Quantum Poincaré covariance of the two-dimensional string

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We show that the free massive quantum string in two space-time dimensions is Poincaré covariant. This is true for both lightlike and timelike gauge choices. When the massless limit is taken, the massless string is also seen to be Poincaré covariant. The set of classical string variables which is chosen to define the quantum theory drastically affects the proof of Poincaré covariance and also the mass spectrum. We therefore suggest that other dynamical systems may exhibit similar phenomena.

The quantum Poincaré covariance of the two-space-time-dimensional string systems treated in Refs. 1 and 2 was discussed only very briefly. Here we present an extended footnote giving the detailed arguments for such systems without string-string interactions.

Lightlike gauge. We choose the metric

$$g^{+-}=g^{-+}=-1, \quad g^{++}=g^{--}=0 \tag{1}$$

and the gauge

$$x^{+}(\tau,\sigma)=-x_{-}(\tau,\sigma)=\frac{1}{\sqrt{2}}\,(x^0+x^1)=\tau\ . \tag{2}$$

The N masses at the end points and folds joined by the string are then described by the variables p_n^+, x_n^- ($n=1,\ldots,N$), which obey the commutation relations

$$[p_n^+, x_m^-]=i\delta_{nm}\,. \tag{3}$$

Note that $p_n^+=m_n[-(\partial x_n/\partial\tau)^2]^{-1/2}$ is classically positive-definite. Here we treat the quantum-mechanical operator p_n^+ as a Hermitian operator acting on a space of states with only positive eigenvalues of p_n^+. Following the usual arguments,[1,2] we find the Poincaré-group generators

$$H\equiv P^-=\sum_{n=1}^{N}\frac{m_n^{\,2}}{2p_n^+}+\gamma\sum_{n=1}^{N-1}\left|x_{n+1}^- - x_n^-\right|\ , \tag{4a}$$

$$P^+=\sum_{n=1}^{N}p_n^+\ , \tag{4b}$$

$$M^{+-}=\tau P^- -\tfrac{1}{2}\sum_{n=1}^{N}(x_n^-p_n^+ + p_n^+x_n^-)\ , \tag{4c}$$

where the operators are ordered in M^{+-}.

The commutation rules of the boost operator M^{+-} with the individual variables x_n^-, p_n^+ follow from Eq. (3) and are complicated because, for arbitrary τ, M^{+-} generates a gauge transformation on the variables in addition to a naive boost. However, the gauge-invariant variables $M^{+-}, P^{\pm}$ have the following simple commutation rules:

$$\begin{aligned}&[P^\mu, P^\nu]=0\ ,\\ &[M^{+-}, P^{\pm}]=\pm iP^{\pm}\ .\end{aligned} \tag{5}$$

The quantum Poincaré algebra is therefore satisfied; all ordering ambiguities are resolved by the choice made in Eq. (4).

We find it useful to make the canonical transformation

$$\begin{aligned}&\rho_n=(x_n^- - x_{n+1}^-)P^+,\quad n=1,\ldots,N-1\\ &\frac{1}{N}+\kappa_n-\kappa_{n-1}=p_n^+/P^+,\quad n=2,\ldots,N-1\\ &\frac{1}{N}+\kappa_1=p_1^+/P^+,\ \ \frac{1}{N}-\kappa_{N-1}=p_N^+/P^+\ ,\end{aligned} \tag{6}$$

so the invariant mass squared is

$$M^2\equiv 2P^+P^-=\frac{m_1^{\,2}}{1/N+\kappa_1}+\sum_{n=2}^{N-1}\frac{m_n^{\,2}}{1/N+\kappa_n-\kappa_{n-1}}+\frac{m_N^{\,2}}{1/N-\kappa_{N-1}}+2\gamma\sum_{n=1}^{N-1}|\rho_n|\ . \tag{7}$$

Note that P^+ commutes with $x_n^- - x_{n+1}^-$, so no ordering problem occurs. The new variables obey the commutation rules $[\kappa_n, \rho_m]=i\delta_{nm}$.

We deduce from Eq. (7) that if we use the lightlike-gauge variables of Eq. (6) we cannot take the massless limit until the spectrum is calculated. As shown in the semiclassical and quantum treatments of Refs. 1–4, the theory possesses a nontrivial spectrum in the zero-mass limit. We may use the treatment of 't Hooft[3] to argue that the solutions of the integral equation defined by Eq. (7) are regular in the massless limit, where all the (renormalized) masses $m_n \to 0$. In this limit, Eq. (5) continues to hold and the theory remains Poincaré-covariant.

