# Improved effective-potential formalism for composite fields*

T. Banks and S. Raby
*Department of Physics and Astronomy, Tel-Aviv University, Tel-Aviv, Israel*
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We develop an effective-potential formalism for studying dynamical symmetry breaking. The potential that we calculate is single-valued and bounded from below. Our formalism incorporates a stability criterion for deciding whether the broken-symmetry solution to the theory is the physical one. In lowest-order calculations in gauge theories we find that the asymmetric theory will be stable *if and only if* a composite Goldstone boson can be bound. Our conclusion is that in the weak-coupling approximation there is no dynamical spontaneous breakdown in gauge theories. We then use the renormalization group to argue that if spontaneous breakdown occurs at all, it must also occur for arbitrarily weak coupling. The renormalization group also provides us with evidence that dynamical symmetry breakdown does not occur in infrared-stable theories.

## I. INTRODUCTION

The term, dynamical symmetry breaking (DSB) refers to the occurrence of the Goldstone phenomenon in a field theory whose Lagrangian contains no elementary scalar fields. Goldstone's theorem requires that massless bosons accompany any spontaneous symmetry breakdown[1,2]; in theories where DSB occurs these bosons must be bound states.

There are two principal advantages to DSB as compared to spontaneous symmetry breaking with elementary scalars. Firstly, a theory with DSB has fewer parameters and would enable us to calculate, for example, the mass of the $W$ boson (of weak interactions) rather than fixing it in terms of unmeasured parameters in the scalar-boson part of the Lagrangian. Secondly, the question of whether a given symmetry is broken or not is a dynamical one. In theories with Higgs scalars we can break any symmetry by manipulating the Higgs Lagrangian.

The possibility of realistic calculations of the $W$-boson mass seems quite remote at this time, but the question of which symmetries suffer DSB is sufficiently qualitative that we may hope to answer it. In this paper we set up a formalism for attacking this problem.

There is a simple intuitive necessary condition for the occurrence of DSB in a given channel: The forces in that channel must be sufficiently attractive to bind a zero-mass Goldstone boson. Thus, most discussions of DSB have followed the seminal work of Nambu and Jona-Lasinio, and used the Bethe-Salpeter equation to try to establish the existence of such a bound state. The existence of a solution of the Bethe-Salpeter equation is equivalent to the existence of symmetry-breaking solutions to the Schwinger-Dyson (SD) equations of the theory.[2] However, this method apparently gives no way for determining which of the symmetric or nonsymmetric solutions to the SD equations is the correct one. What we need is an analog of the stability criterion obtained from the effective potential for elementary scalar fields. We develop such a stability criterion, and find surprisingly enough that (at least in our approximation), the existence of attractive forces is also a *sufficient* condition for the occurrence of DSB.

Effective potentials for local composite operators have been used in the literature[3,4] to study dynamical symmetry breaking. These potentials were multiple-valued functions which were unbounded from below on some sheets. They also suffer from ultraviolet problems which in some cases destroy the interpretation of the potential as an energy density[5] (see Sec. I).

A potential for nonlocal operators was introduced by Cornwall, Jackiw, and Tomboulis.[6] This no longer has ultraviolet problems but it still cannot be interpreted as an energy density. In fact, in free-field theory this potential is not bounded from below. The problem is that this is a potential for operators which are nonlocal in *time* in the Heisenberg picture. As a result the corresponding Schrödinger picture operator is explicitly time-dependent. If we add this operator to the Hamiltonian according to the prescription ($H \to H + J\phi$) then for nonzero $J$ the system does not have any stationary states. Thus the argument that $V$ is the expectation value of the Hamiltonian in a state for which $\langle\phi\rangle = \phi$ cannot be carried through. To solve this problem we define an effective potential for operators which have only spatial point separation. We show that this potential is single valued and bounded from below (see Appendix). We then calculate it to order $\alpha$ in a non-Abelian gauge theory and show that for reasonable values of $\alpha$ (i.e., $\alpha$ small enough so that we can believe the perturbative approximation to the potential) the potential

has positive curvature at the origin. Thus in this approximation we find no evidence for dynamical symmetry breakdown.

The plan of the rest of this paper is as follows: In Sec. II we briefly discuss effective potentials for local composite operators and their ultraviolet problems. We also introduce potentials for various types of nonlocal composite operators and discuss the relations between them. In Sec. V we discuss the effective potential for operators with spacelike point separation to order $\alpha$ in a general gauge theory. The actual calculation of the potential is carried out in the Appendix. Our results show that the lowest-order potential does not manifest spontaneous symmetry breaking for small coupling. We also show that the curvature of the potential is related to the attractive or repulsive nature of the forces in the Goldstone-boson channel.

