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Variational approach to bound states in scalar-gluon field theory

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Two variational approaches are employed to attack the bound-state problem of a charged scalar field interacting with an Abelian gauge field. The resulting variational equations allow for a qualitative discussion of all possible physical situations. Boundary conditions play a crucial role in their interpretation. A specially developed perturbation scheme yields hydrogenlike spectra. "Self-trapping" solutions and configurations with complete screening of the long-range force are discussed and are shown not to be obtainable by perturbation for small coupling. Metastable states appear for strong coupling.

I. INTRODUCTION

The problem of finding bound states in a quantum field theory is an old and still unsolved one. The related perspective of identifying particles as bound states with permanently bound constituents or as spatially bounded configurations of an underlying field theory has amplified the problem in its range and urgency.

In particular, Vinciarelli[1] and Bardeen *et al.*[2] have set up field-theoretical models which display quarks strongly bound in a scalar-gluon field. The problem of low-lying bounded configurations is treated as a variation problem in which the trial states are coherent with respect to the gluon field and few-particle states with respect to the constituent field. The essential binding mechanism in these papers is the trapping in a kink produced by a nonvanishing vacuum expectation value of the gluon field.

As a simple and at the same time physically interesting system we consider charged scalar fields coupled to neutral massless gauge gluons in four space-time dimensions without[3] spontaneous symmetry breaking. We have chosen this theory since for small coupling it is evidently identical with scalar electrodynamics for which our intuition is best supported by experimental and theoretical knowledge. Contrary to the kink mechanism, bound states are here produced by the matter field acting as a source of the gluon field.

Two conceptually very different variational approaches are proposed. Firstly, we use the energy component of the gauge-invariant energy-momentum tensor of scalar-gluon dynamics and minimize within a set of trial states similar to the ones of Ref. 2. In a second approach, we take the energy functional of a generalized Hartree-Fock *Ansatz* which consists of the energy of an effective gluon field and the energy of a scalar field in the latter. The effective gluon field is to reflect the interaction of a matter quantum with the rest of the system; unlike in conventional Hartree-Fock theory, it is not a known functional of the matter field, but is varied independently.

The resulting differential equations for the functions characterizing the trial states for either approach differ from one another and from the classical Euler-Lagrange equations of gluon dynamics by essential signs with interesting physical consequences.

The aim of our considerations is to give a classification and interpretation of the various bounded solutions predicted by the variational equations and to compare our findings with intuitive physical expectations. The bounded solutions of our variational equations are fully characterized only after specifying boundary conditions for the gluon trial function. Different boundary conditions result in completely different physical situations described by the solutions. For instance, it is possible to prescribe either the strength of an effective point charge at the origin or the total charge of the system. Both turn out to be related by a sum rule. It is amusing to note that in our generalized Hartree-Fock *Ansatz* this sum rule, just as in asymptotically free theories, acts in such a way that the effective coupling increases with the distance, although, of course, for general configurations the underlying quantum field theory is not asymptotically free.

Furthermore, we establish the existence of solutions with no effective point charge at the origin, which correspond to "self-trapping" (i.e., binding without any driving term), and the possibility of solutions with a completely screened long-range force.

A special perturbation theory is constructed in the sequel. The essential input there is the identification of the total energy of the system with the levels of an associated eigenvalue problem. Relativistic hydrogenlike spectra are found and next-to-leading-order corrections are calculated. Self-trapping solutions are shown not to be attainable by perturbing around small coupling constants.

For strong coupling we are able to discuss the

qualitative features of the system. Interesting configurations arise: simultaneous self-trapping and long-range screening, radial charge-density oscillations and metastable states.

Although it is not at all clear whether the various bounded states found in the described variational scheme survive in the full quantum-field theory, the approach seems interesting enough to be pursued further.

In detail we proceed as follows. In Sec. II we give a short description of our variational procedure. The solutions of the variational equations under the possible boundary conditions are discussed in general in Sec. III A, the perturbation theory is outlined in Sec. III B, and Sec. III C deals with strong coupling. Our main results are collected in Sec. IV.

II. VARIATIONAL EQUATIONS

As a relatively simple and at the same time physically relevant system, we choose scalar electrodynamics in four dimensions or better, since we do not confine ourselves to small coupling, scalar-gluon dynamics, i.e., a complex scalar field ψ, gauge invariantly coupled to the (massless) gluon field A_μ. Because of the gauge degree of freedom, some care has to be taken in the derivation of variational equations from varying an energy functional.

