple two-particle decays of vector mesons. Specifically, we wish to calculate the width of a vector meson which decays into two pseudoscalars, such $\rho\to\pi\pi$. In general, such a width can be written as

$$\Gamma(\tilde V_a^b\tilde\phi_b^c\tilde\phi_c^a)=\frac{|\vec{k}|}{8\pi m^2(V_a^b)}\frac{1}{3}\sum_{S_i}|T(\tilde V_a^b\tilde\phi_b^c\tilde\phi_c^a)|^2, \quad (3.8)$$

where $\vec{k}$ is the momentum of a daughter particle in the rest frame of the parent. For our specific Lagrangian

$$\sum_{S_i}|T(\tilde V_a^b\tilde\phi_b^c\tilde\phi_c^a)|^2=\frac{g^2|\vec{k}|^2}{16(Z_{bc})^2(Z_{ca})^2}\left\{\frac{[1-(Z_\pi)^2]m^2(\rho_A)m^2(V_a^b)(\alpha_b+\alpha_c)(\alpha_a+\alpha_c)}{2m^2(A_c^a)m^2(A_b^c)\alpha^2} + \frac{[m^2(\rho_A)+m^2(\rho)](\alpha_a-\alpha_b)(\alpha_a+\alpha_c)-4m^2(\rho)(\alpha_a)^2}{2m^2(A_c^a)\alpha^2} + \frac{[m^2(\rho_A)+m^2(\rho)](\alpha_b-\alpha_a)(\alpha_b+\alpha_c)-4m^2(\rho)(\alpha_b)^2}{2m^2(A_b^c)\alpha^2}\right\}^2. \quad (3.9)$$

Using the values in Tables I and II, and also (2.13), (2.20), (2.32), (3.4), and (3.6), we see that

$$\frac{g^2}{4\pi}=\frac{m^2(\rho_A)}{\pi F_\pi^2}Z_\pi^2(1-Z_\pi^2) \cong 4.56 \quad (3.10)$$

and thus

$$\begin{aligned} \Gamma_{\text{total}}(\rho\to\pi\pi) &= 124 \text{ MeV}, \\ \Gamma_{\text{total}}(K^*\to K\pi) &= 50.7 \text{ MeV}, \\ \Gamma_{\text{total}}(\phi\to KK) &= 4.3 \text{ MeV}, \end{aligned} \quad (3.11)$$

which are close to the experimental values[9]

$$\begin{aligned} \Gamma_{\text{total}}(\rho\to\pi\pi) &= (150\pm10) \text{ MeV}, \\ \Gamma_{\text{total}}(K^*\to K\pi) &= (49.8\pm1.1) \text{ MeV}, \\ \Gamma_{\text{total}}(\phi\to KK) &= (3.4\pm0.4) \text{ MeV}. \end{aligned} \quad (3.12)$$

Assuming, for example, that the $m(\eta'')\simeq2.80$ GeV, we then calculate that

$$\begin{aligned} \Gamma_{\text{total}}(D^*\to D\pi) &= 135 \text{ MeV}, \\ \Gamma_{\text{total}}(F^*\to DK) &= 22.1 \text{ MeV}. \end{aligned} \quad (3.13)$$

The suppression of the $F^*$ decay as compared to the $D^*$ rate is almost entirely due to phase space. The same holds true for the $\rho$ and $K^*$ rates. Since the $\phi$ is pure $(\tilde{V}_\mu)^3_3$, there is no $\phi\to\pi\eta$ decay.

Note added. A candidate for the $D$ meson has recently been found at $m(D)\simeq 1.87$ GeV [G. Goldhaber, Berkeley colloquium (unpublished)]. Since we only need one new pseudoscalar-meson mass to specify the other meson masses, we have inverted the previous procedure and used $m(D)$ as input. Thus with $m(D)=1.87$ GeV, we obtain

$$\begin{aligned} &m(\eta'')=2.78\text{ GeV},\quad m(F)=1.94\text{ GeV},\\ &m(F_S)=2.84\text{ GeV},\quad m(D_S)=2.79\text{ GeV},\\ &m(\kappa)=1.11\text{ GeV},\quad m(\delta)=0.965\text{ GeV},\end{aligned} \tag{3.14}$$

and $x=-11.9°$, $y=0.29°$, $z=0.92°$

At this point it may be interesting to look at the symmetry-breaking parameters: the $A$'s, $B$'s, and $U$. We thus have

$$\begin{aligned} &A_1/\alpha=0.196m^2(\pi^0),\\ &A_3/\alpha=6.53m^2(\pi^0),\\ &A_4/\alpha=328m^2(\pi^0),\\ &B_1=16.53m^2(\pi^0),\\ &B_3=13.22m^2(\pi^0),\\ &B_4=-46.28m^2(\pi^0),\\ &U_{\alpha 2}=-0.587m^2(\pi^0).\end{aligned} \tag{3.15}$$