Timelike gauge. The timelike-gauge case turns out to be somewhat more complicated. We choose the metric

$$g^{00}=-1,\ g^{11}=1,\ g^{01}=g^{10}=0 \tag{8}$$

and the gauge

$$x^0(\tau,\sigma)=\tau\ . \tag{9}$$

Let us first examine the classical theory. The N masses at end points and folds joined by the string are described by the canonical variables p_n, x_n $(n=1,\ldots,N)$ obeying the Poisson brackets

$$\{p_n, x_m\} = -\delta_{nm}\ . \tag{10}$$

The classical Poincaré-group generators in the timelike gauge (9) are[1,2]

$$H=\sum_{n=1}^{N}(p_n^2+m_n^2)^{1/2}+\gamma\sum_{n=1}^{N-1}|x_n-x_{n+1}|\ , \tag{11a}$$

$$P=\sum_{n=1}^{N}p_n\ , \tag{11b}$$

$$M^{01}=\tau P-B\ , \tag{11c}$$

where

$$B=\sum_{n=1}^{N}x_n(p_n^2+m_n^2)^{1/2}+\tfrac{1}{2}\gamma\sum_{n=1}^{N-1}(x_n+x_{n+1})|x_n-x_{n+1}|\ . \tag{12}$$

One can verify that the classical Poincaré algebra is satisfied. However, the ordering of the terms in B is ambiguous if we try to quantize the x_n, p_n variables. In particular, the quantum Poincaré algebra is not satisfied if one simply replaces $x_n(p_n^2+m_n^2)^{1/2}$ by $\frac{1}{2}[x_n(p_n^2+m_n^2)^{1/2}+(p_n^2+m_n^2)^{1/2}x_n]$. There may, of course, exist some ordering prescription which works, but its physical interpretation would not be as straightforward as the method we shall present now.

It is well known that the success of canonical quantization may depend crucially upon the particular choice of classical canonical variables used to describe the system.[5] Therefore we search for variables which separate the dynamics of the system in a convenient way. The best choice for the overall center-of-mass coordinate is clearly the two-dimensional Newton-Wigner coordinate[6]

$$Q=B/H\ , \tag{13}$$

where

$$\{Q,H\}=P/H \tag{14}$$

and P/H is the velocity of the center of mass. Since

$$\{P,Q\}=-1\ , \tag{15}$$

Q is canonically conjugate to the translation generator P.

The well-known example of two free particles illustrates the utility of this transformation. We have classically

$$\begin{aligned} H&=(p_1^2+m_1^2)^{1/2}+(p_2^2+m_2^2)^{1/2}\ ,\\ P&=p_1+p_2\ ,\\ Q&=B/H\\ &=[x_1(p_1^2+m_1^2)^{1/2}+x_2(p_2^2+m_2^2)^{1/2}]/H\ . \end{aligned} \tag{16}$$

The canonical transformation from $(x_1p_1x_2p_2)$ to the new variables $(QPkr)$ is generated by

$$F(x_1x_2;Pk)=x_1\frac{P\omega_1-kW}{\omega_1+\omega_2}+x_2\frac{P\omega_2+kW}{\omega_1+\omega_2}\ , \tag{17}$$

where $\omega_n^2=k^2+m_n^2$, and

$$W^2=P^2+(\omega_1+\omega_2)^2\ . \tag{18}$$

As usual, we have $p_1=\partial F/\partial x_1$, $p_2=\partial F/\partial x_2$, $Q=\partial F/\partial P$, $r=\partial F/\partial k$. These variables were chosen so that p_1 and p_2 are obtained by boosting the variable k to a new Lorentz frame,

$$p_1=(-k+\beta\omega_1)/(1-\beta^2)^{1/2}\ ,$$

$$p_2=(k+\beta\omega_2)/(1-\beta^2)^{1/2}\ ,$$

where $\beta=P/W$ is the velocity of the center of mass. Substituting p_1 and p_2 into the form of H given in Eq. (16) shows that in terms of the new variables

$$H(P,k,Q,r)=W\ , \tag{19}$$

with W defined by Eq. (18). The invariant mass is therefore given by

$$M(k,r)=(H^2-P^2)^{1/2}=\omega_1+\omega_2\ . \tag{20}$$

All dependence of the invariant mass on the center-of-mass coordinates (P,Q) has disappeared. *For other choices of canonical variables this might not have been true*.[7]

The quantum Lorentz invariance of two free particles is now proven by canonically quantizing the *new* variables (P,Q,k,r),

$$i[P,Q]=1,\quad i[k,r]=1\ , \tag{21}$$

and expressing the Poincaré-group generators as

$$H=[P^2+(\omega_1+\omega_2)^2]^{1/2}\ , \tag{22a}$$

$$P=P\ , \tag{22b}$$

$$M^{01} = \tau P - \tfrac{1}{2}(QH + HQ) . \tag{22c}$$

One easily finds that the quantum Poincaré-group algebra holds with no ordering problems:

$$\begin{aligned} &i[P,H] = 0 ,\\ &i[M^{01},P] = H ,\\ &i[M^{01},H] = P . \end{aligned} \tag{23}$$