From the Lagrangian

$$\begin{aligned} &\mathfrak{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}|D_\mu\psi|^2 - \tfrac{1}{2}m^2|\psi|^2 , \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \quad D_\mu = \partial_\mu - ieA_\mu , \end{aligned} \tag{2.1}$$

we get the canonical energy-momentum tensor

$$\begin{aligned} T^c_{\mu\nu} = {} & F_\mu{}^\tau F_{\tau\nu} + \tfrac{1}{4}g_{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} - F_{\mu\tau}\partial^\tau A_\nu \\ & + \tfrac{1}{2}[D_\mu\psi(\partial_\nu\psi)^* + (D_\mu\psi)^*\partial_\nu\psi \\ & - g_{\mu\nu}(|D_\tau\psi|^2 - m^2|\psi|^2)] , \end{aligned} \tag{2.2}$$

which is, as is well known, neither gauge invariant nor positive definite.

Using the equations of motion, we recast $T^c_{\mu\nu}$ into

$$\begin{aligned} T_{\mu\nu} = {} & F_\mu{}^\tau F_{\tau\nu} + \tfrac{1}{4}g_{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} + \partial^\tau(F_{\tau\mu}A_\nu) \\ & + \tfrac{1}{2}[D_\mu\psi(D_\nu\psi)^* + (D_\mu\psi)^*D_\nu\psi \\ & - g_{\mu\nu}(|D_\tau\psi|^2 - m^2|\psi|^2)] , \end{aligned} \tag{2.3}$$

which is gauge invariant up to a locally conserved divergence.

To approach the problem of finding the ground state in the quantum field theory defined by (2.1), we follow closely the polaron analogy outlined in great detail in Ref. 2. Essentially, this method consists in minimizing the total energy

$$U = \int d^3x\, T_{00} , \tag{2.4}$$

naively normal ordered, with respect to coherent trial states

$$|\varphi, \phi_\mu\rangle = C_0^\dagger \exp\left(i\int d^3x\, \dot{A}^\mu \phi_\mu\right)|0\rangle . \tag{2.5}$$

The creation operator $C_0^\dagger$ is defined by the expansion of the matter-field operator

$$\psi(x) = \sum_n [\varphi_n(x)C_n + \bar{\varphi}_n(x)B_n^\dagger] \tag{2.6}$$

in terms of Klein-Gordon eigenfunctions for the motion in the self-consistent external field to be determined; for the photons, Gupta-Bleuler quantization is adopted.

We find, with $\varphi \equiv \varphi_0$,

$$\begin{aligned} \bar{U} &= \langle \varphi, \phi_\mu | U | \varphi, \phi_\mu \rangle \\ &= \tfrac{1}{2}\int d^3x\, [\vec{E}^2 + \vec{B}^2 + |\dot{\varphi}|^2 + |(\vec{\nabla} + ie\vec{\phi})\varphi|^2 \\ &\quad + m^2|\varphi|^2 + e^2|\varphi|^2\phi_0{}^2] . \end{aligned} \tag{2.7}$$

The trial function φ by definition corresponds to a stationary state. Hence, with ω as the frequency, we get

$$|\dot{\varphi}|^2 = \omega^2|\varphi(\vec{x})|^2 ,$$

and the normalization condition is

$$\int d^3x\, |\varphi|^2 = \frac{1}{2\omega} . \tag{2.8}$$

Varying under the latter constraint, we find

$$\Delta\phi_\mu = e^2|\varphi|^2\phi_\mu \quad (\partial^\mu\phi_\mu = 0) , \tag{2.9a}$$

$$(\vec{\nabla} + ie\vec{\phi})^2\varphi = (\lambda + e^2\phi_0{}^2)\varphi , \tag{2.9b}$$

where the mass and the frequency ω have been absorbed in the Lagrange parameter λ which takes account of the constraint (2.8).

It is important to note a sign difference when Eqs. (2.9) are compared with the Euler-Lagrange equations corresponding to (2.1),

$$\Delta A_\mu^{\text{class}} = e^2|\psi^{\text{class}}|^2 A_\mu^{\text{class}} , \tag{2.10a}$$

$$(\vec{\nabla} + ie\vec{A}^{\text{class}})^2\psi^{\text{class}} = [m^2 - e^2(A_0^{\text{class}})^2]\psi^{\text{class}} , \tag{2.10b}$$

which we wrote for the sake of comparison for time-independent classical fields. A similar sign difference, the importance of which will be discussed in the sequel, has also been noted by Vinciarelli[3] in a somewhat different context.