## II. THE MOST EFFECTIVE POTENTIAL

The standard tool for investigating spontaneous symmetry breakdown is the effective-potential formalism.[7] Spontaneous symmetry breakdown occurs when the ground state is not invariant under a unitary transformation $U$ which commutes with the Hamiltonian. One can then find a nonsinglet operator $\varphi$ ($[U, \varphi] \neq 0$) with a nonzero expectation value in the ground state. To investigate this possibility we add a term $J\varphi$ to the Hamiltonian and compute the amplitude for the $J=0$ vacuum to be unchanged by this perturbation. The function $W(J)$ is defined as $-i$ times the logarithm of this amplitude. It is easy to verify that $W'(0)$ is the expectation value of $\varphi$ in the $J=0$ vacuum, so spontaneous symmetry breaking occurs if and only if $W'(0) \neq 0$.

If the Hamiltonian is written as $H_0 + gH$, where $H_0$ has a nondegenerate ground state, then to all orders in perturbation theory in $g$ we will find $W'(0) = 0$. This does not mean that $W'(0) = 0$ but only that the nonsymmetric ground states (if any) do not have a perturbation expansion. It does mean that if spontaneous symmetry breaking occurs the derivative of $W$ is not a single-values function at $J=0$. One way of investigating this question is to introduce the Legendre transform of $W$:

$$\Gamma(\varphi) = W(J) - J\varphi \ , \tag{1}$$

where $J$ is determined as a function of $\varphi$ by inverting the equation

$$W'(J) = \varphi \ . \tag{2}$$

Multiple values of $W'(0)$ will show up as stationary points of $\Gamma$.

When $\varphi$ is an elementary scalar field it is relatively easy to compute $\Gamma$ in perturbation theory, for it can be shown[7] to be the sum of all zero-momentum one-particle irreducible (1PI) $\varphi$ Green's functions. In the theories that we are interested in the symmetry-breaking operator $\varphi$ is a composite like $\bar{\psi}\psi$ or $A_\mu A^\mu$ and no such simple computational method exists (at least for local operators).

There are essentially two procedures that have been used to get around this difficulty. The first, introduced in Ref. 3, is to introduce auxiliary scalar fields into the theory by adding terms to the Lagrangian like

$$(\varphi - \bar{\psi}\psi)^2 \ .$$

Such a term merely equates $\varphi$ to $\bar{\psi}\psi$ and does not change the $\psi$ equations of motion. The effective potential for $\varphi$ can be computed using 1PI diagrams and is related to the potential for $\bar{\psi}\psi$ in a definite way. The relation is transcendental but one can show that simple questions about $V(\bar{\psi}\psi)$ (like the existence of nonzero minima) can be answered by examining $V(\varphi)$.

The method of Ref. 3 is convenient in field theories with local quartic couplings, but it becomes very clumsy when applied to gauge theories.[8] For this reason, we introduced in a previous paper[4] an alternative method for computing the effective potential for composite operators. The idea is very simple: Given a function $W(J)$ which has a perturbation expansion

$$W(J) = W_0(J) + \alpha W_1(J) + \alpha^2 W_2(J) \ , \tag{3}$$

we can easily compute the effective potential (equal to minus the Legendre transform) of $W$ as a power series in $\alpha$

$$V(\varphi) = J_0(\varphi)\,\varphi - W_0(J_0(\varphi)) - \alpha W_1(J_0(\varphi)) - \alpha^2[W_2(J_0) - \tfrac{1}{2}(W_1' W_0''^{-1} W_1')_{J_0}] \ , \tag{4}$$

where $J_0(\varphi)$ is the solution of

$$W_0'(J_0(\varphi)) = \varphi \ .$$

This formula is universal. It can be used for composite or elementary, local or nonlocal fields. It can also be used for effective potentials for several fields if we replace ordinary by partial derivatives and sum over indices.

An important feature of Eq. (4).is that of all orders in $\alpha$, $V(\varphi)$ is expressed as a function of $J_0$. The functions $W_i(J)$ are easily computed (for small $i$) by diagrammatic or functional integral methods, so it is easy to find $V$ as a function of $J_0$. But

$$\frac{\partial V}{\partial \varphi} = \frac{\partial V}{\partial J_0}\frac{\partial J_0}{\partial \varphi} = \frac{\partial V}{\partial J_0}\left(\frac{\partial^2 W_0}{\partial J_0^2}\right)^{-1} \ , \tag{5a}$$

$$\frac{\partial^2 V}{\partial \varphi^2} = \frac{\partial^2 V}{\partial J_0^2}\left(\frac{\partial J_0}{\partial \varphi}\right)^2 + \frac{\partial V}{\partial J_0}\frac{\partial^2 J_0}{\partial \varphi^2} \ . \tag{5b}$$

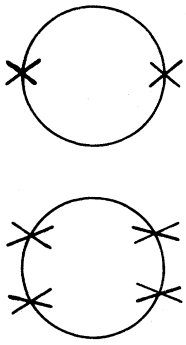

FIG. 1. Divergent contributions to the function $W(J)$ for the operator $\bar{\psi}\psi$ in free Dirac field theory.