For particles interacting via the string, it is much harder to find the analog of the canonical transformation (17), and we shall not carry out an explicit analysis here. We can, however, give a convincing general argument that the crucial features of the timelike free-particle case continue to be valid. We begin by using Eqs. (12)–(14) to define the properties of the classical interacting center-of-mass coordinate $Q = B/H$ conjugate to P. For some set of variables k_n, r_n, $n = 1, \ldots, N-1$, there must exist a canonical transformation from the variables

$$(p_n, x_n), \quad n = 1, \ldots, N$$

to the new variables

$$(P, Q) ,$$
$$(k_n, r_n) .$$

In principle, the variables k_n, r_n can be related to the p_n, x_n variables by a boost, just as we found in the case of two free particles. However, the form of the *finite* boost generated by Eqs. (11) and (12) is more complex because of the gauge transformation accompanying the naive Lorentz transformation. Without demonstrating the explicit transformation, we may replace the original Poisson brackets (10) by the canonically transformed brackets

$$\{A, B\} = \frac{\partial A}{\partial Q}\frac{\partial B}{\partial P} - \frac{\partial A}{\partial P}\frac{\partial B}{\partial Q} + \sum_{n=1}^{N-1}\left(\frac{\partial A}{\partial r_n}\frac{\partial B}{\partial k_n} - \frac{\partial A}{\partial k_n}\frac{\partial B}{\partial r_n}\right). \tag{24}$$

Then Eq. (14) becomes a differential equation for $H(PQk_nr_n)$,

$$\{Q, H\} = \frac{\partial H}{\partial P} = \frac{P}{H} , \tag{25}$$

along with

$$\{H, P\} = \frac{\partial H}{\partial Q} = 0 . \tag{26}$$

The solution of Eqs. (25) and (26) is simply

$$H^2(P, Q; k_n, r_n) = P^2 + M^2(k_n r_n) , \tag{27}$$

where $M^2(k_n r_n)$ has *no* P, Q dependence.[8] We conclude that even in the interacting case the qualitative features of Eq. (20) are preserved if we choose the right canonical variables.[9]

We now canonically quantize the new variables

$$i[P, Q] = 1, \quad i[k_n, r_m] = \delta_{nm} , \tag{28}$$

and rewrite the boost generator Eq. (11c) as

$$M^{01} = \tau P - \tfrac{1}{2}(QH + HQ) , \tag{29}$$

where H is given by Eq. (27). The quantum Poincaré algebra Eq. (23) continues to hold and the timelike system is Poincaré-covariant. Taking the massless limit presents no further problems, so the massless $D = 2$ string treated in Ref. 1 also possesses a Poincaré-covariant quantum mechanics.

The spectrum. Although we have been able to argue that both the timelike and lightlike gauges give Poincaré-covariant quantum systems, the determination of the mass spectrum presents a puzzling problem. We showed in Refs. 1 and 2 that the invariant masses of the no-fold string in the lightlike and timelike gauges,[8]

$$M^2 = \frac{m_1^{\,2}}{\frac{1}{2} - \kappa} + \frac{m_2^{\,2}}{\frac{1}{2} + \kappa} + 2\gamma|\rho| \quad \text{(lightlike)} ,$$
$$M = (k^2 + m_1^{\,2})^{1/2} + (k^2 + m_2^{\,2})^{1/2} + \gamma|r| \quad \text{(timelike)} ,$$

gave the same semiclassical mass spectra. This means that for large quantum numbers the two systems are identical to order $\hbar$. The quantum spectra of these systems, however, are defined by Schrödinger-like integral equations[1,2]; it is by no means obvious that the quantum spectra of these two equations are identical. Yet if they are not identical, what physical criterion exists for choosing between them? We have seen that quantum Poincaré covariance does not single out any particular set of relative variables such as (k_n, r_n), (κ_n, ρ_n), etc., to be used in defining the quantum theory.

In particular one could transform to classical action-angle variables,[1,2,8,9] (J_n, θ_n), and make a further canonical transformation to harmonic-oscillator variables,

$$a_n = [J_n]^{1/2} e^{-i\theta_n} ,$$
$$a_n^\dagger = [J_n]^{1/2} e^{i\theta_n} .$$

One could then quantize the system by taking a_n and $a_n^\dagger$ to be operators with

$$[a_n, a_m^\dagger] = \delta_{nm}$$

and making the replacement

$$J_n = \tfrac{1}{2}(a_n a_n^\dagger + a_n^\dagger a_n)$$

in the Hamiltonian. The spectrum of the system relative to the vacuum defined by the a_n, $a_n^\dagger$ would then be exactly known. The quantum spectrum determined by Eq. (7), for example, would be completely different, but apparently equally legitimate.