Still another variational model is conceivable. We imagine a matter quantum to move in a Coulomb-type gluon field, which is supposed to simulate the effects of the interaction with the rest of the system. Thus, we take as the total energy to

be minimized the sum of the gluon-field energy and the canonical energy of the matter field in the former (i.e., the energy component of the canonical energy-momentum tensor for a scalar field in a Coulomb-type external gluon field). This can be considered as a generalized Hartree-Fock *Ansatz*. One particle moves in the field generated by the rest of the system. In ordinary Hartree-Fock theory this field is a known functional of the particle fields; in view of the fact that in our case the gluon field is quantized and that there is no sharp number of matter quanta, we propose to vary the effective gluon field as well. The energy functional to be varied is now

$$\overline{V} = \tfrac{1}{2}\int d^3x\,[(\nabla\phi)^2 + |\nabla\varphi|^2 + m^2|\varphi|^2 + |\dot{\varphi}|^2 - e^2\phi^2|\varphi|^2]. \tag{2.11}$$

$\phi(x)$ and $\varphi(x)$ are gluon- and matter-field trial functions. We now get

$$\Delta\phi = -e^2|\varphi|^2\phi\,, \tag{2.12a}$$

$$\Delta\varphi = (\lambda - e^2\phi^2)\varphi\,, \tag{2.12b}$$

which correspond to Eqs. (2.9) if a solution with $\vec{\phi}=0$ is chosen, up to another significant sign difference. Again these equations differ from the Euler-Lagrange equations (2.10).

As we shall see in the next section, these sign differences lead to different solutions which simulate even qualitatively different physical properties, and the following question arises: Which of the *Ansätze* is the appropriate one to describe properties of the physical system given by the Lagrangian (2.1)?

In this paper we shall arrive at only a partial answer to this question. We shall show that to lowest order in e^2 all three equations give the same answer. For strong coupling we merely establish the important physical properties predicted by these three *Ansätze* in the hope that at least some features will be shared by the fully quantized theory.

III. DISCUSSION AND INTERPRETATION

A. General discussion

To arrive at a physical interpretation of Eqs. (2.9), (2.10), and (2.12), we still have to specify boundary conditions. Since the trial states (2.5) are normalizable, Eq. (2.8) follows as one boundary condition. For the gluon fields $\phi(\vec{x})$ (to avoid unnecessary and awkward semantics, the classical fields, trial functions, and Hartree-Fock-type fields introduced in the previous section are all referred to as "fields") boundary conditions have to be imposed. Different assumptions lead to radically different physical situations. In the following we shall emphasize two circumstances: In one solution the long-range component of the gluon field is completely shielded;[4] the other one, rather, corresponds to a particle which is bound in a Coulomb potential modified at small distances.

That these two situations will occur can be anticipated from our equations. If there exist solutions with a normalizable matter field at all, then Eqs. (2.9a), (2.10a), and (2.12a) imply that the gluon field ϕ fulfills Laplace's equation $\Delta\phi=0$ at large distances.

For the sake of simplicity we choose rotationally invariant configurations (other assumptions lead, in general, to infinitely many coupled ordinary differential equations). Furthermore, for Eqs. (2.9b) and (2.10b) we take the trivial solutions $\vec{\phi}\equiv 0$ and $\vec{A}^{\text{class}}=0$.

Hence, Eqs. (2.12) assume the form

$$v_1'' + e^2\frac{u_1^2}{r^2}v_1 = 0\,, \tag{3.1a}$$

$$u_1'' + \left(\frac{v_1^2}{r^2} - \lambda\right)u_1 = 0\,, \tag{3.1b}$$

where we set $v_1 = er\phi(r)$ and $u_1 = r\varphi(r)$. Similarly, Eqs. (2.9) and (2.10) give

$$v_2'' - e^2\frac{u_2^2}{r^2}v_2 = 0\,, \tag{3.2a}$$

$$u_2'' - \left(\frac{v_2^2}{r^2} - \lambda\right)u_2 = 0\,, \tag{3.2b}$$

and

$$v_3'' - e^2\frac{u_3^2}{r^2}v_3 = 0\,, \tag{3.3a}$$

$$u_3'' + \left(\frac{v_3^2}{r^2} - \lambda\right)u_3 = 0\,, \tag{3.3b}$$

respectively. The equations determining the "potentials" v_i and the "Schrödinger" equations for the matter fields shall be referred to as (a) and (b) equations for greater convenience. Normalizability means

$$u_i(\infty) = 0 \tag{3.4}$$

and $u_i(r)$ not too singular at the origin.

A possible strategy for solving these sets of equations is to specify boundary conditions for the v_i and to solve the (a) equations to get functionals

$$v_i = v_i[u_i; r]\,, \tag{3.5}$$

which inserted into the (b) equations result in very complicated nonlinear "eigenvalue" problems

$$u_1'' + (r^{-2}v_1^2[u_1; r] - \lambda)u_1 = 0\,, \tag{3.6}$$

etc.

