Description of unstable particles in quantum field theory*

H. A. Weldon
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 23 December 1975)

The consequences of a pole on the second sheet of the S matrix are investigated under the assumption that a certain Green's function has the same pole. It is shown that corresponding to each such pole is an eigenstate of the Hamiltonian with a complex energy. These eigenstates lie in a natural extension of the physical Hilbert space. Because it is the vector space that is modified and not the Hamiltonian, unstable particle states transform covariantly. They have complex energy and momentum but real integer or half-integer spin. Scattering amplitudes involving unstable particles are expressed as residues of poles in a reduction formula of the Lehmann-Symanzik-Zimmermann type. An example from potential theory is worked out in detail.

I. INTRODUCTION

Of the known elementary particles only five are stable; all others eventually decay into combinations of $e, p, \gamma, \nu_e, \nu_\mu$ and their antiparticles. For the stable particles field theory provides a beautiful description based on three principles:

(a) Stable particles correspond to eigenstates of energy-momentum in a linear vector space.
(b) They transform as irreducible representations of the Poincaré group labeled by mass and spin.[1]
(c) Scattering amplitudes are matrix elements of Heisenberg operators taken between these states.

Unfortunately most real particles are not stable. The purpose of this paper is to show that even for unstable particles these same principles apply. The essential difference is that stable particles lie in a Hilbert space and unstable ones do not.

Conventional treatments of unstable particles are all based on perturbation theory in that the unstable states are always eigenstates of an effective Hamiltonian which is chosen to contain the essentials of the spectrum. The difference between the total Hamiltonian and the effective Hamiltonian is a small perturbation that induces decays of the eigenstates of the effective Hamiltonian. This approach was the very basis of theoretical atomic physics. There the effective Hamiltonian includes the Coulomb field of the nucleus but neglects the electron-photon interaction. This perturbation results in the radiative decays summarized by the Balmer formula. Dirac[2] showed how to calculate the effect of such unstable states on the scattering amplitude

$$T(E) = H_1 + H_1 \frac{1}{E - H_0} H_1 + \cdots \tag{1.1}$$

when E is near an eigenvalue of H_0. Later, in nuclear physics, the abundance of resonances required more sophistication. A number of approaches were developed based on the energy dependence of the scattering wave function at large distances. Kapur and Peierls[3] expanded the wave function in terms of states with complex eigenvalues corresponding to resonance energies and widths. Wigner and Eisenbud[4] expanded in different states that led to the reactance matrix and the many-level formula for overlapping resonances (see Ref. 5 for complete reviews.) Feshbach[6] and also Fonda and Newton[7] generalized the earlier approaches by abandoning coordinate space in favor of an abstract Hamiltonian and projection operators that distinguish open and closed channels. The resonant part of the perturbation series (1.1) is expressed in terms of a general effective Hamiltonian that is both energy dependent and non-Hermitian. Feshbach showed that the Kapur-Peierls and Wigner-Eisenbud solutions result from different choices of projection operators. Common to all these approaches is the identification of an unstable state with a vector in the usual Hilbert space of stable states that is an eigenstate of a modified Hamiltonian.

The S-matrix theory of unstable particles naturally makes no statement about Hamiltonians or state vectors. It began with the suggestion by Møller[8] that an unstable particle corresponds to a pole on the second Riemann sheet of the analytically continued S matrix. This observation is based on the idea that the scattering wave function outside the range of the potential should have only outgoing waves when the energy is exactly that of the unstable state. The presence of a pole on the second sheet is then taken as a Lorentz-invariant characterization of an unstable particle. Peierls[9] brought this idea into field theory by suggesting that the pole should occur in the one-particle propagator. Both the relation of this pole to the experimentally measured mass and the dependence of the pole position upon the choice of field type was questioned by

Stapp,[10] who formulated a pure S-matrix theory of unstable particles.

This paper employs features of both approaches. The most important step is to use the existence of second-sheet poles to construct a state vector for unstable particles. The construction is begun in Sec. II. The off-mass-shell Green's functions are assumed to possess the same second-sheet pole as the corresponding scattering amplitude. It is shown that whenever there are such poles, then there is an eigenstate of the full, Hermitian Hamiltonian with a complex energy. The usual prohibition against this happening is that

$$\langle E|E\rangle E = \langle E|H|E\rangle = E^*\langle E|E\rangle \tag{1.2}$$

forbids E being complex. However, Sec. III shows that the unstable states lie not in the usual Hilbert space but in a natural extension that corresponds to the second sheet of the S matrix. In this larger space unstable particles automatically have zero norm and thus escape the prohibition (1.2). Therefore, in contrast to all earlier approaches, it is the vector space and not the Hamiltonian that is modified. The unstable states so constructed are eigenstates of $(H,\vec{\mathcal{P}})$ but do not transform as irreducible representations of the Poincaré group. Therefore, in Sec. IV the Casimir operator corresponding to spin is expressed in a convenient manner that is used in Sec. V to construct unstable particle states that do transform irreducibly under real Lorentz transformations. The spin is automatically Lorentz invariant and can take on only real integral or half-integral values. The most interesting irreducible representations are the helicity states, which have the particularly simple one-dimensional transformation law

$$U(\Lambda)|k,j,\sigma\rangle = |\Lambda k,j,\sigma\rangle e^{i\Theta w^\sigma}. \tag{1.3}$$

Here, in contrast to the familiar massless case, the helicity σ may take on all integrally spaced values from $-j$ to $+j$. The practical reason for considering irreducible representations is given in Sec. VI. There it is shown that precisely these representations produce poles in partial-wave amplitudes. The Lorentz transformation law (1.3) is, however, not observable because of the way in which the complex energy-momentum must be continued to the real axis. The observed transformation law is shown to be just the same as for stable particles. Next, wave functions are investigated. Because unstable particles correspond to eigenstates of energy-momentum, they possess genuine Bethe-Salpeter wave functions. These are used to calculate scattering amplitudes, which are expressed in a reduction formula of the LSZ (Lehmann-Symanzik-Zimmermann) type.

To defray any mystery about extending the Hilbert space to include zero-norm states, a simple nonrelativistic example is worked out in Sec. VII. The Schrödinger wave functions are displayed and used in calculations. This last section is essentially self-contained and may profitably be read immediately after Sec. II.

II. CONSTRUCTION OF AN INTERPOLATING STATE VECTOR

Suppose that when the S matrix is analytically continued in the total energy k^0 in a clockwise direction around a particular n-particle branch point there is a pole in k^0. The first step in constructing a state vector for the corresponding unstable particle is to find an eigenstate of momentum $|\phi(\vec{k})\rangle$ such that the function

$$\left\langle\phi(\vec{k}')\left|\frac{1}{k^0-H}\right|\phi(\vec{k})\right\rangle \tag{2.1}$$

has that same pole in k^0 on the second sheet. It is essential for the later analytic continuations that $|\phi(\vec{k})\rangle$ itself does not depend on k^0. Therefore the similarity of (2.1) to the familiar expression

$$T_{fi} = \left\langle f\left|V + V\frac{1}{k^0-H}V\right|i\right\rangle \tag{2.2}$$

is of no use; $|\phi\rangle$ cannot be chosen to be $V|i\rangle$ because $|i\rangle$ in (2.2) depends on k^0. In other words, (2.2) is an on-shell amplitude.

It is only at this point that field theory enters. Its Green's functions are off-shell analogs of (2.2). In particular, the S matrix of a field theory has a pole whenever the Fourier transform of the corresponding time-ordered product of Heisenberg fields also has the same pole. It follows that the state

$$\int d^3X e^{i\vec{k}\cdot\vec{X}}\, T[\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)]|0\rangle\,, \tag{2.3}$$

with

$$X \equiv \frac{1}{n}\sum_1^n x_i\,,$$

is a suitable choice for $|\phi(\vec{k})\rangle$. Appendix A contains a more complete argument. It is convenient to sum the Lorentz indices in (2.3) to form eigenstates of $\vec{J}^2$ and J_3 (see also Appendix A) and to integrate the relative coordinates against some smearing function to achieve the normalization

$$\langle\phi(\vec{k}')j',j_3'|\phi(\vec{k})j,j_3\rangle = \delta^3(\vec{k}'-\vec{k})\delta_{j'j}\delta_{j_3'j_3}\,. \tag{2.4}$$

It should be emphasized that the $|\phi\rangle$ so constructed is not an eigenstate of the Hamiltonian and does not transform into anything simple under boosts. In fact, a boosted $|\phi\rangle$ is not even an eigenstate of $\vec{J}^2$ and J_3.

Given such a $|\phi\rangle$, define projection operators

$$A \equiv \int d^3k \sum_{j_3} |\phi(\vec{k})j, j_3\rangle\langle\phi(\vec{k})j, j_3|,$$
$$B \equiv 1 - A, \tag{2.5}$$

with the multiplication properties

$$A^2 = A, \quad B^2 = B, \quad AB = BA = 0.$$

Split the full Hamiltonian into diagonal and off-diagonal parts with respect to $|\phi\rangle$ by defining

$$H = H' + H'' ,$$

where

$$H' \equiv AHA + BHB$$
$$H'' \equiv AHB + BHA. \tag{2.6}$$

Define the full resolvent by

$$R(k^0) \equiv \frac{1}{k^0 - H}$$

and a reduced resolvent diagonal with respect to $|\phi\rangle$ by

$$r(k^0) \equiv \frac{1}{k^0 - H'} \ .$$

Let the matrix element (2.1) be expressed as

$$\langle\phi(\vec{k}')j', j_3'|R(k^0)|\phi(\vec{k})j, j_3\rangle \equiv \frac{\delta^3(\vec{k}' - \vec{k})\delta_{j'j}\delta_{j_3' j_3}}{D(k)} , \tag{2.7}$$

where $D(k)$ depends on both k^0 and $\vec{k}$ and vanishes whenever the S matrix has a pole. [The j dependence of $D(k)$ is suppressed.]

A state vector for the unstable particle corresponding to the pole in (2.1) can now be built from $|\phi\rangle$. The connection between the full resolvent and the matrix element (2.1) is given by the identity

$$R(k^0) = Br(k^0)B + [1 + r(k^0)H'']AR(k^0)A[H''r(k^0) + 1], \tag{2.8}$$

which is proved in Appendix B. Rewrite this as

$$R(k^0) = Br(k^0)B + \int d^3k \sum_{j_3} \frac{|\psi(k)j, j_3\rangle\langle\psi(k)j, j_3|}{D(k)} , \tag{2.9}$$

where

$$|\psi(k)j, j_3\rangle \equiv [1 + r(k^0)H'']|\phi(\vec{k})j, j_3\rangle. \tag{2.10}$$

This may also be written as

$$|\psi(k)j, j_3\rangle = \frac{1}{k^0 - H} |\phi(\vec{k})j, j_3\rangle D(k). \tag{2.11}$$

(From now on the spin indices j and j_3 will be suppressed. They will be displayed again in Sec. V and subsequently.) Equation (2.9) suggests that $|\psi\rangle$ interpolates between eigenstates of H. To demonstrate this, use the definition of $|\psi\rangle$:

$$(k^0 - H')|\psi(k)\rangle = [k^0 - H' + H'']|\phi(\vec{k})\rangle,$$
$$H''|\psi(k)\rangle = [H'' + H''r(k^0)H'']|\phi(\vec{k})\rangle.$$

Subtract to get

$$(k^0 - H)|\psi(k)\rangle = [k^0 - H' - H''r(k^0)H'']|\phi(\vec{k})\rangle.$$

The right-hand side of this is orthogonal to B and is therefore just $|\phi(\vec{k})\rangle$ multiplied by a function of k. This function is just $D(k)$ because

$$\langle\phi(\vec{k}')|k^0 - H' - H''r(k^0)H''|\phi(\vec{k})\rangle = \delta^3(\vec{k}' - \vec{k})D(k) \tag{2.12}$$

as proved in Appendix B. Hence

$$(k^0 - H)|\psi(k)\rangle = |\phi(\vec{k})\rangle D(k). \tag{2.13}$$

This equation contains the principal result: If the S matrix has a pole at some $\tilde{k}^0$, then there is an eigenstate of the Hamiltonian with that energy. The state vector $|\psi\rangle$ therefore interpolates between all eigenstates of H with the same quantum numbers as $|\phi\rangle$. When the pole of S is on the physical Riemann sheet, $|\psi\rangle$ lies in the physical Hilbert space. When the pole is reached by continuing in k^0 to another sheet, the state $|\psi(k)\rangle$ must also be "continued" and this is the topic of Sec. III.