For the massless $D=2$ string described in Sec. V of Ref. 1, this procedure gives the mass operator

$$M^2/\gamma=\sum_{n=1}^{\infty} n a_n^\dagger a_n+\alpha_0\ ,$$

where an infinite vacuum mass has been absorbed in the definition of α_0. This spectrum resembles that expected from a dual model, but, as we have noted, is not unique and is not dictated by Poincaré covariance. We further note that the vacuum defined relative to one set of canonically quantized variables may be quite different from that defined relative to another set. This may have implications for the energy of the ground state.

Comments. Our demonstration that the free N-fold $D=2$ string can be formulated so that its quantum mechanics is Poincaré-covariant is to be compared with the Goddard-Goldstone-Rebbi-Thorn result[10] for the free $D=26$ string. However, we have not dealt with interacting $D=2$ strings, and thus have shed no light on the $D=2$ analog of Mandelstam's treatment[11] of interacting-string Poincaré covariance in $D=26$. It may be necessary to include arbitrary numbers of folds simultaneously in order to demonstrate interacting Poincaré covariance for the general $D=2$ string.[12]

We conclude that a careful choice of canonical variables is necessary to demonstrate that the string systems described in Refs. 1 and 2 possess a Poincaré-covariant quantum mechanics. The quantum spectrum, however, may not be unique; the classical variables which one chooses to quantize canonically determine the spectral properties of the quantum system. *Our observations suggest that similar phenomena may occur in the Hamiltonian formulation of any dynamical theory in any dimension.*

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[1]W. A. Bardeen, I. Bars, A. J. Hanson, and R. D. Peccei, Phys. Rev. D 13, 2364 (1976).

[2]I. Bars and A. J. Hanson, Phys. Rev. D 13, 1744 (1976).

[3]G. 't Hooft, Nucl. Phys. B75, 461 (1975).

[4]I. Bars, Phys. Rev. Lett. 36, 1521 (1976).

[5]For example, expressing the simple harmonic oscillator in polar coordinates gives a different quantum system from that found using rectangular coordinates. In some constrained systems, one may even need to make a noncanonical transformation of the original coordinates [see, for example, E. Del Giudice, P. Di Vecchia, and S. Fubini, Ann. Phys. (N.Y.) 70, 378 (1972), or A. J. Hanson and T. Regge, *ibid*. 87, 498 (1974)].

[6]F. Rohrlich, Phys. Rev. Lett. 34, 842 (1975), has advocated the use of Newton-Wigner coordinates for the string. However, as argued in Ref. 1, Rohrlich's simple spectrum does not necessarily follow.

[7]For example, the choice $P=p_1+p_2$, $Q=\frac{1}{2}(x_1+x_2)$, $k=\frac{1}{2}(p_2-p_1)$, $r=x_2-x_1$ fails this requirement.

[8]An example of the expected form for $M(kr)$ in the two-mass no-fold case is found by going to the $P=0$ frame and setting $p_2=-p_1=k$, $x_2-x_1=r$. Then

$$H(P=0)=M=(k^2+m_1^2)^{1/2}+(k^2+m_2^2)^{1/2}+\gamma|r|\ .$$

Another example follows from the canonical transformation to the action-angle variables given in Ref. 1 for vanishing masses:

$$H=(P^2+\gamma J)^{1/2}\ .$$

It was explicitly shown in Ref. 1 that the massless string Hamiltonian can be expressed in this latter form for any number of folds.

[9]Note that the canonical Schrödinger equation defined by $M^2(k_n,r_n)$ can still be altered by making further canonical transformations among the k_n and r_n variables. In particular, there should exist a canonical transformation from the k_n,r_n variables to the κ_n,ρ_n variables of Eq. (6). In Refs. 1 and 2, we showed that the canonical transformations from timelike or lightlike canonical variables to action-angle variables gave the same classical invariant mass. (In the zero-mass limit, we found simply $M^2=\gamma J$.) Thus the action-angle variables provide an intermediate step in the explicit construction of a canonical transformation from timelike-gauge variables to lightlike-gauge variables.

[10]P. Goddard, J. Goldstone, C. Rebbi, and C. Thorn, Nucl. Phys. B56, 109 (1973).

[11]S. Mandelstam, Nucl. Phys. B64, 205 (1973); B69, 77 (1974).

[12]The interacting no-fold string is apparently Poincaré-covariant by itself, since it was shown in Ref. 4 to be equivalent to 't Hooft's model of Ref. 3.