Nonetheless, it is possible to discuss the physically important behavior at the origin and at infinity under quite general conditions.

At $r=\infty$, we see from (3.4) and the (a) equations that v_i has to be asymptotically linear

$$v_i \underset{r\to\infty}{\sim} Er+Z\alpha . \tag{3.7}$$

The quantity $Z\alpha$ ($\alpha=e^2/4\pi$) obviously determines the total charge with which the matter field interacts; E is related to the exponential falloff of the fields u_i, as is evident from the (b) equations and is thus connected to the total energy, as we will discuss in some detail below.

The discussion of the behavior of u_i and v_i at the origin can be established in the standard manner. With the *Ansätze*

$$v\sim r^{\mu},\quad u\sim r^{\nu}\ \text{for}\ r\to 0,$$

we get from our equations

$$\mu>0\Rightarrow\nu=0\ \text{or}\ \nu=1,$$

$$\mu=0\Rightarrow\begin{cases}\nu=\frac{1}{2}\pm[\frac{1}{4}-v_i^{\,2}(0)]^{1/2} & (i=1,3),\\ \nu=\frac{1}{2}\pm[\frac{1}{4}+v_2^{\,2}(0)]^{1/2},\end{cases} \tag{3.8}$$

where $\mu<0$ is inconsistent. Further information is obtained from the requirement that the total energy

$$\overline{U}=\tfrac{1}{2}e^2\int d^3x\,\varphi^2\phi^2-ZE$$

$$=2\pi\int_0^{\infty}dr\,u_i^{\,2}v_i^{\,2}r^{-2}-ZE, \tag{3.9}$$

expressed in terms of solutions of the (a) and (b) equations [the boundary term due to the asymptotic behavior (3.7) is included], be finite. This excludes some of the cases listed in (3.8).

We now turn to the discussion of the important question of which data are to be specified to fix the energy of the ground state we are looking for. As external parameters e^2 and λ appear in our equations, the coupled eigenvalue and boundary value problem requires, of course, three boundary values. Two of them, $Z\alpha$ and E, are introduced via the asymptotics of $v_i(r)$; the norm of φ figures as the third parameter. The norm (2.8) tells us

$$\int_0^{\infty}dr\,u_i^{\,2}(r)=\frac{1}{8\pi\omega}. \tag{3.10}$$

However, from the (a) equations we see that only

$$e^2\int_0^{\infty}dr\,u_i^{\,2}=\frac{\alpha}{2\omega} \tag{3.11}$$

serves as a parameter. Hence, the spectrum is given as functions

$$\lambda=\lambda\left(\frac{\alpha}{2\omega},E,Z\alpha\right)$$

$$=E^2\tilde{\lambda}\left(\frac{\alpha E}{2\omega},Z\alpha\right), \tag{3.12}$$

and the total energy is

$$\overline{U}=\overline{U}\left(\frac{\alpha}{2\omega},E,Z\alpha\right)$$

$$=EV\left(\frac{\alpha E}{2\omega},Z\alpha\right), \tag{3.13}$$

where the last equations result because of dimensional reasons.

A further reduction of the number of parameters can be achieved only if additional physical input is added. To exemplify this, we write the (a) equations as integral equations

$$v_i(r)=\tilde{E}r+\tilde{Z}\alpha$$

$$\mp4\pi\alpha\int_0^{r}ds\,(r-s)s^{-2}u_i^{\,2}(s)v_i(s) \tag{3.14}$$

or

$$v_i(r)=Er+Z\alpha$$

$$\mp4\pi\alpha\int_r^{\infty}ds\,(s-r)s^{-2}u_i^{\,2}(s)v_i(s); \tag{3.15}$$

the upper (lower) signs apply for $i=1$ ($i=2,3$).

The inhomogeneous term in (3.15) exhibits the behavior for large r; the behavior at the origin is then

$$v_i(0)=\tilde{Z}\alpha=Z\alpha\mp4\pi\alpha\int_0^{\infty}ds\,s^{-1}u_i^{\,2}(s)v_i(s) \tag{3.16}$$

and

$$v_i'(0)=\tilde{E}=E\pm4\pi\alpha\int_0^{\infty}ds\,s^{-2}u_i^{\,2}(s)v_i(s). \tag{3.17}$$

Equation (3.14), on the contrary, prescribes the behavior at the origin and allows one to compute the behavior at infinity.