Before proceeding, however, it is a useful comparison to recall the usual applications of the same formulas. Equation (2.12) is essentially Dirac's method[2] of summing (1.1) near an eigenvalue of H_0. Without translation invariance there is no $\vec{k}$ dependence. As the energy approaches the real axis from above

$$\lim_{k \to 0^+} D(k^0) = k^0 - E_\phi - \Delta(k^0) + \frac{i}{2}\gamma(k^0),$$

where the unperturbed energy, level shift, and width are given by

$$E_\phi = \langle\phi|H|\phi\rangle,$$
$$\Delta(k^0) = \left\langle\phi\left|H'' \frac{\mathcal{P}}{k^0 - H'} H''\right|\phi\right\rangle,$$
$$\gamma(k^0) = \langle\phi|H'' 2\pi\delta(k^0 - H')H''|\phi\rangle.$$

In the Dirac approach H' is chosen as the unperturbed Hamiltonian and $|\phi\rangle$ is required to be an eigenstate of H'. Feshbach,[6] on the other hand, let the projection operator (i.e., $|\phi\rangle$ determine H' as done here in (2.6). His effective Hamiltonian is the energy-dependent, non-Hermitian operator

$$H_{\text{eff}} \equiv H' + H''r(k^0)H''$$

occurring in (2.12)

In all such approaches the final results are dependent upon the choice of the effective Hamiltonian. Write (2.7) with spin indices suppressed:

$$\frac{\delta^3(\vec{k}'-\vec{k})}{D(k)}=\left\langle\phi(\vec{k}')\left|\frac{1}{k^0-H}\right|\phi(\vec{k})\right\rangle. \tag{2.14}$$

Clearly $D(k)$ is a functional of $|\phi\rangle$ and $|\phi\rangle^\dagger$. Choosing an effective Hamiltonian is equivalent to choosing a particular $|\phi\rangle$. The variation of $D(k)$ when $|\phi\rangle^\dagger$ is varied independently of $|\phi\rangle$ is

$$\frac{\delta D(k)}{\delta|\phi\rangle^\dagger}=-\frac{1}{k^0-H}|\phi(\vec{k})\rangle D(k)^2,$$

which by (2.11) is just

$$\frac{\delta D(k)}{\delta|\phi\rangle^\dagger}=-|\psi(k)\rangle D(k). \tag{2.15}$$

This variation vanishes only when $D(k)$ vanishes. Thus if a $|\phi\rangle$ has been chosen that produces a pole in (2.14), then the location of that pole is independent of the choice of $|\phi\rangle$ because of (2.15). It is essential to continue k^0 to the pole. All effective Hamiltonian treatments focus on the real value of k^0 that produces a bump in (2.14). The location of such a maximum is usually called the resonance energy. However, (2.15) shows that this energy clearly depends on the choice made for $|\phi\rangle$ and depends on it even though all orders of the perturbation are summed.

Note that it is the zeros of $D(k)$ that are significant for stable particles too. Equations (2.10) and (2.12) are then the usual formulas of Wigner-Brillouin perturbation theory for the eigenvectors and eigenvalues of H.[11] (The exact formulas do not depend on H' being small, of course.)

III. EXTENSION OF THE HILBERT SPACE

To apply (2.13) to unstable particles an extension of the physical Hilbert space $\mathfrak{h}$ must be found that corresponds to different sheets of $D(k)$. Always the appropriate sheet of $D(k)$ is determined by the resonance and not by the particular scattering process (i.e., $|\phi\rangle$) under consideration. For example, to reach the ρ^0 in the e^+e^- scattering amplitude requires going around the very same branch points as for the $\pi^+\pi^-$ amplitude (see Fig. 1). Regardless of the amplitude, the pole is always reached by going around the heaviest decay threshold.

A. Inner products in $\mathfrak{h}\uparrow$

For complex k^0 the state

$$|\psi(k)\rangle=[1+r(k^0)H'']|\phi(\vec{k})\rangle \tag{3.1}$$

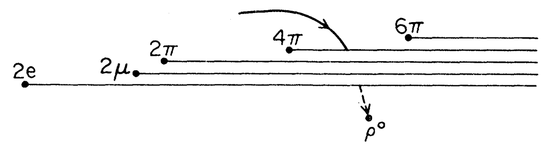

FIG. 1. Example showing that the continuation to the unstable-particle pole is always around the same branch point (viz. the heaviest decay mode) independently of the scattering process considered.

is an ordinary state in the physical Hilbert space $\mathfrak{h}$ as long as the cuts along the real k^0 axis are avoided. A new state vector may always be added to $\mathfrak{h}$ by specifying the inner product of the new state with every vector of $\mathfrak{h}$. The means of specifying that inner product here will be analytic continuation and the enlarged space will be called $\mathfrak{h}\uparrow$. In particular, define the continuation of $|\psi(k)\rangle$ as a new state whose inner product with an arbitrary state $|f\rangle$ is the analytic continuation in k^0 of $\langle f|\psi(k)\rangle$. The inner product to be continued is then

$$\langle f|\psi(k)\rangle=\langle f|1+r(k^0)H''|\phi(\vec{k})\rangle.$$

Because of (2.11) this is just

$$\langle f|\psi(k)\rangle=\left\langle f\left|\frac{1}{k^0-H}\right|\phi(\vec{k})\right\rangle D(k). \tag{3.2}$$

The new states of special interest are those that correspond to a second-sheet pole in the S matrix. Let the pole occur at $\tilde{k}^0$ on the sheet reached by clockwise continuation around a particular n-particle branch point and let $D_n(k)$ be the continuation of $D(k)$ around this branch point. The existence of a pole means that

$$D_n(\tilde{k})=0. \tag{3.3}$$

The corresponding state is $|\psi_n(\tilde{k})\rangle$. For any real-energy eigenstate $|E\rangle$ of $\mathfrak{h}$, (3.2) gives

$$\langle E|\psi(k)\rangle=\langle E|\phi(\vec{k})\rangle\frac{D(k)}{k^0-E}.$$

Analytically continuing to $\tilde{k}^0$ yields

$$\langle E|\psi_n(\tilde{k})\rangle=0. \tag{3.4}$$

The unstable particle is therefore orthogonal to all real-energy eigenstates.

At first, this result seems paradoxical because $|\phi\rangle$ itself is in $\mathfrak{h}$ and is therefore just a superposition of real-energy eigenstates. But $|\phi\rangle$ is certainly not orthogonal to the unstable state because

$$\langle\phi(\vec{k}')|\psi(k)\rangle=\delta^3(\vec{k}'-\vec{k}), \tag{3.5}$$

independently of k^0. The explanation lies in the difference between a continuous and a discrete superposition and is discussed further in Appendix C.

The inner product of two new states is the analytic continuation in two variables of their inner product in $\mathfrak{h}$. Thus the analog of (3.2) is

$$\langle\psi(k)|\psi(k')\rangle = D(k)^*\langle\phi(\vec{k})|R(k^0)^\dagger R(k^{0\prime})|\phi(\vec{k}')\rangle D(k').$$

Using

$$R(k^0)^\dagger R(k^{0\prime}) = \frac{R(k^{0\prime}) - R(k^0)^\dagger}{k^{0*} - k^{0\prime}} \tag{3.6}$$

and the definition of $D(k)$, this may be written as

$$\langle\psi(k)|\psi(k')\rangle = \delta^3(\vec{k} - \vec{k}')\,\frac{D(k)^* - D(k')}{k^{0*} - k^{0\prime}}\ . \tag{3.7}$$

[Of course, the same result may be obtained directly from (3.1) by using an identity like (3.6) for the reduced resolvent $r(k^0)$ and the alternate expression (2.12) for $D(k)$.] The inner product of an unstable state with itself results from continuing both k^0 and $k^{0\prime}$ in (3.7) clockwise around the same branch point. Because

$$D_n(\tilde{k})^* = D_n(\tilde{k}) = 0,$$

$$\tilde{k}^{0*} - \tilde{k}^0 \neq 0$$

it follows that

$$\langle\psi_n(\tilde{k})|\psi_n(\tilde{k})\rangle = 0. \tag{3.8}$$

Obviously (3.7) also implies that two different unstable particles (i.e., with different energies) are orthogonal. The vanishing of (3.8) is a direct consequence of H being Hermitian in (3.6). It will now be shown that since H is Hermitian in $\mathfrak{h}$ it is also Hermitian in $\mathfrak{h}\dagger$ so that the argument (1.2) for a zero-norm state was correct.

All the poles of S discussed so far lie in the lower half k^0 plane. However, it is well known[12] that Hermitian analyticity requires S to have poles in complex conjugate pairs. From the definition of $D(k)$ in (2.14) it is clearly Hermitian analytic for $\vec{k}$ real:

$$D(k^0, \vec{k})^* = D(k^{0*}, \vec{k}). \tag{3.9}$$

If (3.9) is continued in k^0 clockwise around an n-particle branch point and into the lower half plane, then k^{0*} goes counterclockwise around that branch point and into the upper half plane (see Fig. 2).

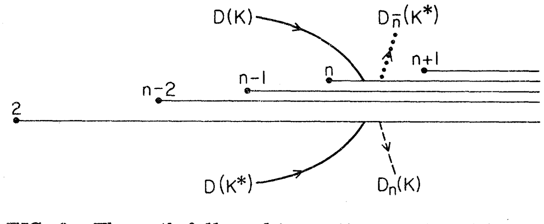

FIG. 2. The path followed in continuing Hermitian analyticity away from the physical sheet to obtain $D_n(k)^* = D_{\bar{n}}(k^*)$.

Since $D_n(k)$ denotes the clockwise continuation of $D(k)$, let $D_{\bar{n}}(k)$ denote the counterclockwise continuation of $D(K)$. The continuation of (3.9) is then

$$D_n(k^0, \vec{k})^* = D_{\bar{n}}(k^{0*}, \vec{k}). \tag{3.10}$$

Hence if D_n vanishes at $\tilde{k}^0$ the $D_{\bar{n}}$ vanishes at $\tilde{k}^{0*}$. Thus unstable particles come in complex-conjugate pairs. This again reflects the fact that H is Hermitian even in the larger space $\mathfrak{h}\dagger$. Denote the state resulting from counterclockwise continuation into the upper half plane with a subscript $\bar{n}$. Thus

$$H|\psi_n(\tilde{k})\rangle = \tilde{k}^0|\psi_n(\tilde{k})\rangle,$$
$$H|\psi_{\bar{n}}(\tilde{k}^*)\rangle = \tilde{k}^{0*}|\psi_{\bar{n}}(\tilde{k}^*)\rangle. \tag{3.11a}$$

The adjoint states are defined by

$$\langle\psi_n(\tilde{k})| \equiv |\psi_n(\tilde{k})\rangle^\dagger.$$

Thus

$$\langle\psi_n(\tilde{k})|H = \langle\psi_n(\tilde{k})|\tilde{k}^{0*},$$
$$\langle\psi_{\bar{n}}(\tilde{k}^*)|H = \langle\psi_{\bar{n}}(\tilde{k}^*)|\tilde{k}^0. \tag{3.11b}$$

It is not surprising that

$$\langle\psi_{\bar{n}}(\tilde{k}^*)|\psi_{\bar{n}}(\tilde{k}^*)\rangle = 0,$$

just like (3.8). A more interesting calculation is the inner product between the two conjugate partners. This requires continuing both the $k^{0\prime}$ in (3.7) clockwise from the upper half plane around the n-particle branch point to $\tilde{k}^0$ and k^0 in (3.7) counterclockwise from the lower half plane around the same threshold to the value $\tilde{k}^{0*}$. Performing the k^0 continuation first gives

$$\langle\psi_{\bar{n}}(\tilde{k}^*)|\psi(k')\rangle = \delta^3(\vec{k} - \vec{k}')\,\frac{-D(k')}{\tilde{k}^0 - k^{0\prime}}\ .$$

Because $D(k')$ vanishes at $\tilde{k}^0$, the continuation in $k^{0\prime}$ then gives

$$\langle\psi_{\bar{n}}(\tilde{k}^*)|\psi_n(\tilde{k}')\rangle = \delta^3(\vec{k} - \vec{k}')N(\tilde{k}),$$

where

$$N(k) \equiv \left.\frac{dD_n(k)}{dk^0}\right|_{k=\tilde{k}}\ . \tag{3.12}$$

If instead, the continuation in $k^{0\prime}$ is done before that in k^0 the result is

$$\langle\psi_{\bar{n}}(\tilde{k}^*)|\psi_n(\tilde{k}')\rangle = \delta^3(\vec{k} - \vec{k}')\left[\frac{dD_{\bar{n}}(\tilde{k}^*)}{d\tilde{k}^{0*}}\right]^*\ .$$

Because of Hermitian analyticity this is the same as (3.12). By assumption the pole in the S matrix is of first order so that the zero of $D(k)$ is simple. Thus $N(\tilde{k})$ is a finite but nonvanishing constant. It is useful to absorb this factor by defining

$$|\Psi(k)\rangle\!\rangle \equiv |\psi_n(\tilde{k})\rangle\!\rangle \frac{1}{[N(\tilde{k})]^{1/2}} ,$$
$$|\Psi(k^*)\rangle\!\rangle \equiv |\psi_n(\tilde{k}^*)\rangle\!\rangle \frac{1}{[N(\tilde{k})^*]^{1/2}} \tag{3.13}$$

so that

$$\langle \Psi(k^*)|\Psi(k')\rangle\!\rangle = \delta^3(\vec{k}-\vec{k}'). \tag{3.14}$$

[The momentum k in $|\Psi(k)\rangle\!\rangle$ will always be on the mass shell so there is no need to call it $\tilde{k}$.]

Because the unstable-particle states have zero norm, it is necessary to find a new definition of norm that is positive-definite. This is necessary so that the equality of two vectors

$$|F_1\rangle = |F_2\rangle \tag{3.15}$$

will have a precise meaning. In a Hilbert space the meaning of (3.15) is that

$$\langle \Delta F|\Delta F\rangle = 0, \tag{3.16}$$

where

$$|\Delta F\rangle \equiv |F_1\rangle - |F_2\rangle .$$

Obviously (3.16) is not good enough for $\mathfrak{h}\!\uparrow$ because it will still allow $|F_1\rangle$ and $|F_2\rangle$ to differ by an arbitrary number of unstable states. In Appendix C a positive-definite-norm operator Ω is constructed. The precise meaning of (3.13) is then that

$$\langle \Delta F|\Omega|\Delta F\rangle = 0. \tag{3.17}$$

Goldberger and Watson[13] have emphasized that the S matrix may have higher-order poles on the second sheet. (The decay law then is the familiar $e^{-\Gamma t}$ multiplied by a polynomial in t.) In such a case $N=0$ in (3.12). Appendix D discusses the additional states that are then present in $\mathfrak{h}\!\uparrow$.