Since in particular

$$\frac{\partial V_0}{\partial \varphi} = \frac{\partial V_0}{\partial J_0}\frac{\partial J_0}{\partial \varphi} \quad [V_0 = V(\alpha = 0)] \,, \tag{6}$$

we see that $\partial J_0/\partial\varphi$ never vanishes except at $\varphi = 0$ (because the only stationary points of the potential for a free-field theory are at $\varphi = 0$). Furthermore, since there is no symmetry breaking in free-field theory we have $J_0(0) = 0$,[9] so we can investigate symmetry breaking by looking for stationary points of $V$ *as a function of* $J_0$, without worrying about the (sometimes complicated) relation between $J_0$ and $\varphi$. Equation (5b) tells us that the curvature of $V$ at the stationary point can also be easily expressed (as a function of $J_0$) in terms of the $W_i$.

For local operators the results of this method are of course equivalent to those of the auxiliary-field technique. Unfortunately, these results are invalidated by ultraviolet divergences. The introduction of composite operators into a field theory introduces divergences above and beyond those that are dealt with in the ordinary renormalization program. If we are dealing with an operator of canonical dimension less than 4 and looking only at matrix elements of $T$ products of this operator and the fundamental fields in the theory, then all of the new divergences can be absorbed into a wave-function renormalization of the composite operator. However, when we look at vacuum matrix elements of $T$ products of the operator with no fundamental fields, new divergences arise. In fact, these new divergences arise even in free-field theory. For example, the graphs of Fig. 1 contribute divergences to the two- and four-point vacuum matrix elements of the operator $\bar{\psi}\psi$ in a free Dirac theory. Since they contribute only to vacuum matrix elements, these divergences may be eliminated by $c$-number counterterms. Moreover, one can easily argue[4] that to all orders in perturbation theory, these counterterms are local polynomials (of degree $2 \leq d \leq 4$) in the source $J$. Thus they can contribute neither to scattering amplitudes (because of their locality) nor to $W'(0)$.

However, in any finite order of perturbation theory, these new counterterms do contribute to the structure of the approximate effective potential. Thus the effective potential will appear to depend on a new arbitrary parameter, the finite piece of the vacuum counterterm. By a suitable choice of this parameter we can make minima of $V$ appear or disappear at will. On the other hand, as we have mentioned above, the stationary points of the exact $V$ must be independent of this parameter.

This is bad enough, for it makes nonsense of any perturbative calculation of $V$, but the vacuum counterterms cause an even more serious problem with the effective potential. The argument[5] that the effective potential is an energy density depends crucially on the fact that we add a term to the Hamiltonian which is linear in the source $J$. The vacuum counterterms, by forcing us to add higher-order terms in $J$ to the Hamiltonian, destroy the possibility of interpreting $V$ as an energy density and thus rob us of our stability criterion.[10]

It seems to us that all of these ultraviolet problems must be irrelevant to the physics of spontaneous symmetry breakdown, which is really a low-energy, infrared phenomenon. The simplest way to avoid these problems is to separate the points in the composite operator, that is, to study nonlocal operators. These are perfectly good signals for spontaneous symmetry breaking: If a nonsinglet nonlocal operator has an expectation value then the vacuum cannot be invariant under the symmetry.

Now that we have decided to investigate nonlocal operators, we are faced with an embarrassment of riches. Even if we do not consider boson operators, we can study the vacuum expectation value of any operator of the form

$$\varphi_f = \int d^4\epsilon \bar{\psi}(x) f(\epsilon) \psi(x+\epsilon) \,, \tag{7}$$

where, for each $\epsilon$, $f(\epsilon)$ is a matrix in spin and internal-symmetry space.

If we were solving the theory exactly, the exact choice of symmetry-breaking operator would not be very important, but obviously the accuracy of a perturbative approximation to $V(\varphi_f)$ may depend crucially on the choice of the function $f$. How then, do we decide on the best choice for $\varphi_f$?

A possible strategy to follow would be to compute a multivariable effective potential for *all* the $\varphi_f$, which is to say a potential for the fermion propagator. This is the approach of Cornwall, Jackiw, and Tomboulis.[6] They show that the effective potential for the propagator $G$ is given by

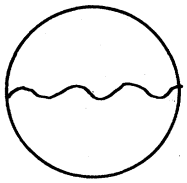

FIG. 2. The lowest-order graph for the effective potentials for fermion-antifermion operators in gauge theories. The Feynman rules for this graph depend on the operator being studied.

$$V(G) = -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}[\ln S^{-1}(p)G(p) - S^{-1}(p)G(p) + 1] + V_2(G)\,, \tag{8}$$

where $V_2(G)$ is the sum of all two-particle irreducible vacuum graphs with internal fermion lines equal to $G$. [This result, by the way, is easily seen using our perturbative formula for the Legendre transform Eq. (4).] If we include only the graph of Fig. 2 in $V_2$, then the stationarity condition $\delta V/\delta G = 0$ is equivalent to the Schwinger-Dyson equation for the electron propagator in an approximation in which the vector-meson propagator and the vertex function are given by their lowest-order values.