Equations (3.16) are of special interest. They are sum rules which relate the apparent charge Z experienced at infinity and the strength of a point charge $\tilde{Z}$ at the origin. Either Z or $\tilde{Z}$ enters as a free parameter; the other one is given by the sum rule. Physically speaking, setting $Z=0$, we have a complete screening of the long-range force, a necessary condition for confinement; $\tilde{Z}=0$ gives a self-trapping without a pointlike driving singularity.

In view of this, the sign differences in the (a) equations are of particular importance. Let us start with Eqs. (3.2a) and (3.3a). Since the curvature of v_i has in this case always the same sign as v_i itself, the qualitative shape is as depicted in Fig. 1. For a better visualization we also plot the

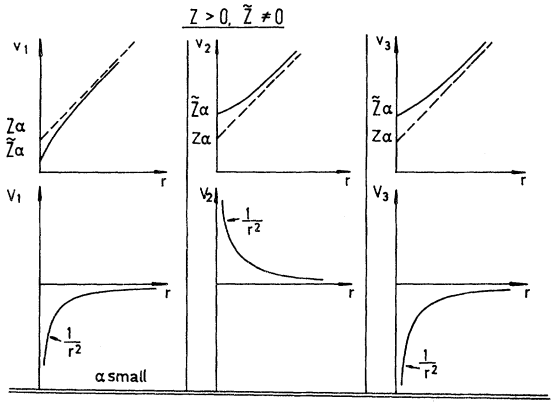

FIG. 1. Behavior of v_i [Eqs. (3.1a), (3.2a), (3.3a)] and V_i [Eq. (3.18)] for $Z>0$, $\tilde{Z}\neq 0$, where α is arbitrary for $i=2,3$ and assumed to be small for $i=1$.

self-consistent Schrödinger potentials V_i appearing in the (b) equations;

$$V_i = \pm\left(\frac{v_i^2}{r^2} - E^2\right), \quad + \text{ for } i=2 \text{ and } - \text{ for } i=1,3; \tag{3.18}$$

negative potentials then mean attraction. We have

$$\tilde{Z} > Z \quad \text{for } Z \geq 0,$$

and v_i $(i=2,3)$ has at most one zero; $Z>0$ means repulsion in Eqs. (3.2) and attraction in Eqs. (3.3). Hence, in Eq. (3.3b) the potential will be increasingly attractive as we go to smaller distances. Self-trapping, i.e., $\tilde{Z}=0$, thus never occurs. For $Z<0$, i.e., asymptotic attraction for Eq. (3.2b), the magnitude of $\tilde{Z}$ is solely determined by the detailed dynamics (see Fig. 2). For Eq. (3.1a), the situation is more complicated since $v_i(r)$ will, in general, show an arbitrary but finite number of

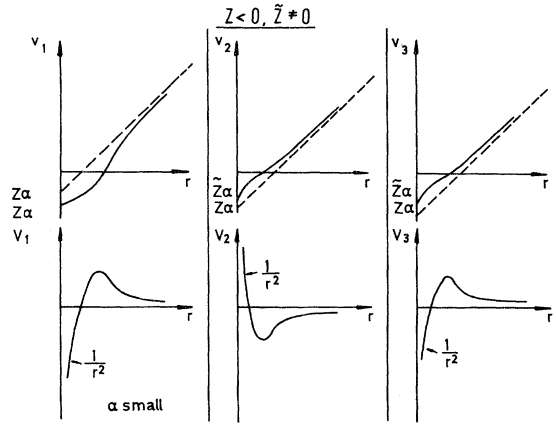

FIG. 2. Behavior of v_i [Eqs. (3.1a), (3.2a), (3.3a)] and V_i [Eq. (3.18)] for $Z<0$, $\tilde{Z}\neq 0$, where α is arbitrary for $i=2,3$ and assumed to be small for $i=1$.

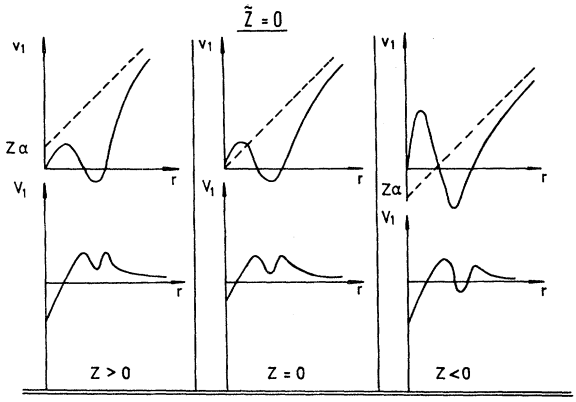

FIG. 3. Behavior of v_1 [Eq. (3.1a)] and V_1 [Eq. (3.18)] for $\tilde{Z}=0$ and Z unrestricted and α not small.