B. A Lee-model example

A simple example of an unstable particle occurs in the Lee model.[14] The Hamiltonian for a static N and V is

$$H = m_V\psi_V^\dagger\psi_V + m_N\psi_N^\dagger\psi_N + \int d^3p\, \omega a^\dagger(\vec{p})a(\vec{p}) + \left[\frac{g_0}{(4\pi)^{1/2}}\psi_V^\dagger\psi_N\int d^3p\,\frac{f(\omega)}{(2\omega)^{1/2}}\,a(\vec{p}) + \text{H.c.}\right], \tag{3.18}$$

where $\omega = (m_\theta{}^2 + \vec{p}^2)^{1/2}$ and $f(\omega)$ is a real cutoff function. The interaction produces a dressed V state that is a simple combination of a bare V and a bare $N+\theta$,

$$|V(E)\rangle\!\rangle = \left[\psi_V^\dagger + \frac{g_0}{(4\pi)^{1/2}}\psi_N^\dagger \times\int \frac{d^3p}{(2\omega)^{1/2}}\,\frac{f(\omega)}{E-m_N-\omega}\,a^\dagger(\vec{p})\right]|0\rangle . \tag{3.19}$$

This state is an eigenstate of H for those values of E that cause

$$D(E) = E - m_V - \frac{g_0{}^2}{4\pi}\int\frac{d^3p}{2\omega}\,\frac{|f(\omega)|^2}{E-m_N-\omega} \tag{3.20}$$

to vanish.

The formula for $|V\rangle$ in (3.19) is just a special case of that for $|\psi\rangle\!\rangle$ in (3.1) in which $|\phi\rangle = \psi_V^\dagger|0\rangle$, H' is the free part of (3.18), and H'' is the interacting term of (3.18). The expression for $D(E)$ is a special case of (2.12). The zeros of $D(E)$ can only lie on the real axis below the branch point at $m_N + m_\theta$ or in the complex plane on another sheet. The s-wave $N+\theta\to N+\theta$ scattering amplitude is[15]

$$e^{2i\delta(E)} = \frac{D_{\rm II}(E)}{D_{\rm I}(E)} ,$$

where I and II indicate the physical and unphysical sheets of D. The second-sheet poles of the scattering amplitude are therefore the zeros of D on the second sheet. These poles come in complex conjugate pairs and have zero norm:

$$\langle V(E)|V(E)\rangle\!\rangle = \langle V(E^*)|V(E^*)\rangle\!\rangle = 0.$$

The inner product of the two is the analytic continuation of

$$\langle V(E^*)|V(E)\rangle\!\rangle = 1 + \frac{g_0{}^2}{4\pi}\int\frac{d^3p}{2\omega}\,\frac{|f(\omega)|^2}{(E-m_N-\omega)^2} , \tag{3.21}$$

as obtained from (3.19). Clearly

$$\langle V(E^*)|V(E)\rangle\!\rangle = \frac{dD(E)}{dE} ,$$

as shown generally in (3.12). There is no real benefit in choosing a particular cutoff function $f(\omega)$ and then calculating these integrals explicitly. A more illustrative example is worked out in detail in Sec. VII.

It should be noted that this version of the Lee model is nothing fancy despite the zero-norm states. In particular, it is not the indefinite metric quantization of Källén and Pauli.[15] They showed that the Lee model violates unitarity in the point-source limit [i.e., $f(\omega)\to 1$]. To preserve unitarity it is necessary to make g_0 imaginary and to let the bare V state have negative norm. Such a modification changes the sign of m_V in (3.18) and (3.20) but, more importantly, makes $g_0{}^2$ negative in (3.20) and (3.21). This allows $D(E)$ to have a second-order zero on the real axis[16] or zeros at complex E that lie on the physical sheet.[17] These theories are quite different from the simple proposal made here of taking a well-defined cutoff theory, quantized in a conventional Hilbert space with positive-definite metric, and investigating the consequences of a pole on the second sheet of the scattering amplitude.

Another contrast can be made to the work of Glaser and Källén.[18] They also treat the cutoff theory with conventional quantization. In order to discuss resonances they do not continue to the zeros of D on the second sheet, however, but instead define a new D by replacing the integral in (3.20) with its principal value. The new D has zeros on the real axis above $m_N + m_\theta$ that give approximate eigenvalues and approximate eigenstates of H. Of course, their answers depend on this prescription.

IV. POINCARÉ-GROUP PRELIMINARIES

Once the eigenstates of energy and momentum are constructed, all the transformation properties can be derived by imitating Wigner.[1] The present section introduces the covariance notions that will be needed.

A. Complex momentum

The unstable particle state $|\Psi(k)\rangle$ was constructed with a real momentum $\vec{k}$ and a complex energy k^0 such that

$$\begin{aligned} \mathcal{P}^\mu |\Psi(k)\rangle &= k^\mu |\Psi(k)\rangle, \\ \langle\Psi(k^*)|\Psi(k')\rangle &= \delta^3(\vec{k}' - \vec{k}). \end{aligned} \tag{4.1}$$

Lorentz transformations are generated by six operators $M^{\mu\nu}$ satisfying

$$[M^{\mu\nu}, M^{\alpha\beta}] = i(g^{\mu\alpha}M^{\nu\beta} - g^{\mu\beta}M^{\nu\alpha} - g^{\nu\alpha}M^{\mu\beta} + g^{\nu\beta}M^{\mu\alpha}).$$

Corresponding to each Lorentz transformation matrix Λ^μ_ν is a linear operator

$$U[\Lambda] = \exp(i\lambda_{\mu\nu}M^{\mu\nu}).$$

The commutation relations of the Poincaré-group generators alone imply

$$U^{-1}[\Lambda]\mathcal{P}^\mu U[\Lambda] = \Lambda^\mu{}_\nu \mathcal{P}^\nu .$$

From this and (4.1) it follows that states with complex three-momentum $\vec{k}'$ are automatically generated:

$$\mathcal{P}^\mu\{U[\Lambda]|\Psi(k)\rangle\} = k'^\mu\{U[\Lambda]|\Psi(k)\rangle\}, \tag{4.2}$$

where $k'^\mu = \Lambda^\mu{}_\nu k^\nu$.

Complex momenta suggest wave functions which grow exponentially in some spatial direction. In the soluble example of Sec. VII this is exactly the case. Such exponential growth is, in fact, characteristic of second-sheet singularities. It is no problem because inner products are always calculated by analytically continuing in momentum space. (Heuristically this corresponds to integration along a complex path in position space.) To analytically continue in momentum space requires continuing the δ function in (4.1). This is discussed in (5.35) and Ref. 24.

At present, it is necessary to introduce some covariant notation. Because of (4.2) the momentum of an unstable particle has the general form

$$k^\mu = p^\mu + iq^\mu , \tag{4.3}$$

where p and q are real four-vectors. Since k^2 is Lorentz invariant, so are its real and imaginary parts. Thus put

$$k^2 = M^2 - iM\Gamma , \tag{4.4}$$

where M and Γ are real constants that characterize the unstable particle. In terms of p and q this means

$$\begin{aligned} M^2 &= p^2 - q^2, \\ M\Gamma &= -2p\cdot q. \end{aligned} \tag{4.5}$$

It is also useful to define the complex number

$$\mathfrak{M} \equiv \frac{M}{\sqrt{2}}\left\{1 + \left[1 + \left(\frac{\Gamma}{M}\right)^2\right]^{1/2}\right\}^{1/2} - i\frac{\Gamma}{\sqrt{2}}\left\{1 + \left[1 + \left(\frac{\Gamma}{M}\right)^2\right]^{1/2}\right\}^{-1/2}, \tag{4.6}$$

in terms of which the mass-shell condition (4.4) is

$$k^2 = \mathfrak{M}^2.$$

Note that Γ may be positive or negative because of Hermitian analyticity.

The analytic continuation of Sec. III yields poles with $\vec{k}$ real, i.e.,

$$\begin{aligned} p &= (\mathrm{Re}k^0, \vec{k}), \\ q &= (\mathrm{Im}k^0, \vec{0}). \end{aligned} \tag{4.7}$$

Obviously $q^2 > 0$. Because of (4.5) $p^2 > 0$. Any state whose momentum is related to (4.7) by a real Lorentz transformation must have the same values of p^2 and q^2. Conversely, any such state may be transformed by a real Lorentz transformation into one with the standard momentum

$$\begin{aligned} \overline{p} &\equiv \left(\frac{p\cdot q}{(q^2)^{1/2}}, 0, 0, \left[\frac{(p\cdot q)^2 - p^2q^2}{q^2}\right]^{1/2}\right), \\ \overline{q} &\equiv ((q^2)^{1/2}, 0, 0, 0). \end{aligned} \tag{4.8}$$

This is just (4.7) with $\vec{k}$ rotated into the z axis.

B. The spin Casimir operator

In order to find unstable particle states that transform irreducibly under the Poincaré group it is necessary to diagonalize the Casimir operators $\mathcal{P}^2$ and W^2, where

$$W_\alpha \equiv \tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}\mathcal{P}^\beta M^{\mu\nu} \tag{4.9}$$

in the notation of Gasiorowicz.[19] This is easily done for a state with real momentum by transforming to the rest frame. This section will show that this technique is the only way to diagonalize W^2 even for states with complex momentum.

The application of W_α to the eigenstates of energy-momentum constructed in Sec. III replaces $\mathcal{P}^\beta$ by its eigenvalue:

$$W_\alpha |\Psi(k)\rangle = W_\alpha(k)|\Psi(k)\rangle,$$

with (4.10)

$$W_\alpha(k) \equiv \tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}k^\beta M^{\mu\nu}.$$

Now evaluate (4.10) when k has the special form

$$\begin{aligned} \bar{k}^\mu &= (\alpha, 0, 0, \beta), \\ \alpha^2 - \beta^2 &= \mathfrak{M}^2 \end{aligned} \tag{4.11}$$

introduced in (4.8). The result is

$$\begin{aligned} W_0(\bar{k}) &= -\beta J_3, \\ W_1(\bar{k}) &= \alpha J_1 - \beta K_2, \\ W_2(\bar{k}) &= \alpha J_2 + \beta K_1, \\ W_3(\bar{k}) &= \alpha J_3, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} J_i &\equiv -\tfrac{1}{2}\epsilon_{ijk}M^{jk}, \\ K_i &\equiv M_{i0} \end{aligned} \tag{4.13}$$

are the generators of rotations and boosts, respectively.[19] The Casimir operator is then

$$[W(\bar{k})]^2 = -(\alpha J_1 - \beta K_2)^2 - (\alpha J_2 + \beta K_1)^2 - \mathfrak{M}^2(J_3)^2. \tag{4.14}$$

This suggests defining

$$\begin{aligned} S_1(\bar{k}) &= \frac{1}{\mathfrak{M}}(\alpha J_1 - \beta K_2), \\ S_2(\bar{k}) &= \frac{1}{\mathfrak{M}}(\alpha J_2 + \beta K_1), \\ S_3(\bar{k}) &= J_3, \end{aligned} \tag{4.15}$$

in order that the Casimir operator be

$$[W(\bar{k})]^2 = -\mathfrak{M}^2[\vec{S}(\bar{k})]^2. \tag{4.16}$$

The operators $S_a(\bar{k})$ in (4.15) are the particular combination of rotations and boosts that leave the vector $\bar{k}$ invariant, i.e., they are the generators of the little group. [In the special case of stable massless particles the appropriate combination provided by (4.14) is obviously $J_1 - K_2$ and $J_2 + K_1$ rather than (4.15). It is well known that these two combinations plus their closure, J_3, generate the little group of massless particles.[20]]

To find the eigenvalues of W^2, we observe that (4.15) gives the familiar commutation relations

$$[S_a(\bar{k}), S_b(\bar{k})] = i\epsilon_{abc}S_c(\bar{k}). \tag{4.17}$$

Obviously both S_1 and S_2 in (4.15) are non-Hermitian because α, β, and $\mathfrak{M}$ are complex. However, because SU(2) is compact, solutions to the commutation relations (4.17) are equivalent to Hermitian operators,[21] i.e., there exists a Hermitian operator $\vec{V}$ and a transformation T such that

$$\vec{S}(\bar{k}) = T\vec{V}T^{-1}. \tag{4.18}$$

Hence the eigenvalues of $[\vec{S}(\bar{k})]^2$ are, as usual, real nonnegative integers or half-integers and label irreducible representations of the Poincaré group because of (4.16). That these integers or half-integers actually are related to rotations will not be shown until Sec. V.

To diagonalize the spin Casimir operator when k is arbitrary requires constructing a general $\vec{S}(k)$. It is useful for this and later sections to define complex Lorentz transformation matrices L^c by

$$L^c(k', k)^\alpha{}_\beta k^\beta = k'^\alpha, \tag{4.19}$$

where k and k' are arbitrary complex momenta with the same mass. [Note that (4.19) does not completely specify L^c.] Now consider the transformation

$$L^c(k, \mathfrak{M})^\alpha{}_\mu \mathfrak{M}^\mu = k^\alpha, \tag{4.20}$$

where $\mathfrak{M}$ denotes the momentum vector

$$\mathfrak{M}^\mu = (\mathfrak{M}, 0, 0, 0). \tag{4.21}$$

Use the columns of this matrix to define three complex vectors,

$$e^\alpha(k, a) \equiv L^c(k, \mathfrak{M})^\alpha{}_a, \quad a = 1, 2, 3. \tag{4.22}$$

These vectors are orthonormal,

$$e^\alpha(k, a)e_\alpha(k, b) = -\delta^{ab}, \tag{4.23}$$

and orthogonal to k^α itself. Together the four vectors form a complete basis in that

$$-\sum_{a=1}^{3} e^\alpha(k, a)e^\beta(k, a) + \left(\frac{1}{\mathfrak{M}}\right)^2 k^\alpha k^\beta = g^{\alpha\beta}. \tag{4.24}$$

This relation is useful in calculating the Casimir operator

$$[W(k)]^2 = W_\alpha(k)g^{\alpha\beta}W_\beta(k). \tag{4.25}$$

From the definition (4.10)

$$k^{\alpha} W_{\alpha}(k)=0$$

so that (4.25) reduces to

$$[W(k)]^2=-\mathfrak{M}^2[\vec{S}(k)]^2, \tag{4.26}$$

where

$$S_a(k)\equiv\frac{1}{\mathfrak{M}}e^{\alpha}(k,a)W_{\alpha}(k), \quad a=1,2,3. \tag{4.27}$$

A different expression for $\vec{S}(k)$ may be obtained by using

$$e^{\alpha}(k,a)\frac{k^{\beta}}{\mathfrak{M}}\epsilon_{\alpha\beta\mu\nu}=-\epsilon_{abc}e_{\mu}(k,b)e_{\nu}(k,c)$$

in (4.27) to get

$$S_a(k)=-\tfrac{1}{2}\epsilon_{abc}e_{\mu}(k,b)e_{\nu}(k,c)M^{\mu\nu}. \tag{4.28}$$

From (4.28) the commutation relations

$$[S_a(k),S_b(k)]=i\epsilon_{abc}S_c(k) \tag{4.29}$$

follow easily.