There are several serious problems with the use of the full potential $V(G)$. Recall that our principal reason for desiring an effective-potential formalism for DSB is that it gives us a stability criterion for deciding which solution of the equations of the theory is the correct one. The basis for this criterion is the argument[5] that $V$ is an energy density. This argument does not go through for the operator $\bar{\psi}(x)\psi(y)$ because it is nonlocal in time in the Heisenberg picture and thus explicitly time-dependent in the Schrödinger picture. Thus $V(G)$ is not an energy density, and there is no reason for the sign of its curvature at stationary points to have any physical meaning. In fact, one can easily check in free-field theory that $V(G)$ is not even bounded from below.

If we want to keep the interpretation of $V$ as an energy density, then we must restrict our attention in Eq. (7) to functions $f(\epsilon)$ which have support only for spacelike $\epsilon$. It will also be convenient to restrict the matrix form of $f$ so that the only possible symmetric value for the vacuum expectation value of $\varphi_f$ is zero. For example to study chiral symmetry breaking in QED we would take $f$ to be proportional to the unit Dirac matrix.

Our proposal then is to study the multivariable effective potential for all operators $\varphi_f$ satisfying the above restrictions. We will see in Sec. III that the task of finding the stationary points of this potential is even more difficult than the corresponding problem for $V(G)$. However, since the only symmetric stationary point is the one at the origin, we can learn a lot about spontaneous symmetry breaking just by studying the curvature of this new potential near the origin. (Remember that for $f$'s with spacelike support, the potential is an energy density.)

To conclude this section we will discuss the gauge transformation properties of our formalism. Clearly the nonlocal operators that we are interested in are not gauge-invariant. There are two reasons to worry about this. The first is a question of accuracy: In which gauge will our perturbative calculations be most accurate? The second is more serious. In any covariant gauge, gauge theories are formulated in Hilbert spaces with non-positive-definite metric, and non-gauge-invariant operators can create negative-metric states from the vacuum. Thus, if we work with a non-gauge-invariant operator, we have no guarantee that its effective potential will really satisfy the positivity requirements that are needed for the stability criterion.

The solution is simple: Work only with gauge-invariant operators. But gauge-invariant nonlocal operators are generally complicated nonpolynomial functions of the fields. Consider, however, the operator

$$\bar{\psi}(x)\left\{\exp\left[ie\int d^4z A_\mu(z)J^\mu(z)\right]\right\}\psi(x+\epsilon) \tag{9}$$

in an Abelian gauge theory. $J_\mu$ is a $c$-number current. This operator is invariant under gauge transformations with gauge functions that vanish at infinity as long as

$$\partial_\mu J^\mu(y) = \delta^4(x-y) - \delta^4(x-y-\epsilon)\,. \tag{10}$$

In particular, we can pick a purely longitudinal current

$$J_\mu = \partial_\mu \chi\,. \tag{11}$$

If we now choose to work in the Landau gauge where $\partial_\mu A^\mu = 0$ is an operator statement, we find that the gauge-invariant operator (9) is simply equal to

$$\bar{\psi}(x)\,\psi(x+\epsilon)\,.$$

Thus, in the Landau gauge the non-gauge-invariant operators we are interested in coincide with gauge-invariant operators, and their effective potentials should behave perfectly properly.[11]

We have not been able to generalize the above argument to non-Abelian gauge theories, but we believe that a generalization exists. The analogous gauge, where gauge-non-invariant operators will be equal to gauge-invariant ones, will cer-

tainly coincide with the Landau gauge in lowest order in perturbation theory because the non-Abelian part of a gauge transformation is of higher order in the coupling constant. Thus, for the present paper, where we will compute the effective potential only to lowest order, it will be permissible to work in the Landau gauge even in non-Abelian theories.

## III. CALCULATION OF THE EFFECTIVE POTENTIAL IN GAUGE THEORIES

In Sec. III we decided that the best strategy for studying DSB was to construct the effective potential for operators of the form

$$\int d^3\epsilon\,\bar{\psi}(\vec{x})M\psi(\vec{x}+\vec{\epsilon})J(\vec{\epsilon})\equiv\varphi_J,$$

where $M$ is some symmetry-breaking matrix. To do this we must first construct the functions $W_0$ and $W_1$ and then use formula (4). As explained in Sec. III we work in the Landau gauge.