oscillations before it tends to a straight line. For small $\alpha/2\omega$ [the coupling strength in (3.1a)], however, no oscillations are present and we will have the qualitative behavior as in Fig. 1. In particular, we notice that in this case

$$Z > \tilde{Z} \quad \text{for } \tilde{Z} \geq 0. \tag{3.19}$$

Here, we have the tendency that the attracting effective charge increases as we move away from the origin, a situation reminiscent of what is called "infrared slavery" in asymptotically free theories. Moreover, self-trapping $\tilde{Z}=0$ is possible; that means there are attraction and bounded solutions without any driving point charge at the origin. For larger $\alpha/2\omega$, oscillations of $v_1(r)$ are expected and the interesting case $Z=0$, $\tilde{Z}=0$; that is, self-trapping and simultaneously complete screening of the long-range force is possible. This situation is shown in Fig. 3. It is very suggestive to interpret these oscillations as alternating shells of charge density generated in the strong spherically symmetric electric field. Furthermore, in Fig. 3 we display the cases $\tilde{Z}=0$, $Z>0$, and $Z<0$ for large coupling $\alpha/2\omega$ in Eq. (3.1a).

B. Perturbation theory

Next, we are going to develop a systematic perturbation theory for small α. At the same time an identification of the parameters ω, λ, E appearing in our equations is achieved. A central point in the procedure is a postulated relation of the total energy $\bar{U}$ to the ground-state level [obtained from the (b) equations] of the motion in the self-consistent field [obtained from the (a) equations].

The perturbation theory is centered around an iteration of the (a) equations written as integral equations (3.15). Each iteration step gives from the (b) equations a spectrum $E^i(\lambda)$ and normalizable eigenfunctions $u^{(i)}(r)$, which then, in turn, allow

for a computation of $\overline{U}$ from Eq. (3.9). The latter results as a function of $E^{(i)}$, ω, and α [see Eq. (3.13)]. Iterative identification of $\overline{U}$ with the ground-state level (up to an irrelevant additive term) determines ω and, hence, gives an *a posteriori* justification of the iteration of the integral equation (3.15) which, in turn, contains ω via the normalization of the $u_i(r)$.

Consider Eq. (3.1a) and Eq. (3.15); as the first step we take

$$v_1^{(1)}(r) = Er + Z\alpha , \tag{3.20}$$

which corresponds to a Coulomb potential of strength $Z\alpha$. Inserting into (3.1b) obviously yields the stationary, S-wave Klein-Gordon equation in an external field, the spectrum of which is well known

$$E = \pm\sqrt{\lambda}\left(1 + \frac{Z^2\alpha^2}{n'^2}\right)^{-1/2} , \tag{3.21}$$

with

$$n' = \tfrac{1}{2} + (\tfrac{1}{4} - Z^2\alpha^2)^{1/2}$$

in the ground state. From (3.9) and (3.10) we compute $\overline{U}$ to lowest order in α,

$$\overline{U} = -Z\sqrt{\lambda} + \frac{\lambda}{4\omega} . \tag{3.22}$$

Following our prescription, we now equate

$$\overline{U} = -(Z-1)\sqrt{\lambda} - E^{(1)} , \tag{3.23}$$

where $E^{(1)}$ is the ground-state level (3.21) to lowest significant order in α. Thus,

$$-\sqrt{\lambda}\left(1 - \frac{Z^2\alpha^2}{2}\right) = -\sqrt{\lambda} + \frac{\lambda}{4\omega^{(1)}} ,$$

and we find

$$\omega^{(1)} = \frac{M}{2Z^2\alpha^2} , \quad M = E^{(0)} = \sqrt{\lambda} . \tag{3.24}$$

Physically speaking, the energy $\overline{U}$ represents the total energy of a particle of mass M bound in field of Z holes, which explains the Z behavior and the sign of the first term in (3.24). The second term then, of course, has the correct sign for a binding energy. Actually, $-ZM$ is an irrelevant shift of the origin of the energy scale and could be trivially omitted. Notice that the frequency ω has a singular α^{-2} behavior which is not unfamiliar in the theory of extended solutions of classical field theories.[5]