Again because of the compactness of SU(2) the eigenvalues of $[\vec{S}(k)]^2$ are real integers or half-integers.[21] The equivalence transformation from the non-Hermitian $\vec{S}(k)$ to three Hermitian operators can now be explicitly constructed. The definitions (4.22) and (4.27) give

$$S_a(k)=\frac{1}{\mathfrak{M}}L^c(k,\mathfrak{M})^{\alpha}{}_{a}W_{\alpha}(k). \tag{4.30}$$

The definition (4.10) of $W_{\alpha}(k)$ guarantees its covariance under a real Lorentz transformation,

$$\Lambda^{\alpha}{}_{\mu}W_{\alpha}(k)=U[\Lambda]W_{\mu}(\Lambda^{-1}k)U^{-1}[\Lambda], \tag{4.31}$$

where

$$U[\Lambda]=\exp(i\lambda_{\mu\nu}M^{\mu\nu}) \tag{4.32}$$

as always. When the transformation matrix Λ is complex the six parameters $\lambda_{\mu\nu}$ are complex, but (4.31) still holds because it depends only on the commutators of the $M^{\mu\nu}$ with themselves. Thus using (4.31) in (4.30) gives

$$S_a(k)=\frac{1}{\mathfrak{M}}U[L^c(k,\mathfrak{M})]W_a(\mathfrak{M})U^{-1}[L^c(k,\mathfrak{M})]. \tag{4.33}$$

Because

$$W_a(\mathfrak{M})=\mathfrak{M}J_a,$$

(4.33) may be rewritten as

$$\vec{S}(k)=U^{-1}[L^c(k,\mathfrak{M})]\vec{J}U[L^c(\mathfrak{M},k)]. \tag{4.34}$$

Note that U here is not unitary because the parameters $\lambda_{\mu\nu}$ in (4.32) are complex. This is the promised equivalence relation between some Hermitian operator and the non-Hermitian $\vec{S}(k)$. It shows, furthermore, that to construct unstable-particle states that transform irreducibly under real Lorentz transformations necessarily requires the use of complex Lorentz transformations.

V. LORENTZ TRANSFORMATIONS OF UNSTABLE PARTICLES

Sections II and III showed that corresponding to every pole in a scattering amplitude there is an eigenstate of energy-momentum in a space $\mathfrak{H}^{\dagger}$ larger than the physical Hilbert space. This section will show that corresponding to every pole is a state which transforms irreducibly under the Poincaré group.

A. J_3 states

Because of (4.26) a state $|k\rangle$ will transform irreducibly under real Lorentz transformations only if

$$[\vec{S}(k)]^2|k\rangle=j(j+1)|k\rangle, \tag{5.1}$$

with j a real integer or half-integer. Right away, the states constructed in Sec. III therefore cannot be irreducible. They are given in (3.21) by the analytic continuation of

$$|\Psi(k)j,j_3\rangle=\left[1+\frac{1}{k^0-H'}H''\right]|\phi(\vec{k})j,j_3\rangle\frac{1}{[N(k)]^{1/2}}. \tag{5.2}$$

Recall that $|\phi(\vec{k})j,j_3\rangle$ are the states constructed in Appendix A for an arbitrary real momentum $\vec{k}$. They are eigenstates of $\vec{J}^2$ and J_3 because the discrete field indices are summed against appropriate Clebsch-Gordan coefficients. Both H' and H'' commute with $\vec{J}^2$ and J_3 since they are just the projections (2.6) of H determined by $|\phi(\vec{k})j,j_3\rangle$. Therefore, (5.2) and its analytic continuation satisfy

$$\begin{aligned}\vec{J}^2|\Psi(k)j,j_3\rangle&=j(j+1)|\Psi(k)j,j_3\rangle,\\ J_3|\Psi(k)j,j_3\rangle&=j_3|\Psi(k)j,j_3\rangle\end{aligned} \tag{5.3}$$

and do not satisfy (5.1).

The $|\Psi\rangle$ are, however, irreducible representations when $\vec{k}=0$ because (5.1) and (5.3) are then identical owing to

$$S_a(k)|_{\vec{k}=0}=J_a. \tag{5.4}$$

An irreducible representation with momentum k then automatically results from boosting such a $|\Psi\rangle$ from rest to k. With the rest momentum again denoted by

$$\mathfrak{M}^{\mu}=(\mathfrak{M},0,0,0), \tag{5.5}$$

the irreducible representation is just

$$|k,j,j_3\rangle\equiv U[L^c(k,\mathfrak{M})]|\Psi(\mathfrak{M})j,j_3\rangle(2\mathfrak{M})^{1/2}. \tag{5.6}$$

[The $(2\mathfrak{M})^{1/2}$ is included to give a covariant normalization in Sec. V D.] The proof of irreducibility is that

$$\mathcal{P}^{\mu}|k,j,j_3\rangle = k^{\mu}|k,j,j_3\rangle,$$

$$[\vec{S}(k)]^2|k,j,j_3\rangle = j(j+1)|k,j,j_3\rangle$$

In addition, (5.6) satisfies

$$S_3(k)|k,j,j_3\rangle = j_3|k,j,j_3\rangle. \qquad (5.7)$$

The difference between (5.2) and (5.6) has nothing to do with complex momentum; the difference is equally present for stable particles in the physical Hilbert space. Also note that the precise meaning of (5.6) is that its inner product with an arbitrary state $|f\rangle$ is given by the analytic continuation in k^0 of

$$\langle f|k,j,j_3\rangle = \left\langle f\left|e^{i\lambda_{\mu\nu}M^{\mu\nu}}\frac{1}{k^0-H}\right|\phi(\vec{0})j,j_3\right\rangle \times D(k^0)(2\mathfrak{M})^{1/2} \qquad (5.8)$$

to the value $k^0=\mathfrak{M}$. Here the $\lambda_{\mu\nu}$ are six complex parameters corresponding to a boost from momentum $\mathfrak{M}^{\mu}$ to k^{μ}. Equation (5.8) is an obvious parallel to (3.2). From now on Lorentz transformations will be applied at will but the precise meaning will always be in terms of inner products like (5.8).

The behavior of (5.6) under a real Lorentz transformation Λ now follows easily:

$$U[\Lambda]|k,j,j_3\rangle = U[L^c(\Lambda k,\mathfrak{M})]U[R_W^c]|\mathfrak{M},j,j_3\rangle, \qquad (5.9)$$

where

$$R_W^c \equiv L^c(\Lambda k,\mathfrak{M})^{-1}\Lambda L^c(k,\mathfrak{M}).$$

Clearly R_W^c is a complex Lorentz transformation that leaves $\mathfrak{M}^{\mu}$ invariant. It is therefore just a complex Wigner rotation whose effect on the rest state is

$$U[R_W^c]|\mathfrak{M},j,j_3\rangle = \sum_{j_3'}|\mathfrak{M},j,j_3'\rangle D_{j_3'j_3}(R_W^c). \qquad (5.10)$$

Here D is the usual rotation matrix for spin j. The three rotation angles are complex. The complete transformation law is then

$$U[\Lambda]|k,j,j_3\rangle = \sum_{j_3'}|\Lambda k,j,j_3'\rangle D_{j_3'j_3}(R_W^c). \qquad (5.11)$$

This result is quite similar to the usual transformation law for stable particles. It has been suggested, in fact, that the only momentum k^{μ} allowable for an unstable particle are those for which $(1/\mathfrak{M})k^{\mu}$ is purely real.[22] Such a restriction makes the Wigner rotation in (5.11) purely real. The explicitly constructed states in Sec. III, however, show that other momenta do occur and R_W^c is therefore complex.

B. Other states

The states constructed in Sec. III by analytic continuation have $p^2>0$ and $q^2>0$. [See Eq. (4.7).] Once complex boosts are allowed, states with any momentum k satisfying $k^2=\mathfrak{M}^2$ can be constructed from these. (The J_3 states of the previous subsection, for example, have no restriction on the sign of p^2 or q^2.) Out of this myriad of possible states the physically important ones are those that can be excited in physical scattering processes. All states used in scattering experiments have a total energy-momentum that is timelike. Therefore, the only unstable-particle states that can be excited in physical scattering processes must have a four-momentum k that becomes timelike when continued back to the real axis. In short, they must have $p^2>0$ to produce resonances in scattering amplitudes (see Sec. VI A for more details). For a given timelike p the associated q may be either timelike, lightlike, or spacelike. In each case the momentum k is related by a real Lorentz transformation to a standard momentum $\bar{k}$ defined as follows:

If $p^2>0$, $q^2>0$,

$$\bar{p} \equiv \left(\frac{p\cdot q}{(q^2)^{1/2}},0,0,\left[\frac{(p\cdot q)^2-p^2q^2}{q^2}\right]^{1/2}\right),$$
$$\bar{q} \equiv ((q^2)^{1/2},0,0,0); \qquad (5.12a)$$

if $p^2>0$, $q^2=0$,

$$\bar{p} \equiv \left(\frac{p^2A^2+(p\cdot q)^2}{2A(p\cdot q)},0,0,\frac{p^2A^2-(p\cdot q)^2}{2A(p\cdot q)}\right),$$
$$\bar{q} \equiv (A,0,0,A); \qquad (5.12b)$$

if $p^2>0$, $q^2<0$,

$$\bar{p} \equiv \left(\left[\frac{(p\cdot q)^2-p^2q^2}{-q^2}\right]^{1/2},0,0,\frac{p\cdot q}{(-q^2)^{1/2}}\right),$$
$$\bar{q} \equiv (0,0,0,(-q^2)^{1/2}). \qquad (5.12c)$$

Obviously (5.12a) is the prototype (4.8) that resulted from analytic continuation in the variable k^0.

The J_3 states constructed previously did not take advantage of the fact that only states with $p^2>0$ are excited in physical scattering processes. Their transformation law (5.11) is based only upon the invariance of momentum (5.5) under arbitrary rotations. The little group thus has three generators. On the other hand, the only real Lorentz transformations that leave any of the $\bar{k}$ in (5.12) invariant are rotations around the z axis. The little group thus has only one generator and its irreducible representations are one dimensional. It might therefore be possible to construct states which are irreducible representations both of the full Poincaré group and of this one-parameter little group. Because the representation of the translations is not unitary there is no guarantee of finding such states. In Sec. V C, however, they are successfully constructed and their transformation law (5.21) is derived.

For completeness, states with momenta other than $p^2 > 0$ are discussed in Appendix E. Beltrametti and Luzzatto[23] have shown that for any complex momentum k a standard $\bar{k}$ can be chosen. The corresponding little group is always one dimensional, but depends on the sign of

$$\Delta \equiv (p \cdot q)^2 - p^2 q^2 . \tag{5.13}$$

The momenta in (5.12) all have $\Delta > 0$ and the states which are irreducible representations of both the Poincaré group and the little group are constructed in Sec. V C. In Appendix E, states with $\Delta < 0$ on which both groups are represented irreducibly are also constructed. For the case $\Delta = 0$, however, such representations are not possible.