$W_0$ is given by the expression

$$\left[\int d^4x\right]W_0(J)=\frac{1}{i}\ln\det[i\not\partial\delta(x-x')-MJ(x-x')],$$

$$W_0(J)=\frac{1}{i}\int\frac{d^4p}{(2\pi)^4}\operatorname{tr}\ln[\not p-M\bar{J}(p)] \tag{12}$$

$$=2\int\frac{d^4p}{(2\pi)^4}\operatorname{tr}\ln\left(1+\frac{M^2\bar{J}^2}{p^2}\right).$$

In writing (12) we have introduced the notation $J(x)=\delta(x_0)J(\vec{x})$ and dropped an infinite term independent of $J$. $\bar{J}$ is the 4-dimensional Fourier transform of $J$, and the trace in the last line is over the internal-symmetry space.

$W_0$ is given by the diagram in Fig. 2. Its explicit value is

$$W_1=\frac{-e^2}{2(2\pi)^8}\int\frac{d^4pd^4q}{(p-q)^2+i\epsilon}\operatorname{tr}\tau_a\gamma_\mu\frac{1}{\not p-M\bar{J}(p)+i\epsilon}\tau_a\gamma_\nu\frac{1}{\not q-M\bar{J}(q)+i\epsilon}\left[\frac{(p-q)_\mu(p-q)_\nu}{(p-q)^2}-g_{\mu\nu}\right]$$

$$=\frac{2e^2}{(2\pi)^8}\int\frac{d^4pd^4q}{(p-q)^2}\left\{\left[3p\cdot q+2\frac{(p\cdot q)^2-p^2q^2}{(p-q)^2}\right]\operatorname{tr}\tau_a\frac{1}{q^2+M^2\bar{J}^2(q)}\tau_a\frac{1}{p^2+M^2\bar{J}^2(p)}\right.$$

$$\left.+3\bar{J}(p)\bar{J}(q)\operatorname{tr}\tau_a\frac{M}{q^2+M^2\bar{J}^2(q)}\tau_a\frac{M}{p^2+M^2\bar{J}^2(p)}\right\}, \tag{13}$$

where the last integral is Euclidean and the trace runs over internal-symmetry space. Notice that we have not assumed that $[\tau_a,M]=0$ so that we can study the case of spontaneous breakdown of the gauge symmetry itself. Our approximation for the vector propagator is not a very good one when the Higgs phenomenon takes place. However, it is reasonable to assume that one can make a better approximation by simply adding a mass term to the denominator of the propagator. We will see shortly that such a term will have little effect on our conclusions.

At this point we could remember that $\bar{J}(p)$ is independent of $p_0$, do the energy integrations, and obtain the effective potential for $\varphi_J$. This program is carried out in the Appendix. The resulting effective potential is single-valued and bounded from below. However, it is quite complicated, and the equation for its stationary points is a nonlinear integral equation which appears to be harder to solve than the Schwinger-Dyson equations.

Acting on the time-honored principle that discretion is the better part of valor, we have decided to restrict ourselves to an investigation of the effective potential near the point $\varphi_J=0$. This may, of course, be a dangerous procedure, for Coleman[5] has argued that spontaneous symmetry breaking can occur even if the curvature of the effective potential near the origin remains positive. All that is necessary is that there exist another minimum with an effective potential value lower than $V(\varphi_J=0)=0$. On the other hand, in all extant models of spontaneous symmetry breakdown, the origin does become unstable. We feel that it is worth the risk to assume that this also happens in our model.

The effective potential $V(\varphi_J)$ near the origin is computed, in the Appendix. Instead of examining it directly, however, we want to present a method for determining its curvature which we feel has a better chance of being useful in higher-order calculations. The form of the potential near the origin is

$$\int d^3pd^3q\varphi(\vec{p})K_3(\vec{p},\vec{q})\varphi(\vec{q}).$$

If we had looked at operators that were nonlocal in time, the corresponding potential would look like

$$\int d^4pd^4q\varphi(p)K_4(p,q)\varphi(q),$$

where $K_3$ and $K_4$ are related by[12]

$$K_3 = \frac{1}{\pi^2}|\vec{p}||\vec{q}| \int dp_0 dq_0 \frac{K_4(p,q)}{p^2 q^2}.$$

We now make the observation that a *sufficient* condition for $K_3$ to be positive definite is simply that $K_4$ be positive definite. Thus, *near the origin* it is reasonable to look at the curvature of the effective potential for operators that are nonlocal in time. Of course, if $K_4$ has a negative eigenvalue, we cannot be sure that $K_3$ also has one. Nonetheless, we feel that the relative simplicity of $K_4$ makes it a useful first test for DSB.