The identification (3.24) allows for a consistent perturbation theory and justifies our first iteration step. Indeed, if we compute the next order of v_1 with the normalization (3.10) and ω given by (3.24) by iterating Eq. (3.15) we get

$$v_1^{(2)} = Er + Z\alpha - \tfrac{1}{4}Z^2\alpha^3 e^{-2\epsilon r}(1 + \epsilon r) , \tag{3.25}$$

where ϵ^{-1} is the Bohr radius,

$$\epsilon = MZ\alpha .$$

In the next iteration we determine the perturbed spectrum from $v_1^{(2)}$,

$$E^{(2)} = \sqrt{\lambda}\,(1 - \tfrac{1}{2}Z^2\alpha^2 - \tfrac{5}{8}Z^4\alpha^4 + \tfrac{3}{128}\pi Z^3\alpha^4 + \cdots) , \tag{3.26}$$

and the total energy to next significant order is

$$\overline{U}(E) = -ZE + \frac{E^2}{4\omega} + \frac{E^2}{N\omega}Z^2\alpha^2 ; \tag{3.27}$$

N is an irrelevant numerical factor. Now, $\overline{U}(E)$ has to be taken at $E^{(1)}$ and related to the new ground-state level $E^{(2)}$ by

$$\overline{U}(E_1) = -(Z-1)E^{(1)} - E^{(2)} . \tag{3.28}$$

As in (3.23) the observed mass M is given by the energy level of the former iteration step

$$M = E^{(1)} = \sqrt{\lambda} . \tag{3.29}$$

This equality is to be understood as an expansion of our parameter λ in terms of α and only formally means a mass renormalization. The frequency is then determined from (3.28),

$$\omega^{-1} = \frac{2Z^2\alpha^2}{M}(1 - \overline{\omega}Z^2\alpha^2) , \tag{3.30}$$

where $\overline{\omega}$ is a known number, and we could go on to the next iteration.

Equations (3.2) and (3.3) can be treated in the same way. The last term in Eq. (3.26), which is a correction of the usual relativistic effects and comes from the first iteration of the self-consistent potential, acquires a different sign. Furthermore, Eq. (3.2b) generates in the first-iteration step the spectrum first discussed by Bergmann.[6] Taking

$$v_2^{(1)} = Mr - Z\alpha , \tag{3.31}$$

we find

$$\sqrt{\lambda} = M\left(1 - \frac{Z^2\alpha^2}{n'^2}\right)^{1/2} ,$$
$$n' = \tfrac{1}{2} + (\tfrac{1}{4} + Z^2\alpha^2)^{1/2} , \tag{3.32}$$

for the ground state.

We have thus established the existence of perturbative solutions[7] with normalizable u_i for small enough α. For an intuitive summary of our detailed findings we again refer to the "Schrödinger" potentials shown in Figs. 1 and 2.

C. Strong coupling

Finally, we should like to discuss the cases $\tilde{Z} = 0$ and $Z = 0$. Let us first consider $\tilde{Z} = 0$, which was referred to as self-trapping. The qualitative behavior of the potentials as found from the (a) equa-

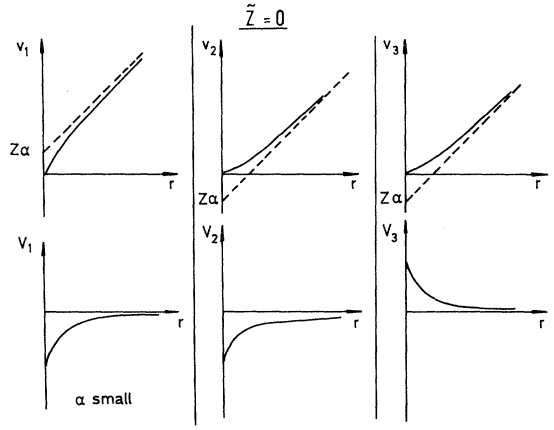

FIG. 4. Behavior of v_i [Eqs. (3.1a), (3.2a), (3.3a)] and V_i [Eq. (3.18)] for $\tilde{Z}=0$ (self-trapping), where α is arbitrary for $i=2,3$ and assumed to be small for $i=1$.

tions is displayed in Fig. 4. At first sight we see that binding is possible in (3.1) and (3.2) but not for the classical field equations. A moment's thought will suffice to convince oneself that range and depth of the binding potential will increase with increasing $Z\alpha$. For small $Z\alpha$, binding will not occur because of insufficient attraction. Hence, we cannot expect to find bound solutions by perturbing in $Z\alpha$. Indeed, if we try to satisfy the sum rule (3.16) with $\tilde{Z}=0$ in a perturbative way, we find by inserting the lowest order for v_i and u_i,

$$|Z|\alpha^2(1+Z^2\alpha^2)=16\,,$$

which is inconsistent with $Z\alpha \ll 1$.