C. Helicity states

The unstable particle states that produce resonances in scattering amplitudes all have a momentum k that is related by a real Lorentz transformation to one of the $\bar{k}$ of (5.12). All three possibilities in (5.12) have the form

$$\bar{k} = (\alpha, 0, 0, \beta), \qquad \mathfrak{M}^2 = \alpha^2 - \beta^2 . \tag{5.14}$$

Define a new state by

$$|k, j, \sigma\rangle \equiv U[L(k, \bar{k}) L^c(\bar{k}, \mathfrak{M})] \, |\Psi(\mathfrak{M}) j, j_3\rangle (2\mathfrak{M})^{1/2} \big|_{j_3 = \sigma} . \tag{5.15}$$

It should be noted that although (5.15) is very different from (5.6) the only notational distinction will be the use of an index σ (later related to helicity) rather than a j_3. In spite of the two-stage boost in (5.15) it is easy to check irreducibility by

$$[\vec{S}(k)]^2 |k, j, \sigma\rangle = U[L^c(k, \mathfrak{M})] \vec{J}^2 U[X] \times |\Psi(\mathfrak{M}) j, j_3\rangle (2\mathfrak{M})^{1/2} \big|_{j_3=\sigma} , \tag{5.16}$$

where

$$X \equiv L^c(k, \mathfrak{M})^{-1} L(k, \bar{k}) L^c(\bar{k}, \mathfrak{M}) .$$

Because X leaves the four-vector $\mathfrak{M}$ invariant it must be some complex rotation. Therefore

$$\vec{J}^2 U[X] = U[X] \vec{J}$$

so that (5.16) becomes

$$[\vec{S}(k)]^2 |k, j, \sigma\rangle = j(j+1) |k, j, \sigma\rangle . \tag{5.17}$$

The behavior of (5.15) under a real Lorentz transformation Λ follows from

$$U[\Lambda] |k, j, \sigma\rangle = U[L(\Lambda k, \bar{k}) L^c(\bar{k}, \mathfrak{M})] U[\bar{R}] \times |\Psi(\mathfrak{M}) j, \sigma\rangle (2\mathfrak{M})^{1/2} , \tag{5.18}$$

where

$$\bar{R} \equiv L^c(\bar{k}, \mathfrak{M})^{-1} [L(\Lambda k, \bar{k})^{-1} \Lambda L(k, \bar{k})] L^c(\bar{k}, \mathfrak{M}) . \tag{5.19}$$

Again $\bar{R}$ leaves $\mathfrak{M}^\mu$ invariant and must therefore be a complex rotation at worst. Closer inspection shows that the quantity in brackets in (5.19) is a real Lorentz transformation that leaves $\bar{k}$ of (5.14) invariant. Therefore it can only be a rotation about the z axis:

$$U[L(\Lambda k, \bar{k})^{-1} \Lambda L(k, \bar{k})] = e^{i\theta_W J_3} , \tag{5.20}$$

where θ_W is a real Wigner rotation angle that depends on k and Λ. Now $L^c(\bar{k}, \mathfrak{M})$ is just a boost along $\hat{z}$. Because

$$[J_3, K_3] = 0 ,$$

it follows that $\bar{R}$ itself is simply a rotation around the z axis,

$$U[\bar{R}] = e^{i\theta_W J_3} .$$

Therefore (5.18) becomes

$$U[\Lambda] |k, j, \sigma\rangle = U[L(\Lambda k, \bar{k}) L^c(\bar{k}, \mathfrak{M})] e^{i\theta_W J_3} \times |\Psi(\mathfrak{M}) j, \sigma\rangle (2\mathfrak{M})^{1/2} .$$

Because $|\Psi\rangle$ is an eigenstate of J_3 this collapses to

$$U[\Lambda] |k, j, \sigma\rangle = |\Lambda k, j, \sigma\rangle e^{i\theta_W \sigma} . \tag{5.21}$$

This is the analog of (5.11). The transformation law for J_3 states in (5.11) was more complicated just because those states are an awkward linear combination of the helicity states. The precise combination is

$$|k, j, j_3\rangle = \sum_\sigma |k, j, \sigma\rangle D_{\sigma, j_3}(Y) , \tag{5.22}$$

where Y is another complex rotation given by

$$Y \equiv L^c(\mathfrak{M}, \bar{k}) L(\bar{k}, k) L^c(k, \mathfrak{M}) . \tag{5.23}$$

The transformation law (5.21) is exactly the same as for massless particles because in both cases there is no real Lorentz transformation that takes the three-momentum to zero. The only difference is that here σ may take on all integrally spaced values from $-j$ to $+j$ rather than just the two extreme values allowed in the massless case. It should be emphasized that the restriction of j to integers or half-integers comes from the Poincaré group itself in Sec. IV B. The construction of $|\phi, j, j_3\rangle$ and then $|\Psi, j, j_3\rangle$ by combining field indices (as discussed in Appendix A) merely capitalized on that result.

Calling (5.15) helicity states is as yet unjustified. The real justification is the demonstration in Sec. VI A that these states occur as resonances in helicity amplitudes. At the present this nomenclature may even seem inappropriate. Clearly (5.15) is an eigenstate of $\vec{\mathcal{P}}\cdot\vec{J}$ when the momentum is $\bar{k}$, and hence along $\hat{z}$

$$J_3|\bar{k},j,\sigma\rangle=\sigma|\bar{k},j,\sigma\rangle. \tag{5.24}$$

For arbitrary complex $\vec{k}$, however, it is not likely that (5.15) will be an eigenstate of $\vec{\mathcal{P}}\cdot\vec{J}$ as the massless case might suggest. The correct generalization of the helicity operator is obtained from boosting (5.24) to get

$$\Sigma(k)|k,j,\sigma\rangle=\sigma|k,j,\sigma\rangle, \tag{5.25}$$

where

$$\Sigma(k)\equiv U[L(k,\bar{k})]J_3U^{-1}[L(k,\bar{k})]. \tag{5.26}$$

An alternative expression for this operator that will be derived shortly is

$$\Sigma(k)=\frac{\epsilon_{\alpha\beta\mu\nu}p^\alpha q^\beta M^{\mu\nu}}{2[(p\cdot q)^2-p^2q^2]^{1/2}}. \tag{5.27}$$

This helicity operator is manifestly a Lorentz scalar,

$$U[\Lambda]\Sigma(k)U^{-1}[\Lambda]=\Sigma(\Lambda k). \tag{5.28}$$

Furthermore, whenever $\vec{p}$ and $\vec{q}$ are parallel so that k has the special form

$$k_S\equiv(p^0+iq^0,\hat{n}|\vec{p}|\pm i\hat{n}|\vec{q}|) \tag{5.29}$$

then (5.27) reduces to

$$\Sigma(k_S)=\hat{n}\cdot\vec{J}.$$

This result at present is only a plausibility argument for the name helicity but it will be very important in Sec. VI A.

Naturally $\Sigma(k)$ is just the generator which leaves a general complex vector k invariant. As an example take

$$p=(a,0,0,b),$$
$$q=(c,0,d,0).$$

Then p is left invariant by the three generators

$$J_3,\quad bK_1+aJ_2,\quad aJ_1-bK_2,$$

just as in (4.15). On the other hand, q is left invariant by

$$J_2,\quad cJ_3-dK_1,\quad cJ_1+dK_3.$$

There is, however, only one combination of generators that leaves both p and q invariant. It is given by

$$\epsilon_{\alpha\beta\mu\nu}p^\alpha q^\beta M^{\mu\nu}=bcJ_3-d(bK_1+aJ_2)$$
$$=b(cJ_3-dK_1)-adJ_2.$$

This combination of generators is grouped in two different ways to show how it leaves both p and q invariant.

Now to show that (5.27) is correct, explicitly carry out the transformations in (5.26) to get

$$\Sigma(k)=x^{\mu\nu}(k,\bar{k})M_{\mu\nu} \tag{5.30}$$

where

$$x^{\mu\nu}(k,\bar{k})\equiv-\tfrac{1}{2}L(k,\bar{k})^\mu{}_aL(k,\bar{k})^\nu{}_b\epsilon^{ab3}.$$

Apply a real Lorentz transformation Λ,

$$\Lambda^\xi{}_\mu\Lambda^\eta{}_\nu x^{\mu\nu}(k,\bar{k})=-\tfrac{1}{2}[L(\Lambda k,\bar{k})R]^\xi{}_a[L(\Lambda k,\bar{k})R]^\eta{}_b\epsilon^{ab3},$$

where

$$R\equiv L^{-1}(\Lambda k,\bar{k})\Lambda L(k,\bar{k}). \tag{5.31}$$

Obviously, R is a real transformation that leaves $\bar{k}$ invariant. Hence it is only a rotation about the z axis. This means

$$R^{a'}{}_aR^{b'}{}_b\epsilon^{ab3}=\epsilon^{a'b'3}.$$

Therefore (5.31) is just

$$\Lambda^\xi{}_\mu\Lambda^\eta{}_\nu x^{\mu\nu}(k,\bar{k})=x^{\xi\eta}(\Lambda k,\bar{k}). \tag{5.32}$$

This means that $x^{\mu\nu}$ is a tensor function of k independently of $\bar{k}$. Furthermore, from the definition (5.30) it also satisfies

$$x^{\mu\nu}=-x^{\nu\mu},$$
$$p_\mu x^{\mu\nu}=q_\mu x^{\mu\nu}=0.$$

The only such tensor is

$$x^{\mu\nu}(k)=c\epsilon^{\alpha\beta\mu\nu}p_\alpha q_\beta.$$

The constant c may be evaluated in a particular frame (like $k=\bar{k}$) and gives (5.27) as claimed.

Note that this argument fails, as it should, for stable particles. Then the R in (5.31) that leaves a real vector $\bar{p}$ invariant will contain boosts, though in a special combination like (4.15). Therefore (5.32) fails for massive stable particles and their helicity operator cannot be Lorentz invariant.

D. Covariance of inner products and norms

Both the J_3 states and the helicity states are related to $|\Psi(\mathfrak{M})\rangle$ by complex, nonunitary boosts. The inner product of either with itself, therefore, might not be covariant. However, because H is Hermitian the inner product vanishes and hence is trivially covariant.

The complex-conjugate partner of (5.6) is

$$|k^*,j,j_3\rangle\equiv U[L^c(k^*,\mathfrak{M}^*)]|\Psi(\mathfrak{M}^*)j,j_3\rangle(2\mathfrak{M}^*)^{1/2}, \tag{5.33}$$

and the results are similar for helicity states. Because the Lorentz generators $M^{\mu\nu}$ are Hermitian

$$[e^{i\lambda^*_{\mu\nu}M^{\mu\nu}}]^\dagger = e^{-i\lambda_{\mu\nu}M^{\mu\nu}},$$

or equivalently

$$U[L^*]^\dagger = U[L]^{-1}.$$

This means that the inner product of (5.33) with its partner (5.6) is

$$\langle j_3, j, k^* | k', j', j_3'\rangle = 2k^0\delta^3(\vec{k}-\vec{k}')\delta_{j,j'}\delta_{j_3,j_3'}. \quad (5.34)$$

The covariance of this product may be explicitly verified. From the definition (4.19)

$$L^c(k'^*, k^*) = L^c(k', k)^*.$$

Consequently the transformation law of (5.33) is

$$U[\Lambda]|k^*, j, j_3\rangle = \sum_{j_3'} |\Lambda k^*, j, j_3'\rangle D_{j_3', j_3}(R_W^{c*}),$$

where R_W^{c*} is just the complex conjugate of the rotation occurring in (5.11). Because

$$[D_{j_3', j_3}(R^*)]^* = D_{j_3, j_3'}(R^{-1}),$$

the inner product (5.34) is genuinely covariant.

For helicity states this check is easier. The transformation law for the conjugate state is

$$U[\Lambda]|k^*, j, \sigma\rangle = |\Lambda k^*, j, \sigma\rangle e^{i\theta_W\sigma},$$

where θ_W is the same real angle as in (5.21). The covariance of the inner product just amounts to

$$[e^{i\theta_W\sigma}]^* = e^{-i\theta_W\sigma}.$$

The momenta $\vec{k}$ and $\vec{k}'$ in (5.34) are generally complex and the δ function must therefore be analytically continued. Møller proposed this long ago.[8] The continuation was thoroughly developed by Bremermann and Durand.[24] The usual representation

$$\delta(k'-k) = \frac{-1}{2\pi i}\left[\frac{1}{(k'+i\epsilon)-k} - \frac{1}{(k'-i\epsilon)-k}\right],$$

is valid for k and k' real. For complex variables

$$\int_c dk' f(k')\delta(k'-k) \equiv \int_{c_1+c_2} dk' f(k')\frac{-1}{2\pi i}\left(\frac{1}{k'-k}\right), \quad (5.35)$$

where c_1 is a contour parallel to c that passes above k and c_2 is a contour antiparallel to c that passes below k.

VI. WAVE FUNCTIONS AND S-MATRIX ELEMENTS FOR UNSTABLE PARTICLES

The consequences of the transformation laws derived in Sec. V for experiments carried out on the real axis will now be discussed. The wave functions for unstable particles and their use in calculating scattering amplitudes will be quite analogous to the usual stable-particle results.

A. How the irreducible representations are produced in scattering

Two points will be emphasized in this section: first, that the irreducible representations of Sec. V actually do occur as poles in partial-wave amplitudes; second, the Lorentz invariance of the helicity in (5.21) is not experimentally observable. It will turn out, in fact, that measurements conducted in two different frames will find the $2j+1$ helicity components rotated according to the usual law for stable particles with $m \neq 0$. In short, it is (6.18) that is observed and not (5.21). [It goes without saying that the transformation law (5.11) for J_3 states automatically reduces to that for stable particles when the momentum becomes real.]

For definiteness, consider the scattering of two stable particles that produce a resonance in the s channel. Denote the scattering amplitude by

$$T(1+2\to A) = \langle A | T(p) | \vec{p}_1, \lambda_1; \vec{p}_2, \lambda_2\rangle, \quad (6.1)$$

where A is some multiparticle final state, λ_i are the individual helicities, and $p = p_1 + p_2$. This amplitude may be decomposed into partial waves of definite total angular momentum by following the procedure of Jacob and Wick[25] to construct eigenstates of $\vec{J}^2$ in the center-of-mass frame. These states must be further specified as eigenstates either of J_3 or of helicity. The partial-wave amplitude in any frame is then obtained by boosting these two-particle states. Denote the partial-wave amplitude with s-channel helicity σ by

$$T^j(1+2\to A) = \langle A | T(p) | p, j, \sigma; 1+2\rangle, \quad (6.2)$$

where

$$|j_1 - j_2| \le j \le |j_1 + j_2|,$$
$$-j \le \sigma \le j.$$

The three labels on the two-particle state in (6.2) denote the fact that

$$\mathcal{P}^\mu |p, j, \sigma; 1+2\rangle = p^\mu |p, j, \sigma; 1+2\rangle, \quad (6.3a)$$

$$[\vec{S}(p)]^2 |p, j, \sigma; 1+2\rangle = j(j+1) |p, j, \sigma; 1+2\rangle, \quad (6.3b)$$

$$\hat{p}\cdot\vec{J} |p, j, \sigma; 1+2\rangle = \sigma |p, j, \sigma; 1+2\rangle, \quad (6.3c)$$

A Lorentz transformation rotates the $2j+1$ values of σ according to the usual law

$$U[\Lambda]|p, j, \sigma; 1+2\rangle = \sum_{\sigma'} |\Lambda p, j, \sigma'; 1+2\rangle D_{\sigma',\sigma}(\tilde{R}), \quad (6.4)$$

where

$$\tilde{R} = L(\tilde{p}', W)^{-1}[R(p',\tilde{p}')^{-1}\Lambda R(p,\tilde{p})]L(\tilde{p}, W),$$
$$W = ((p^2)^{1/2}, 0, 0, 0),$$
$$\tilde{p} = (p^0, 0, 0, |\vec{p}|),$$
$$p' = \Lambda p,$$
$$\tilde{p}' = (p^{0\prime}, 0, 0, |\vec{p}'|).$$

It is, of course, obvious that the different helicity components must mix from the observation that the helicity operator in (6.3c) is not Lorentz invariant.