The values of $w$ and $w'$ in (2.8) and (2.20) are not changed. They depend only on vector-meson masses.

It is interesting to note that even though the symmetry of the vacuum as measured by $w$ and $w'$ is not too badly broken, the "quark masses" are in the ratio

$$A_1:A_3:A_4=1:33:1681,$$

while the $B$'s, the coefficients of the quadratic term, are in the ratio

$$B_1:B_3:B_4=1.25:1:-3.50.$$

### IV. COMMENTS

(i) The vector and axial-vector mass spectrum generated by $\mathfrak{L}_I$ is in reasonable agreement with the limited experimental data that is available. Since $\mathfrak{L}_I$ has the full U(4) × U(4) symmetry, we find that the $\rho$ and $\omega$, and $\rho_A$ and $\omega_A$ are degenerate. For the pseudoscalar mesons, this problem is alleviated by the addition of a term proportional to $I_5$. The $I_5$ term breaks U(4) × U(4) to SU(4) × SU(4) and thus creates the "U(1) problem."[11] This suggests a possible method to break the $\rho$-$\omega$ degeneracy: Add a new U(4) × U(4)-breaking term $\mathfrak{L}_I^1$ to $\mathfrak{L}_I$, namely

$$\mathfrak{L}_I^1=-\frac{1}{4!}\epsilon^{abcd}\epsilon_{efgh}\times\Big[E(l^e_{\mu_a}M^f_b r^g_{\mu_c}M^{\dagger h}_d)+\frac{F}{2}(l^e_{\mu_a}l^f_{\mu_b}M^g_c M^{\dagger h}_d+r^e_{\mu_a}r^f_{\mu_b}M^{\dagger g}_c M^h_d)\Big] \tag{4.1}$$

(summation convention).

This only compounds the problem. One would like to keep all the symmetry-breaking terms strictly in the spin-zero-meson potential terms.

(ii) The scalar and pseudoscalar mass spectrum of this new $\sigma$ model seems to be more reasonable than the mass spectrum of the old model. First, in this model we can see from Table III that $m(F)>m(D)$. The old $\sigma$ model gave the opposite result. We get this new result because of the "renormalization" constants $Z_F$ and $Z_D$. From (2.15) we see that the masses are inversely proportional to the $Z$'s, while from (2.32) we see that $Z_F>Z_D$.

Second, from this new model we see that the $\eta$-$\eta'$ mixing angle is always $-11.9°$, while the old $\sigma$ model, the angle was an order of magnitude smaller. This new value of the $\eta$-$\eta'$ mixing angle $x$ is in agreement with the original Gell-Mann–Okubo[12] (GMO) value of $-10°$. This is remarkable since the $\sigma$ model does not generate masses by any GMO-type term. This $-10°$ mixing angle is also important in the observed electromagnetic decays of the $\eta$ and the $\eta'$.[13]

Third, the masses of the new mesons $F$ and $D$ are much lighter in this new model. This is again caused by the renormalization constants $Z_F$ and $Z_D$. Still they are both greater than $\frac{1}{2}m(\psi)$. In particular, if the mass of the $\eta''$ is 2.80 GeV, then Table II tells us that both the $F$ and the $D$ are under 2 GeV. In a recent neutrino-induced dilepton experiment,[14] the results were consistent with the decay of a particle with a new quantum number and a mass between 1.5 and 2 GeV.

(iii) The calculation of the $f_K/f_\pi$ ratio and the vector-meson widths shows the consistency of the model. The only input data were masses. The parameters of the theory, $w$, $w'$, $A_a$, $B_a$, $C$, $D$, and $g$, were all fitted to those masses. The fact that these new quantities $f_K/f_\pi$ and the widths came out so near their experimental value is remarkable. It then seems highly reasonable to assume that the model will also predict the decay constants and widths of the new particles.