The "screening" case $Z=0$ is even more complicated. The qualitative behavior is depicted in Fig. 5. There is no binding for Eqs. (3.2). The potentials V_1 and V_3 have an attractive $\tilde{Z}^2\alpha^2/r^2$

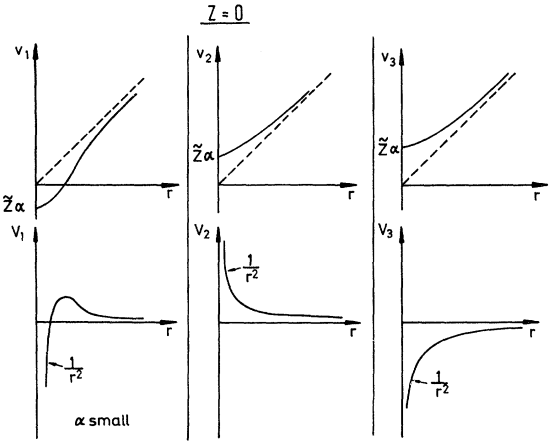

FIG. 5. Behavior of v_i [Eqs. (3.1a), (3.2a), (3.3a)] and V_i [Eq. (3.18)] and $Z=0$ (screening), where α is arbitrary for $i=2,3$ and assumed to be small for $i=1$.

term; $\tilde{Z}$ is given again by the sum rule (3.16), this time with $Z=0$. As long as

$$\tilde{Z}^2\alpha^2<\tfrac{1}{4}\,,$$

there will be no collapse. For Eqs. (3.1) this condition can be fulfilled, in general, also for large α, since oscillations of v_1 are to be expected (c.f. Fig. 3). This is not the case of (3.3).

The interesting possibility of metastable states is revealed in the plot of V_1 in Fig. 5 or Fig. 3.

To conclude this discussion we should like to remark again that $Z=\tilde{Z}=0$, if possible at all, requires strong coupling (see Fig. 3).

IV. CONCLUSIONS

To look for bound configurations in a scalar quantum field theory with gluons, we derived variational equations by varying expectation values of the component T_{00} of the gauge-invariant energy-momentum tensor and of a bona fide effective Hamiltonian for a system with a matter field and a self-consistent Hartree-Fock-type gluon field. The resulting equations for the trial functions both differ by signs from the classical Euler-Lagrange equations.

The possibility of bound solutions has been discussed using only general properties of the equations. In particular, we find self-trapping, i.e., solutions without a driving point charge at the origin of the system, and solutions with complete asymptotic screening of the charge. Furthermore, the possibility of metastable states and charge-density oscillations arises.

We then developed a perturbation procedure for the computation of bound states. Relativistic hydrogenlike spectra of the total energy (including the gluon field) are thus found in a field-theoretical framework. Corrections to the Klein-Gordon spectrum to order $\alpha(Z\alpha)^3$ are computed, which might indicate the limitations of the variational *Ansatz*.

"Self-trapping" and "screening" solutions, which do not exist for the Euler-Lagrange equations, turn out not to be obtainable by perturbing for small α, a situation which is in agreement with experiences with confinement theory and solitons. In spite of this fact, the general properties of our equations give insight into the conditions under which self-trapping and screening occur, and allow one to establish the shape of these solutions.

It is interesting to note that in Eqs. (3.1) (the ones which obtain from our bona fide Hamiltonian) the effective charge behaves as in asymptotically free theories.

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[1]P. Vinciarelli, Lett. Nuovo Cimento 4, 905 (1972); Nucl. Phys. B89, 463 (1975); B89, 493 (1975).
[2]W. A. Bardeen, M. Chanowitz, S. D. Drell, M. Weinstein, and T. M. Yan, Phys. Rev. D 11, 1094 (1975).
[3]P. Vinciarelli, Phys. Lett. 62B, 203 (1976).
[4]Charge shielding has been discussed recently in various physically different contexts; see e.g., Y. Frishman, Weizmann Report No. WIS 74/73 Ph; S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. (N.Y.) 93, 267 (1975) and references contained therein.
[5]For a review of the subject see R. Rajaraman, Phys. Rep. 21C, 227 (1975); for a perturbative expansion around classical solutions see, in particular, J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).
[6]O. Bergmann, Am. J. Phys. 24, 38 (1956).
[7]For other recent approaches to the bound-state problem in QED see, e.g., V. A. Rizov, I. T. Todorov, and B. L. Aneva, Nucl. Phys. B98, 447 (1975); H. W. Crater, Phys. Rev. D 12, 1804 (1975); H. W. Crater and J. Naft, *ibid.* 12, 1165 (1975); H. C. von Baeyer, *ibid.* 12, 3085 (1975); M. H. Partovi, *ibid.* 12, 3887 (1975); M. Bander and P. Thomas, *ibid.* 12, 1798 (1975).