By hypothesis (6.1) has a resonance pole when analytically continued to the second sheet. The amplitude can be related to the Fourier transform of the Green's function

$$\langle 0 | T[\mathcal{A}(y)\psi(x_1)\bar{\psi}(x_2)] | 0\rangle.$$

Just as in Appendix A the pole can then come only from the term

$$G(p) = \left\langle 0 \middle| \mathcal{A}(y) \frac{i(2\pi)^3\delta^3(\vec{p}-\vec{\mathcal{P}})}{p^0 - H} T[\psi(r)\bar{\psi}(-r)] \middle| 0 \right\rangle,$$

with $r = \frac{1}{2}(x_1 - x_2)$. Analytic continuation of $G(p)$ will then yield a pole whose residue may be expressed in terms of state vectors for the unstable particle:

$$G(k) \to \frac{1}{k^2 - \mathfrak{M}^2} \sum_{j',\sigma'} \langle 0 | \mathcal{A}(y) | k, j', \sigma'\rangle \times \langle k^*, j', \sigma' | T[\psi(r)\bar{\psi}(-r)] | 0\rangle. \tag{6.5}$$

The unstable-particle states in (6.5) have been chosen as helicity states:

$$\mathcal{P}^\mu | k, j', \sigma'\rangle = k^\mu | k, j', \sigma'\rangle, \tag{6.6a}$$
$$[\vec{S}(k)]^2 | k, j', \sigma'\rangle = j'(j'+1) | k, j', \sigma'\rangle, \tag{6.6b}$$
$$\Sigma(k) | k, j', \sigma'\rangle = \sigma' | k, j', \sigma'\rangle, \tag{6.6c}$$

where

$$k^\mu = p^\mu + iq^\mu.$$

The contribution of (6.5) to the partial-wave amplitude is obtained by taking $q \to 0$ in the residue of (6.5), projecting out the two-particle formation amplitude $\langle p, j', \sigma' | \vec{p}_1, \lambda_1; \vec{p}_2, \lambda_2\rangle$, and then combining spins to get

$$T^j(1+2 \to A) \approx \frac{1}{p^2 - \mathfrak{M}^2} \sum_{j',\sigma'} \langle A | p, j', \sigma'\rangle \langle p, j', \sigma' | p, j, \sigma; 1+2\rangle. \tag{6.7}$$

Now, the question of interest is: Which values of j' and σ' in (6.5) actually survive the partial-wave projection in (6.7)? Comparing (6.6b) with (6.3b) shows that $j' = j$ because

$$\lim_{q\to 0} \vec{S}(k) = \vec{S}(p).$$

[See (4.27) for example.] However, the helicity operator

$$\Sigma(k) = \frac{\epsilon_{\alpha\beta\mu\nu} p^\alpha q^\beta M^{\mu\nu}}{2[(p\cdot q)^2 - p^2 q^2]^{1/2}} \tag{6.8}$$

has the property that in general

$$\lim_{q\to 0} \Sigma(k) \neq \hat{p}\cdot\vec{J}.$$

Only when $\vec{q}$ is taken to zero along the $\vec{p}$ direction does (6.6c) correspond to (6.3c), for then

$$\lim_{\vec{q}\parallel\vec{p}} \Sigma(k) = \hat{p}\cdot\vec{J}. \tag{6.9}$$

Thus if q is taken to zero along the plane $\vec{q}\parallel\vec{p}$, then $\sigma' = \sigma$ in (6.7). Appendix F shows, furthermore, that the only nonvanishing contribution to (6.7) comes from taking the q limit in this manner. Thus

$$T^j(1+2 \to A) \approx \frac{\langle A | p, j, \sigma\rangle \langle p, j, \sigma | p, j, \sigma; 1+2\rangle}{p^2 - M^2 + iM\Gamma}. \tag{6.10}$$

The discussion leading to (6.10) shows that the helicity states of Sec. V C are actually produced in scattering. In spite of this, the Lorentz invariance of the unstable-particle helicity is not observable because the limit required in (6.9) is not covariant. To see exactly what the observable transformation law for the helicity states will be, consider how to calculate (6.10) in two different frames. In the first frame let

$$p = (p^0, 0, 0, p^3). \tag{6.11a}$$

Then the residue of the pole in (6.5) that contributes to (6.10) must have

$$q = (q^0, 0, 0, q^3). \tag{6.11b}$$

With q in this form q^0 and q^3 may be taken to zero independently.

If the same experiment is viewed from a frame moving along the x axis, the observed momenta are

$$p' = (p^0\cosh\lambda, p^0\sinh\lambda, 0, p^3), \tag{6.12a}$$
$$q' = (q^0\cosh\lambda, q^0\sinh\lambda, 0, q^3). \tag{6.12b}$$

But now $\vec{q}'$ is not parallel to $\vec{p}'$. To calculate the contribution of the unstable-particle pole to the partial-wave amplitude in this frame requires continuing q' to zero in the plane of $\vec{p}'$. Let q'' be this continued value of q' and let $p'' = p'$ be the unchanged value of the real momenta:

$$p'' = (p^0\sinh\lambda, p^0\sinh\lambda, 0, p^3) = p', \tag{6.13a}$$

$$q'' = (q'^0, |\vec{q}'|\hat{u}), \tag{6.13b}$$

where

$$\hat{u} = \frac{(p^0 \sinh\lambda, 0, p^3)}{[(p^0 \sinh\lambda)^2 + (p^3)^2]^{1/2}}.$$

For (6.13) to correspond to the same value of mass and width as (6.11) it is necessary that

$$(p'' + iq'')^2 = (p + iq)^2. \tag{6.14}$$

If (6.14) has solutions of the form (6.13), then k'' and k can still be related by a Lorentz transformation that is complex. Solving (6.14) gives

$$q''^0 = \frac{1}{p^2}\{p'^0(p\cdot q) - |\vec{p}'|[(p\cdot q)^2 - p^2q^2]^{1/2}\},$$
$$|\vec{q}''| = \frac{1}{p^2}\{|\vec{p}'|(p\cdot q) - p'^0[(p\cdot q)^2 - p^2q^2]^{1/2}\}. \tag{6.15}$$

Thus there is a complex Lorentz transformation Λ^c for which

$$p'' + iq'' = \Lambda^c(p + iq).$$

Because k'' is the appropriate momentum in the moving frame to produce a pole in the partial-wave amplitude, the observed transformation law will be determined by this complex Lorentz transformation. To find that law requires applying Λ_c to the $|k,j,\sigma\rangle$ helicity state in (6.5). The result is directly analogous to (5.18) and (5.19):

$$U[\Lambda^c]|k,j,\sigma\rangle = \sum_{\sigma'} |\Lambda^c k, j, \sigma'\rangle D_{\sigma',\sigma}(\overline{R}^c),$$

where (6.16)

$$\overline{R}^c \equiv L^c(\overline{k}, \mathfrak{M})^{-1}[L(k'', \overline{k})^{-1}\Lambda^c L(k, \overline{k})]L^c(\overline{k}, \mathfrak{M}).$$

Recall that in (5.19) $\overline{R}$ is a real rotation around the z axis because Λ there is real. Here, however, Λ^c is complex. As a result $\overline{R}^c$ is neither real nor a z rotation. Consequently all $2j+1$ helicity components are rotated in (6.16). The experimentally observed transformation law is the $q'' \to 0$ limit of (6.16). Hence the helicity states will be observed to transform covariantly, not invariantly. The precise form of this covariance is obtained by taking $q'' \to 0$ in (6.16). It will shortly be shown that

$$\lim_{q''\to 0} \overline{R}^c = \tilde{R}, \tag{6.17}$$

where $\tilde{R}$ is just the usual real rotation (6.4) appropriate to stable helicity states. Thus the observed transformation law following from (6.16) is exactly like (6.4):

$$U[\Lambda]|p,j,\sigma\rangle = \sum_{\sigma'} |\Lambda p, j, \sigma'\rangle D_{\sigma',\sigma}(\tilde{R}). \tag{6.18}$$

It is, of course, essential that

$$\lim_{q\to 0} \overline{R}^c = \tilde{R}, \tag{6.19}$$

also. This guarantees that the transformation law is symmetrical between the two observers.

To demonstrate (6.17) it is necessary to specify the transformation $L(\overline{k}, k)$ more carefully. Consider, for example, the case $q^2 > 0$. Then $\overline{k}$ is given by (5.12a). A real Lorentz transformation from k to $\overline{k}$ may always be accomplished by first boosting q to rest and then rotating into the z axis. Thus

$$L(\overline{k}, k) = R(\hat{z}, B(\overline{q}, q)p)B(\overline{q}, q).$$

With the p and q of (6.11) the boost is in the $\hat{z}$ direction so the rotation is just the identity

$$L(\overline{k}, k) = B_z(\overline{q}, q).$$

On the other hand, for p'' and q'' in (6.13)

$$L(\overline{k}, k'') = R(\hat{z}, \hat{u})B_{\hat{u}}(\overline{q}, q'').$$

Putting this together gives the complex rotation in (6.16) as

$$\overline{R}^c = B_z(\mathfrak{M}, \overline{k})[R(\hat{z}, \hat{u})B_{\hat{u}}(\overline{q}, q'')\Lambda^c B_z(q, \overline{q})]B_z(\overline{k}, \mathfrak{M}). \tag{6.20}$$

It is convenient to rewrite (6.20) by using

$$R(\hat{z}, \hat{u})B_{\hat{u}}(\overline{q}, q'') = B_z(\overline{q}, q'')R(\hat{z}, \hat{u})$$

to get

$$\overline{R}^c = B_z(\mathfrak{M}, \overline{k})B_z(\overline{q}, q'')R(\hat{z}, \hat{u})\Lambda^c B_z(q, \overline{q})B_z(\overline{k}, \mathfrak{M}). \tag{6.21}$$

The boosts are explicitly given by

$$B_z(\overline{k}, \mathfrak{M}) = e^{i\mu K_3},$$
$$B_z(q, \overline{q}) = e^{i\mu' K_3},$$
$$B_z(q'', \overline{q}) = e^{i\mu'' K_3},$$

where

$$\cosh\mu = \frac{(p\cdot q)/(q^2)^{1/2} + i(q^2)^{1/2}}{\mathfrak{M}}$$
$$\cosh\mu' = \frac{q^0}{(q^2)^{1/2}}, \tag{6.22}$$
$$\cosh\mu'' = \frac{q''^0}{(q^2)^{1/2}}.$$

(Note that μ' and μ'' are both real.) Thus (6.21) is

$$\overline{R}^c = e^{-i(\mu+\mu'')K_3}R(\hat{z}, \hat{u})\Lambda^c e^{i(\mu+\mu')K_3}. \tag{6.23}$$

By using (6.11), (6.15), and (6.22) one finds that

$$\cosh(\mu + \mu'') = \frac{p''^0 + iq''^0}{\mathfrak{M}},$$
$$\cosh(\mu + \mu') = \frac{p^0 + iq^0}{\mathfrak{M}}. \tag{6.24}$$

Taking $q'' \to 0$ is now easy:

$$\lim_{q''\to 0} \cosh(\mu+\mu'') = \frac{p''^0}{M} = \frac{p'^0}{M},$$

$$\lim_{q''\to 0} \cosh(\mu+\mu') = \frac{p^0}{M}.$$

Hence

$$\lim_{q''\to 0} \bar{R}^c = B(M,\tilde{p}')R(\hat{z},\hat{u})\Lambda B(\tilde{p},M). \tag{6.25}$$

This is precisely the rotation $\tilde{R}$ given in (6.4) appropriate to these p and p'. Hence $\bar{R}^c$ reduces to $\tilde{R}$ as claimed in (6.17). It is also obvious from (6.24) that the same limit $\tilde{R}$ is obtained if $q \to 0$ instead of q''.

B. Elementary unstable particles

The indicated transformation laws apply to any unstable particle. It may or may not be associated with an elementary field occurring in the Lagrangian. In conventional field theories, for example, the μ has an elementary field but the π, being composed of quarks, does not. Thus a μ will be called elementary and a π composite, though both are unstable.

Since the μ is, in fact, the only candidate for an unstable particle with an elementary field, it will be treated explicitly here. The generalization to other spins is straightforward. The Feynman propagator for a μ field is

$$S_{\alpha\beta}(k) = -i\int d^4x\, e^{ik\cdot x}\langle 0|T[\psi_\alpha(x)\bar{\psi}_\beta(0)]|0\rangle.$$

Performing the Fourier transforms gives

$$S_{\alpha\beta}(k) = \left\langle 0\left|\psi_\alpha(0)\frac{(2\pi)^3\delta^3(\vec{k}-\vec{\mathcal{P}})}{k^0-H+i\epsilon}\bar{\psi}_\beta(0)\right|0\right\rangle - \left\langle 0\left|\bar{\psi}_\beta(0)\frac{(2\pi)^3\delta^3(\vec{k}-\vec{\mathcal{P}})}{k^0+H-i\epsilon}\psi_\alpha(0)\right|0\right\rangle.$$

Continuing in k^0 around the positive-energy cuts to the second-sheet pole gives

$$S_{\alpha\beta}(k) \underset{k^2\to\mathfrak{M}^2}{\sim} \sum_{j_3}(2\pi)^3\frac{\langle 0|\psi_\alpha(0)|k,j_3\rangle\langle k^*,j_3|\bar{\psi}_\beta(0)|0\rangle}{k^2-\mathfrak{M}^2} \tag{6.26}$$

In (6.26) a sum over J_3 states ($j_3=\pm\frac{1}{2}$) has been used. The sum could just as well have been over helicity states.