The reason for these good results in the $\sigma$ model with gauge mesons seems to lie with the vector mesons themselves. The only "hard" evidence of a new quantum lies in the $\psi(3.095)$. If we assume that $V_{SB}$ transforms as $(4,4^*)\oplus(4^*,4)$ only, then $V$ cannot be renormalizable and still fit the $\psi$ mass. If we do not assume renormalizability, then there seems to be little that can be said about the scalar masses. More serious than this is the fact that in such a model $(Z_\pi)^2=0.0946$, and thus $\Gamma(\rho\to\pi\pi)\simeq 0.390$ GeV, and $f_K/f_\pi\simeq 1.21$. Thus the addition of vector mesons to the original $\sigma$ model caused us to reexamine our original $V_{SB}$ and include the necessary extra term.

## APPENDIX

Consider the chiral SU(4) object $H^{ac}_{bd}=T^a_bV^c_d$ $(a,\ldots,d=1,\ldots,4)$, where $T^a_b$ and $V^c_d$ both transform as $(4,4^*)$. The upper (lower) index denote the 4 (4*) SU(4) index.

$H$ is a reducible representation. Using the standard Clebsch-Gordan SU(4) series for the direct product of irreducible representations, we have

$$4\times 4=10\oplus 6,$$
$$4^*\times 4^*=10^*\oplus 6$$

and thus

$$(4,4^*)\times(4,4^*)=(10,10^*)\oplus(6,6)\oplus(10,6)\oplus(6,10^*).$$

Decomposing $H$ into its irreducible parts we have

$$\begin{aligned}H^{ac}_{bd}=&\tfrac{1}{4}(T^a_bV^c_d+T^a_dV^c_b+T^c_bV^a_d+T^c_dV^a_b)\Rightarrow(10,10^*)\\&+\tfrac{1}{4}(T^a_bV^c_d-T^a_dV^c_b+T^c_bV^a_d-T^c_dV^a_b)\Rightarrow(10,6)\\&+\tfrac{1}{4}(T^a_bV^c_d+T^a_dV^c_b-T^c_bV^a_d-T^c_dV^a_b)\Rightarrow(6,10^*)\\&+\tfrac{1}{4}(T^a_bV^c_d-T^a_dV^c_b-T^c_bV^a_d+T^c_dV^a_b)\Rightarrow(6,6)\end{aligned}$$

We now note that for a tensor to be an irreducible representation, it must have certain symmetry properties and be traceless. Thus, for example, to examine the SU(4) properties of the (6, 6) part of $H$, we see that (repeated indices here denote summation)

$$\begin{aligned}(6,6)\Rightarrow&[T^a_bV^c_d-T^a_dV^c_b-T^c_bV^a_d+T^c_dV^a_b-\tfrac{1}{4}\delta^a_b(T^f_fV^c_d-T^f_dV^c_f-T^c_fV^f_d+T^c_dV^f_f)]\Rightarrow\underline{20}\\&+\tfrac{1}{4}\delta^a_b[T^f_fV^c_d-T^f_dV^c_f-T^c_fV^f_d+T^c_dV^f_f-\tfrac{1}{2}\delta^c_d(T^f_fV^e_e-T^f_eV^e_f)]\Rightarrow\underline{15}\\&+\tfrac{1}{8}\delta^a_b\delta^c_d(T^f_fV^e_e-T^f_eV^e_f)\Rightarrow\underline{1},\end{aligned}$$

where 20, 15, and 1 represent the SU(4) representations generated. The same procedure can be applied to the other irreducible components of $H$.

Thus, the quadratic part of $V_{SB}$ in (2.4),

$$\tfrac{1}{2}\sum_{a,b}B_a[(M^\dagger)^b_a(M^\dagger)^a_b+M^b_aM^a_b],$$

transforms as $(10,10^*)\oplus(10^*,10)\oplus(6,6)$ under chiral SU(4). The 15-plet in each of these irreducible parts can be obtained by subtracting the trace. Applying this concept to the symmetry-breaking terms also, we have

$$\sum_a B_a=0.$$

A more general symmetry breaker can be constructed by removing the trace condition. This will give a 16-plet in each of the irreducible parts of the quadratic $V_{SB}$.[15]

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