We begin our study of $K_4$ by noticing that since it is Euclidean invariant we can examine its positivity separately in subspaces of functions with definite O(4) transformation properties. First we look at invariant functions $J(p^2)$. For such functions one can easily show that the first term in $W_1$ vanishes identically.[13] It is then easy to compute $K_4$ using Eqs. (12), (13), and (4). The result is

$$K_4(p,q) = p^2\delta^4(p-q) - \frac{3e^2}{2(2\pi)^4} B \frac{1}{(p-q)^2}, \qquad (14)$$

where

$$B = \frac{\mathrm{tr}\tau_a M \tau_a M}{\mathrm{tr} M^2}.$$

We are therefore interested in eigenvalues of this integral operator,

$$p^2\psi(p) - \frac{3e^2}{2(2\pi)^4} B \int \frac{\psi(q)}{(p-q)^2} d^4q = \lambda\psi(p). \qquad (15)$$

By Fourier transforming we obtain a differential equation in position space,

$$-\nabla^2\psi(x) - \frac{3e^2B}{8\pi^2}\frac{\psi(x)}{x^2} = \lambda\psi(x). \qquad (16)$$

Writing $\psi(x) = r^{-3/2}\phi(x)$, we get

$$-\phi''(x) + \left(\frac{3}{4} - \frac{3e^2B}{8\pi^2}\right)\frac{1}{x^2}\phi(x) = \lambda\phi(x). \qquad (17)$$

Clearly, the behavior of the eigenvalues of this equation depends crucially on the magnitude and sign of $B$. If $B<0$, then for any value of $e$ the "potential" in Eq. (17) is repulsive and we can have no negative eigenvalues. Even if $B$ is positive we can only have negative eigenvalues if $e^2$ is greater than $2\pi^2/|B|$. Notice that in this case the spectrum will be unbounded from below since we can get from any negative to any other by scaling $r$.

For arbitrary values of $e$ the operator (14) has zero eigenvalues corresponding to the two possible power-law solutions of a scale-invariant equation.

If we now look at functions $\chi(p)$ with higher values of the O(4) angular momentum we will find that they also have non-negative eigenvalues as long as $e^2 > 2\pi^2/|B|$. This is because of the repulsive angular momentum barrier for higher partial waves.

We see that for $e^2 < 2\pi^2/|B|$ the kernel $K_4$ (and thus, *a fortiori*, the kernel $K_3$) has only non-negative eigenvalues. Values of $e$ larger than this are clearly outside the range of validity of our approximation. We can therefore conclude that, in this order, the origin remains a stable minimum of the effective potential and no spontaneous symmetry breaking takes place.

We can also argue qualitatively that the addition of a mass to the vector propagator will not substantially alter our conclusions. In fact we can guess that the major effect of such a mass would be to change the $1/r^2$ "potential" into something that falls off faster at large $r$, thus reducing the possibility of negative eigenvalues. (Of course, the mass term also makes the potential nonlocal, so our argument is not rigorous.)

Although we have not found spontaneous symmetry breaking for reasonable values of the coupling constant, there is one important fact to be learned from the preceding equations. If the constant $B$ in (14) is positive, then $K_4$ is unstable for sufficiently large $e$, while if $B$ is negative it is stable for all values of $e$. To put it another way, if $B>0$, then even for small $e$ we have a *tendency* to break the symmetry.

The constant $B$ is, however, exactly the internal-symmetry factor which multiplies the kernel of the Bethe-Salpeter equation (in the ladder approximation) for the $\bar{\psi}M\psi$ channel. The reader may verify for himself (e.g., by examining QED with one positively charged and one negatively charged fermion) that $B<0$ corresponds to repulsion. So we see that the tendency toward instability of the effective potential can be completely correlated with the tendency for binding a Goldstone boson. The curvature of the potential at the symmetric point becomes negative at exactly the value of $e$ which binds the Goldstone particle.

Unfortunately, the criterion of attraction does not help us to understand why (for example) singlet chirality should break down, leaving $SU_3$ (valence) preserved in a gauge theory of strong interactions. The forces in *all* color-singlet channels are known to be attractive.[14] One can even find non-color-singlet channels with attractive forces, for example, by combining two quarks in the $\underline{3}$ representation into a $\bar{\underline{3}}$.[15]

It is clear then that to get even a qualitative understanding of when and how dynamical symmetry breaking occurs in gauge theories, we must go beyond weak-coupling approximations to the effective potential. Nonetheless, we feel that the machinery developed in this paper may be useful even in a

nonperturbative context. We have reduced the problem of DSB to that of determining whether a certain kernel is positive definite. This property of $K_3$ can certainly be tested by more sophisticated methods than actually computing all of its eigenvalues. It may not even be necessary to compute $K_3$ itself (e.g., if we can find an approximate integral equation for $K_3$, the positively of the kernel might be determinable just from the structure of the equation).