The J_3 wave functions will be discussed first. Just from the definition (5.6) of the J_3 states the residue must be

$$\langle 0|\psi_\alpha(0)|k,j_3\rangle = \left[\frac{2\mathfrak{M}z}{(2\pi)^3}\right]^{1/2}u_\alpha(k,j_3),$$

where z is some constant and the dimensionless functions u are defined by

$$u_\alpha(k,\tfrac{1}{2}) \equiv S_{\alpha,1}[L^c(k,\mathfrak{M})], \qquad u_\alpha(k,-\tfrac{1}{2}) \equiv S_{\alpha,2}[L^c(k,\mathfrak{M})]. \tag{6.27}$$

Here $S[\Lambda]$ denotes the usual $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group.[20] A real Lorentz transformation applied to (6.27) gives

$$S[\Lambda]u(k,j_3) = \sum_{j_3'} u(\Lambda k,j_3')S_{j_3',j_3}[R_W^c], \tag{6.28}$$

where R_W^c is the same complex rotation as (5.9). Define adjoint spinors as

$$\bar{u}_\alpha(k,j_3) \equiv [\gamma^0 u(k,j_3)]^*_\alpha. \tag{6.29}$$

The completeness and normalization conditions are

$$\sum_{j_3=-1/2}^{1/2} u_\alpha(k,j_3)\bar{u}_\beta(k^*,j_3) = \frac{1}{2\mathfrak{M}}(\gamma^\mu k_\mu+\mathfrak{M})_{\alpha\beta}, \tag{6.30a}$$

$$\sum_{\alpha=1}^{4}\bar{u}_\alpha(k^*,j_3)u_\alpha(k,j_3') = \delta_{j_3,j_3'}. \tag{6.30b}$$

Using (6.30a) in (6.26) gives

$$S_{\alpha\beta}(k) \underset{k^2\to\mathfrak{M}^2}{\sim} z\frac{\gamma^\mu k_\mu+\mathfrak{M}}{k^2-\mathfrak{M}^2}, \tag{6.31}$$

as might be expected.

The location of k and k^* in (6.30) is important. For example, (6.30b) is clearly a reflection of the normalization

$$\langle j_3,k^*|k',j_3'\rangle = 2k^0\delta^3(\vec{k}-\vec{k}')\delta_{j_3,j_3'}.$$

Note, however, that

$$\langle j_3,k|k',j_3'\rangle = 0$$

has no finite-dimensional counterpart:

$$\sum_{\alpha=1}^{4}\bar{u}_\alpha(k,j_3)u_\alpha(k,j_3') \neq 0. \tag{6.32}$$

In fact, the left-hand side of (6.32) is nothing particularly simple: Nor is it needed in calculations.

The spinors may be explicitly displayed by choosing a particular complex Lorentz transformation for (6.27). For example, let this be a pure boost from $\mathfrak{M}$ to k. The direction and magnitude of the boost are then given by

$$\hat{u} = \frac{\vec{p}+i\vec{q}}{(\vec{p}^2-\vec{q}^2+2i\vec{p}\cdot\vec{q})^{1/2}}, \qquad \cosh\lambda = \frac{p^0+iq^0}{\mathfrak{M}}. \tag{6.33}$$

The 4×4 spinor representation of this boost is

$$S[L^c(k,\mathfrak{M})] = e^{(\lambda/2)\hat{u}\cdot\vec{\alpha}} = \cosh\frac{\lambda}{2}+\hat{u}\cdot\vec{\alpha}\sinh\frac{\lambda}{2}, \tag{6.34}$$

where

$$\vec{\alpha}=\gamma^0\vec{\gamma}.$$

This gives

$$u_\alpha(k,\tfrac{1}{2})=\frac{1}{[2\mathfrak{M}(k^0+\mathfrak{M})]^{1/2}}(\gamma^\mu k_\mu+\mathfrak{M})_{\alpha,1},$$
$$u_\alpha(k,-\tfrac{1}{2})=\frac{1}{[2\mathfrak{M}(k^0+\mathfrak{M})]^{1/2}}(\gamma^\mu k_\mu+\mathfrak{M})_{\alpha,2}. \tag{6.35}$$

Obviously as $q^\mu\to 0$ these spinors go smoothly into the usual spinors for real energy-momentum.

The helicity wave functions are much more interesting. Let

$$\langle 0|\psi_\alpha(0)|k,\sigma\rangle=\left[\frac{2\mathfrak{M}z}{(2\pi)^3}\right]^{1/2}u_\alpha(k,\sigma),$$

where $\sigma=\pm$ to distinguish from $j_3=\pm\frac{1}{2}$. From the definition (5.15) of the helicity states it follows that

$$u_\alpha(k,+)\equiv S_{\alpha,1}[L(k,\bar{k})L^c(\bar{k},\mathfrak{M})],$$
$$u_\alpha(k,-)\equiv S_{\alpha,2}[L(k,\bar{k})L^c(\bar{k},\mathfrak{M})]. \tag{6.36}$$

The adjoint spinors are defined just as in (6.29). Completeness and normalization are then the same as (6.30) with j_3 replaced by σ. The novel feature of these spinors is their transformation law. Write (6.36) as

$$u(k,\pm)=S[L(k,\bar{k})]u(\bar{k},\pm).$$

Then

$$S[\Lambda]u(k,\pm)=S[L(\Lambda k,\bar{k})]S[\bar{R}]u(\bar{k},\pm), \tag{6.37}$$

where

$$\bar{R}=L(\Lambda k,\bar{k})^{-1}\Lambda L(k,\bar{k}). \tag{6.38}$$

Because $\bar{R}$ is a real transformation that leaves $\bar{k}$ invariant it can only be a rotation around z just as in Sec. V C. Thus

$$S[\bar{R}]=e^{i\theta_W\sigma_3/2}, \tag{6.39}$$

where σ_3 is the usual Pauli matrix. This gives for (6.37)

$$S[\Lambda]u(k,\pm)=u(\Lambda k,\pm)e^{\pm i\theta_W/2}. \tag{6.40}$$

Note that (6.40) holds even when $q\to 0$. This is because $L(k,\bar{k})$ does not reduce to the identity even when $q\to 0$.

C. Composite unstable particles

By definition a composite unstable particle is one which does not produce a pole in any two-point function. It does, of course, produce poles in Green's functions with more lines. Thus in quark theories all hadrons are composite. All of these except the proton are unstable when weak and electromagnetic interactions are included.

Composite particles that are stable produce a pole in a Green's function when continued in the total energy-momentum onto the real axis below the production threshold. Composite particles that are unstable produce poles when the continuation is onto the second sheet. The residue of the pole is a product of Bethe-Salpeter wave functions like

$$\chi_k(r_1,\ldots,r_n)\equiv\langle 0|T[\psi(r_1)\ldots\psi(r_n)]|k,j,\sigma\rangle. \tag{6.41a}$$

The only difference between (6.41a) and the usual Bethe-Salpeter wave functions is that here k is complex and the state $|k,j,\sigma\rangle$ does not lie in the physical Hilbert space. The Lorentz transformation property of (6.41a) follows from (5.21). Denote the adjoint wave function by

$$\bar{\chi}_k(r_n,\ldots,r_1)=\langle\sigma,j,k|T[\bar{\psi}(r_n)\ldots\bar{\psi}(r_1)]|0\rangle. \tag{6.41b}$$

As always, the complex conjugate of (6.41a) involves antitime ordering and is not simply related to (6.41b). Note, of course, that either of (6.41) may also have momentum k^*.

The momentum-space wave functions are real Fourier transforms of (6.41) in spite of the fact that k is complex. As Appendix A illustrates, if the Green's function depends on $(x_1,\ldots,x_n)$ and $(x_1',\ldots,x_n')$, the pole occurs in the Fourier transform with respect to the average coordinate X given by

$$x_i=X+r_i,\quad \sum_1^n r_i=0. \tag{6.42a}$$

Therefore only the relative coordinates r_i appear in the residue factors like (6.41). If k_i are the momenta conjugate to x_i, then it is convenient to define

$$k_i=\frac{1}{n}k+l_i,\quad \sum_1^n l_i=0, \tag{6.42b}$$

The total momentum k is then conjugate to the average position X and the relative momenta l_i are conjugate to the relative coordinates r_i:

$$\sum_1^n k_i\cdot x_i=k\cdot X+\sum_1^n l_i\cdot r_i. \tag{6.43}$$

Even though the k_i are complex all the l_i may be taken as real because of (6.43). The momentum-space wave functions are then

$$\chi_k(l_1,\ldots,l_n)=\int(dr)\chi_k(r_1,\ldots,r_n)e^{i\Sigma l_i\cdot r_i} \tag{6.44a}$$

$$\bar{\chi}_k(l_1,\ldots,l_n)=\int(dr)\bar{\chi}_k(r_1,\ldots,r_n)e^{-i\Sigma l_i\cdot r_i}, \tag{6.44b}$$

where

$$\int (dr) \equiv \int d^4r_1 \cdots d^4r_n \delta^4\left(\frac{1}{n}\sum_1^n r_i\right).$$

It is important that all the relative momenta l_i are real for (6.44) to make sense.

The Bethe-Salpeter integral equation and normalization condition may be written either in coordinate space or momentum space. They are displayed graphically in Fig. 3. The derivation of these equations and the graphical notation is the same as in Ref. 26. The only difference is that now the normalization is between χ_k and $\bar{\chi}_{k^*}$. This is because

$$G(k',k) \underset{k^2 \to M^2}{\sim} \frac{\chi_{k'} i(2\pi)^4 \delta^4(k'-k)\bar{\chi}_{k^*}}{k^2 - \mathfrak{M}^2}. \qquad (6.45)$$

[See (6.5) for example.]

D. S-matrix elements

Scattering amplitudes involving unstable particles are residues of poles in the off-shell Green's functions in direct analogy with the stable case. The reduction formula expresses just this relation. Its practical value is that the Green's functions may be expanded in a perturbation series and the poles extracted from the external legs. This has been done for stable particles (elementary or composite) in Ref. 26. The graphical analysis leading to the pole extraction applies equally to unstable particles.

A simple example of the reduction formula for unstable particles is provided by e-μ elastic scattering. The scattering amplitude is

$$^{\text{out}}\langle k_2, p_2 | p_1, k_1\rangle^{\text{in}}$$
$$= \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 [U(k_2,y_2)\bar{U}(p_2,x_2) \times \mathfrak{R}\, U(p_1,x_1)U(k_1,y_1)], \qquad (6.46)$$

where

$$\mathfrak{R} \equiv (\overleftarrow{\Box}_{y_2} + \mathfrak{M}^{*2})(\overleftarrow{\Box}_{x_2} + m^2) \times \langle 0 | T[\psi(y_2)\psi(x_2)\bar{\psi}(x_1)\bar{\psi}(y_1)] | 0\rangle \times (\overrightarrow{\Box}_{x_1} + m^2)(\overrightarrow{\Box}_{y_1} + \mathfrak{M}^2).$$

Here x and p are the coordinates and momenta of the e, y and k are the coordinates and momenta of the μ. The external wave functions are spinors times plane waves:

$$U(p,x) = u(p)e^{ip\cdot x}. \qquad (6.47)$$

Continuing the right-hand side of (6.46) to complex k_i such that

$$(k_1)^2 = \mathfrak{M}^2, \quad (k_2)^2 = \mathfrak{M}^{*2}$$

gives the matrix element as the residue of the Green's-function poles. Note that energy-momentum conservation reads

$$p_2 + k_2^* = p_1 + k_1.$$

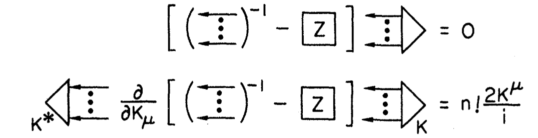

FIG. 3. Bethe-Salpeter equation and normalization condition for a composite unstable particle with complex momentum k^μ.