At present we are not in possession of any such magical nonperturbative methods. We can, however, use the renormalization-group equations to shed a little more light on the properties of $K_3$. Since $K_3$ is one of the coefficients in the effective potential of a multiplicatively renormalizable operator, it satisfies a simple renormalization-group equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-2\gamma\right)K_3=0, \tag{18}$$

where $\gamma$ is twice the anomalous dimension of the fermion field in the Landau gauge.

The solution of (18) of course satisfies

$$K_3(\lambda p,\lambda q,g)=\lambda^{-2}\exp\left[-2\int_0^{\ln\lambda}\gamma(t)dt\right]K_3(p,q,\bar{g}(\ln\lambda)), \tag{19}$$

$$d\bar{g}/dt=\beta(\bar{g}),\quad \bar{g}(0)=g.$$

Now let us suppose that for some value of the coupling $g$, $K_3$ has a negative eigenvalue:

$$-|E|\varphi(\vec{p})=\int d^3q\,K_3(\vec{p},\vec{q},g)\varphi(\vec{q}). \tag{20}$$

Then,

$$-|E|\varphi(\lambda\vec{p})=\lambda^3\int d^3q\,K_3(\lambda\vec{p},\lambda\vec{q},g)\varphi(\lambda\vec{q})$$

$$=\lambda\exp\left[-2\int_0^{\ln\lambda}\gamma(\bar{g}(t))dt\right]\int d^3qK_3(\vec{p},\vec{q},\bar{g}(\ln\lambda))\varphi(\lambda\vec{q}). \tag{21}$$

If the theory is infrared-stable (asymptotically free), then we can make $\bar{g}$ as small as we wish by taking $\lambda\to 0$ ($\lambda\to\infty$). Since $\exp[-2\int_0^{\ln\lambda}\gamma(t)dt]$ is positive, we see that in both types of theory $K_3$ will have negative eigenvalues for arbitrarily small $g$ if it ever has negative eigenvalues at all. In an infrared-stable theory, the value of the eigenvalue goes to $-\infty$ as $g$ gets small, while in the asymptotically free case it goes to zero. The latter behavior is clearly preferable, and this result reinforces previous conclusions[16] about dynamical symmetry breaking in massless infrared-stable theories.

The fact that the renormalization-group equations predict DSB for arbitrarily small coupling does not contradict the explicit results we have obtained previously. In an asymptotically free theory it simply means that no matter how small we make the coupling constant, the effective coupling in the infrared region is large and our weak-coupling approximation invalid. In an infrared-stable theory it probably means that DSB does not occur for *any* value of the coupling.

## IV. CONCLUSION

We have accomplished two tasks in this paper. The first was to demonstrate that DSB in gauge theories is a subject of great potential interest as well as extraordinary difficulty, and that it could not be attacked by weak-coupling methods. The second task was to build a formalism that could be used to extract the qualitative aspects of DSB from a theory with the least possible difficulty. We hope that we have been sufficiently convincing about the first point and sufficiently modest about the second.

The composite-operator effective potential that we have developed is the first to share all of the nice properties of the effective potential for elementary scalar fields. It is single-valued and bounded from below and can be interpreted as an energy density. These properties enabled us to study dynamical symmetry breaking by simply determining the curvature of the potential near the origin. We were then able to show that DSB did not occur in the lowest-order approximation to the potential.

We are convinced that further progress in this subject will come about only through the introduction of a new nonperturbative approximation scheme (perhaps the $1/N$ expansion or the strong-coupling methods of lattice gauge theories), and that in the context of such a scheme, the questions that we have asked will have simple answers. The physics involved here is too fundamental to be as complicated as it looks to us now.

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## APPENDIX

In this appendix we want to construct the effective potential for spacelike composite operators of the form

$$\int d^3\epsilon \bar{\psi}(\vec{x})\psi(\vec{x}+\vec{\epsilon})J(\vec{\epsilon})$$

in massless QED. The extension to other gauge theories is immediate.

We can compute the relevant $W$ functions by simply doing the $p_0$ integrations in the 4-dimensional $W$'s of Eqs. (15) and (16). For the free-field contribution we get

$$\mathcal{W}_0(J(\vec{p})) = \int_{-\infty}^{\infty} dp_0 W_0(J(\vec{p})) = \frac{2}{(2\pi)^3}\int d^3p\,[J^2(\vec{p})+\vec{p}^2]^{1/2}, \tag{A1}$$

$$\frac{\delta \mathcal{W}_0}{\delta J(\vec{p})} = \varphi(\vec{p}) = \frac{2J(\vec{p})}{(2\pi)^3[J^2(\vec{p})+\vec{p}^2]^{1/2}} . \tag{A2}$$

Note that, for real $J$, $\varphi$ is bounded:

$$\varphi(\vec{p}) \leq \frac{2}{(2\pi)^3} . \tag{A3}$$

The inverse relation to (A2) is

$$J(\vec{p}) = \frac{|\vec{p}|\varphi(\vec{p})}{[4/(2\pi)^6 - \varphi^2(\vec{p})]^{1/2}} . \tag{A4}$$

So we can construct the effective potential

$$\mathcal{V}_0(\varphi(\vec{p})) = \int d^3p\varphi(\vec{p})J(\vec{p}) - \mathcal{W}_0(J(\vec{p})) = \int d^3p|\vec{p}|\left\{\left[\frac{4}{(2\pi)^6} - \varphi^2(\vec{p})\right]^{1/2} - \frac{2}{(2\pi)^3}\right\} . \tag{A5}$$

We have added a constant to $\mathcal{V}_0$ to enforce the condition $\mathcal{V}_0(0)=0$. For any $\varphi(\vec{p})$ satisfying the bound (A3) and vanishing sufficiently fast as $|\vec{p}|\to\infty$, $\mathcal{V}_0(\varphi)$ is a finite positive number. Therefore, for values of the vacuum expectation value $\varphi$ which can be attained by applying a real external source to the system, $\mathcal{V}_0(\varphi)$ is bounded from below.

The only minimum of $\mathcal{V}_0$ is $\varphi=0$ [as we already see from (A2)], so there is no spontaneous breakdown of chirality in free-field theory.

The next-order correction is obtained by doing the $p_0$ integration in (16). We obtain

$$\mathcal{W}_1(J) = \frac{e^2}{2(2\pi)^6}\int d^3p\,d^3q\left\{\frac{3}{2}\frac{J(\vec{p})J(\vec{q})+\vec{p}\cdot\vec{q}+\omega_p\omega_q}{\omega_p\omega_q|\vec{p}-\vec{q}|(\omega_p+\omega_q+|\vec{p}-\vec{q}|)} + \frac{2\vec{p}\cdot\vec{q}\,\omega_p\omega_q+p^2\omega_q{}^2+q^2\omega_p{}^2+(\vec{p}\cdot\vec{q})^2-\vec{p}^2\vec{q}^2}{\omega_p\omega_q[|\vec{p}-\vec{q}|^2-(\omega_p+\omega_q)^2]^2} + \frac{(\omega_q{}^2-\vec{q}^2)|\vec{p}-\vec{q}|^2+(\vec{p}\cdot\vec{q})^2-\vec{p}^2\vec{q}^2}{\omega_q|\vec{p}-\vec{q}|^3[\omega_p{}^2-(\omega_q-|\vec{p}-\vec{q}|)^2]} + \frac{2(\omega_q-|\vec{p}-\vec{q}|)^2[\vec{q}^2(\omega_q-|\vec{p}-\vec{q}|)-2\omega_q\vec{p}\cdot\vec{q}]}{\omega_q|\vec{p}-\vec{q}|^2[\omega_p{}^2-(\omega_q-|\vec{p}-\vec{q}|)^2]^2} + \frac{2(\omega_q-|\vec{p}-\vec{q}|)[(\vec{p}\cdot\vec{q})^2-\vec{p}^2\vec{q}^2+\omega_q{}^2\vec{p}^2]}{\omega_q|\vec{p}-\vec{q}|^2[\omega_p{}^2-(\omega_q-|\vec{p}-\vec{q}|)^2]^2} + (\vec{p}\leftrightarrow\vec{q})\right\} , \tag{A6}$$

where

$$\omega_p = [\vec{p}^2+J^2(\vec{p})]^{1/2} .$$

Using Eqs. (A4), (A5), (A6), and (4) one can write the effective potential to $O(\alpha)$ as a function of $\varphi(\vec{p})$.

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[9]Typically $J_0$ will be a multiple-valued function of $\phi$ near $\phi=0$. The branches that do not satisfy $J_0(0)=0$ are unphysical.

[10]This was pointed out to us by D. Gross. T. B. thanks Professor Gross for an enlightening discussion.

[11]We learned this trick from Professor L. Susskind. A similar method has been introduced by Fischler and Brout to deal with the gauge variance of the effective potential in ordinary Higgs theories; see W. Fischler and R. Brout, Phys. Rev. D 11, 905 (1975).

[12]The integrals in the above equation are Euclidean. The rather odd relation between $K_3$ and $K_4$ comes about because $\phi(p)=(|\vec{p}|/\pi p^2)\phi(\vec{p})$.

[13]The computation involved is essentially the same as that done by K. Johnson [*Lectures on Particles and Field Theory*, Brandeis Summer Institute in Theoretical Physics, 1964, edited by S. Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, New Jersey, 1965), Vol. 2, p. 1] when showing that the wave-function renormalization constant of the electron, $Z_2$, vanishes in the Landau gauge.

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