For stable particles the proof of the reduction formula depends on the existence of asymptotic states[27] like

$$|p\rangle^{\text{in}} = \lim_{t\to-\infty} \int d^3y\, \bar{\psi}(y) i \frac{\overleftrightarrow{\partial}}{\partial t} u(p)e^{ip\cdot y}|0\rangle. \qquad (6.48)$$

(This is, of course, only true in the weak-operator sense.) For unstable particles the energy is complex so that this limit either diverges or vanishes. (This reflects the physical reality.) The absence of an asymptotic state has an important consequence for the connectedness of the scattering amplitudes. Consider e-e scattering instead of e-μ. Then the complex k_i in (6.46) become real l_i. The connected amplitude is

$$^{\text{out}}\langle l_2, p_2 | p_1, l_1\rangle_C^{\text{in}} = i \int d^4y\, ^{\text{out}}\langle l_2, p_2 | \bar{\psi}(y) | p_1\rangle^{\text{in}} \times (\overrightarrow{\Box} + m^2) U(l_1, y). \qquad (6.49)$$

Strictly speaking, the external wave functions should really be square integrable solutions of the Klein-Gordon equation that equal (6.47) only in the plane-wave limit. Before this limit is taken, the spatial derivatives in (6.49) may be integrated by parts to get

$$^{\text{out}}\langle l_2, p_2 | p_1, l_1\rangle_C^{\text{in}} = i \int d^4y\, ^{\text{out}}\langle l_2, p_2 | \bar{\psi}(y) | p_1\rangle^{\text{in}} \times \left(\frac{\overrightarrow{\partial}^2}{\partial t^2} - \overleftarrow{\nabla}^2 + m^2\right) U(l_1, y). \qquad (6.50)$$

Because the wave function satisfies the Klein-Gordon equation this is

$$^{\text{out}}\langle l_2, p_2 | p_1, l_1\rangle_C^{\text{in}}$$
$$= -\int d^4y \frac{\partial}{\partial t}\left[^{\text{out}}\langle l_2, p_2 | \bar{\psi}(y) | p_1\rangle^{\text{in}}\, i \frac{\overleftrightarrow{\partial}}{\partial t} U(l_1, y)\right]$$
$$= \left(\lim_{t=-\infty} - \lim_{t=+\infty}\right) \int d^3y\, ^{\text{out}}\langle l_2, p_2 | \bar{\psi}(y) | p_1\rangle^{\text{in}} \times i \frac{\overleftrightarrow{\partial}}{\partial t} U(l_1, y). \qquad (6.51)$$

Because of the asymptotic condition[27] $\bar{\psi}(y)$ creates

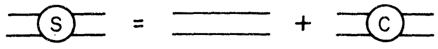

FIG. 4. The usual connectedness structure that results from the LSZ asymptotic condition.

an in state at $t=-\infty$ and creates an out state at $t=+\infty$:

$$^{\rm out}\langle l_2,p_2|p_1,l_1\rangle^{\rm in}_C = {}^{\rm out}\langle l_2,p_2|p_1,l_1\rangle^{\rm in} - \langle l_2|l_1\rangle\langle p_2|p_1\rangle. \qquad (6.52)$$

This shows that stable particles may propagate freely or interact as Fig. 4 illustrates.

It is easy to see that unstable particles do not give this connectedness structure. If $\vec{k}$ is complex, the integration by parts in (6.50) is not possible and the derivation fails. If, however, $\vec{k}$ is actually real then (6.50) is valid, but because k^0 is complex the limits in (6.51) either diverge or vanish. Thus (6.52) does hold for unstable particles. The connectedness structure for the S-matrix elements is then that given in Fig. 5, where the wavy line denotes the unstable particle.

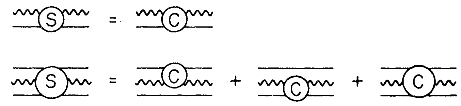

FIG. 5. The connectedness structure that results when one particle (denoted by the wavy line) is unstable.

For unstable particles the proof of the reduction formula must naturally be modified. The method of Appendix A is again the key. The Fourier transform of the three θ functions in (6.46) produce three resolvents. These may be analytically continued to the poles, whose residues are then the unstable state vectors. The right-hand side of (6.46) thus becomes

$$i\int d^4y\ {}^{\rm out}\langle k_2,p_2|\bar\psi(y)|p_1\rangle^{\rm in}(\overleftarrow{\Box}+\mathfrak{M}^2)U(k_1,y),$$

analogously to (6.49). But now there is no fourth θ function to produce a resolvent. Instead, crossing and translational invariance give

$$i\int d^4y\ {}^{\rm out}\langle k_2,p_2,\bar p_1|e^{-i\mathcal{P}\cdot y}\bar\psi(0)|0\rangle(\overleftarrow{\Box}^2+\mathfrak{M}^2)U(k_1,y) = i\ {}^{\rm out}\langle k_2,p_2,\bar p_1|(2\pi)^4\delta^4(k_1-\mathcal{P})\bar\psi(0)|0\rangle(-k_1{}^2+\mathfrak{M}^2)U(k_1). \qquad (6.53)$$

To go to $(k_1)^2=\mathfrak{M}^2$ requires analytically continuing the δ function. Suppose $\vec{k}_1$ is kept real and k_1^0 is taken to be complex. Then, as discussed in Sec. V C, Bremermann and Durand[24] have shown

$$2\pi i\delta(k^0-H)=\begin{cases}\dfrac{-1}{k^0-H}, & \mathrm{Im}(k^0-H)>0\\[2mm] \dfrac{1}{k^0-H}, & \mathrm{Im}(k^0-H)<0.\end{cases} \qquad (6.54)$$

The continuation appropriate to (6.53) starts with k_1^0 in the upper half plane. Thus (6.53) becomes

$$^{\rm out}\left\langle k_2,p_2,\bar p_1\left|(2\pi)^3\delta^3(\vec{k}_1-\vec{\mathcal{P}})\frac{1}{k_1^0-H}\bar\psi(0)\right|0\right\rangle(k_1{}^2-\mathfrak{M}^2)U(k_1) \qquad (6.55)$$

when $\mathrm{Im}k_1^0>0$. Now the resolvent can be continued clockwise onto the second sheet. Because of (2.9) it has a pole whose residue is a product of unstable state vectors. The continuation of (6.55) to the pole is then

$$^{\rm out}\langle k_2,p_2,\bar p_1|k_1\rangle\langle k_1^*|\bar\psi(0)|0\rangle\,U(k_1) = {}^{\rm out}\langle k_2,p_2,\bar p_1|k_1\rangle.$$

Using crossing again shows that this is

$$^{\rm out}\langle k_2,p_2|p_1,k_1\rangle^{\rm in}.$$

This demonstrates the reduction formula (6.46) as claimed. [Note that for stable particles the continuation is to a real energy below the production threshold and the same argument is rather weak because of the double contribution from (6.54). Already knowing the correct answer (6.52) shows that the $\pm i\epsilon$ actually corresponds to the creation of both an in state and an out state.] If the unstable particles are composite rather than elementary, essentially the same proof applies.

VII. A SOLUBLE EXAMPLE

To show how simple it is to describe unstable particles as eigenstates of the Hamiltonian, the familiar problem of elastic scattering from a three-dimensional square-well potential will now be examined. This problem is nonrelativistic, but otherwise contains all the features of unstable-particle states. The exact S matrix is given in (7.7). The wave functions that correspond to the second-sheet poles are displayed in (7.28). These are exponentially growing but nevertheless have zero norm (7.32) when properly calculated by analytic continuation. This method of calculating with unstable particle wave functions is put to an exact test in Sec. VII E.

A. The S matrix

The Schrödinger equation for a constant potential of depth $V>0$ and range R is

$$\left[-\frac{1}{2m}\nabla^2 - V\theta(R-r)\right]\Phi(\vec{r})=E\Phi(\vec{r}). \qquad (7.1)$$

The $l=0$ solution is

$$\Phi_E(r)=\begin{cases}\dfrac{C(E)}{r}\sin Kr, & r\leq R\\[2mm] -\dfrac{D(E)}{r}[e^{-ikr}-\eta(E)e^{ikr}], & r\geq R\end{cases} \tag{7.2}$$

where

$$k=(2mE)^{1/2},\quad K=[2m(E+V)]^{1/2}. \tag{7.3}$$

The three quantities C, D, and η are determined by the continuity of Φ and its derivative at $r=R$ and by the overall normalization.

The scattering amplitude is dètermined by the phase shifts:

$$\langle \vec{p}_2|S|\vec{p}_1\rangle=(2\pi)^3\frac{\delta(p_2-p_1)}{4\pi p_2 p_1}\times\sum_{l=0}^{\infty}(2l+1)e^{2i\delta_l}P_l(\cos\theta), \tag{7.4}$$

where plane waves are normalized as

$$\langle \vec{p}_2|\vec{p}_1\rangle=(2\pi)^3\delta^3(\vec{p}_2-\vec{p}_1). \tag{7.5}$$

The phase shifts may be calculated from the asymptotic behavior of the wave function because

$$\Phi(\vec{r})\underset{r\to\infty}{\sim}e^{i\vec{k}\cdot\vec{r}}+f(\theta)\frac{e^{ikr}}{r},$$

where

$$f(\theta)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta). \tag{7.6}$$

Comparing the asymptotic form of (7.2) with (7.6) gives

$$\eta(E)=e^{2i\delta_0(E)}.$$

Thus η determines the scattering amplitude. To find η explicitly, apply the boundary conditions to (7.2) to get

$$\eta_{\rm I}(E)=\frac{K\cot KR+ik}{K\cot KR-ik}e^{-2ikR}. \tag{7.7a}$$

This is the exact S matrix. It has a cut in E whose location depends on the definition chosen for the square root in (7.3). It is conventional to take this cut along the positive real axis. Thus

$$0<\arg\sqrt{E}\leq\pi,$$

i.e.,

$$\begin{aligned}&\operatorname{Im}k\geq 0\\ &\operatorname{Im}K\geq 0.\end{aligned} \tag{7.8}$$

The analytic continuation of $\eta_{\rm I}(E)$ onto the second sheet is then

$$\eta_{\rm II}(E)=\frac{K\cot KR-ik}{K\cot KR+ik}e^{2ikR}, \tag{7.7b}$$

where k and K are still the principal square roots given in (7.8).

The two-sheeted S matrix $\eta(E)$ explicitly satisfies extended unitarity and Hermitian analyticity. Physical unitarity is the statement

$$|\eta_{\rm I}(E)|^2=1\quad (E \text{ positive real}).$$

The extension of this to all values of E is

$$\eta_{\rm I}(E)\eta_{\rm II}(E)=1\quad (E\text{ complex}). \tag{7.9}$$

Hermitian analyticity follows from the principal square-root definition in (7.8) which satisfies

$$(\sqrt{E})^*=-\sqrt{E^*}.$$

Consequently

$$\begin{aligned}\eta_{\rm I}(E)^*&=\eta_{\rm I}(E^*),\\ \eta_{\rm II}(E)^*&=\eta_{\rm II}(E^*).\end{aligned} \tag{7.10}$$

It is well known that resonance bumps occur along the real axis at energies for which

$$\cot K_B R=0.$$

This means

$$K_B R=(n+\tfrac{1}{2})\pi$$

or equivalently

$$E_B=\frac{(n+\frac{1}{2})^2\pi^2}{2mR}-V. \tag{7.11a}$$

Precisely at this energy

$$\eta(E_B)=-e^{-2ik_BR}.$$

The width of the resonance is obtained by writing

$$\eta_{\rm I}(E)=\frac{f(E)-E}{f(E)+E}e^{-2ikR},$$

where

$$f(E)\equiv i\frac{kK}{2m}\cot KR. \tag{7.12}$$

When E is near E_B

$$f(E)=(E-E_B)f'(E_B)+O((E-E_B)^2).$$

Therefore

$$\eta(E)\approx\frac{E-E_B-\frac{1}{2}i\gamma_B}{E-E_B+\frac{1}{2}i\gamma_B}e^{-2ik_BR},$$

where

$$-\frac{i}{2}\gamma_B\equiv\frac{E_B}{f'(E_B)}.$$

Explicitly calculating this gives

$$\gamma_B=\frac{2[(n+\frac{1}{2})^2\pi^2-2mVR^2]^{1/2}}{mR^2}. \tag{7.11b}$$

The resonances will be distinct only if the width is much smaller than the energy spacing. This requires that

$$\frac{2\left[(n+\frac{1}{2})^2\pi^2 - 2mVR^2\right]^{1/2}}{(n+1)\pi^2} \ll 1. \qquad (7.13)$$

B. Location of the poles

The discussion of these resonance bumps is quite standard and is given in many textbooks. The emphasis of the present work, however, is not on these bumps but rather on second-sheet poles. In this example, as always, there is no guarantee that real axis bumps are a manifestation of second-sheet poles. This section will show that, in fact, there are second-sheet poles. These locations will turn out to agree with the energies and widths in (7.11). Let the poles occur at a complex energy $\tilde{E}$ on the second sheet. Put

$$\alpha = R\left[2m(\tilde{E}+V)\right]^{1/2}, \quad \beta = R(2m\tilde{E})^{1/2}, \qquad (7.14)$$

where

$$\mathrm{Im}\,\alpha > 0, \quad \mathrm{Im}\,\beta > 0.$$

Then the S matrix (7.7b) has a second-sheet pole only if

$$\alpha\cot\alpha = -i\beta, \quad \alpha^2 - \beta^2 = 2mVR^2. \qquad (7.15)$$

Now eliminate β to get

$$\frac{\alpha}{\sin\alpha} = \pm(2mVR^2)^{1/2}. \qquad (7.16)$$

The real part of the left-hand side must vanish and the imaginary part must equal the right-hand side. Let

$$\alpha = \alpha_1 + i\alpha_2, \quad \beta = \beta_1 + i\beta_2. \qquad (7.17)$$

Then (7.16) gives

$$\frac{\tanh\alpha_2}{\alpha_2} = \frac{\tan\alpha_1}{\alpha_1} \qquad (7.18a)$$

$$\alpha_2 = \pm\cos\alpha_1\left[\left(\frac{\alpha_1}{\sin\alpha_1}\right)^2 - 2mVR^2\right]^{1/2}. \qquad (7.18b)$$

The existence of solutions to these equations is proven graphically in Fig. 6. There the solutions of (7.18a) and of (7.18b) are both plotted. The intersections of the plots give the α_1 and α_2 of the corresponding unstable-particle pole. Figure 6 is plotted for a weak potential. As V is increased the widths become smaller and α_1 approaches $(n+\frac{1}{2})\pi$. If V is increased so much that $2mVR^2 > \alpha_1$, the unstable particles become stable. Then $\alpha_2 = 0$ and (7.18b) determines α_1. As $V \to \infty$, the stable-particle energies approach the usual value $\alpha_1 = n\pi$.

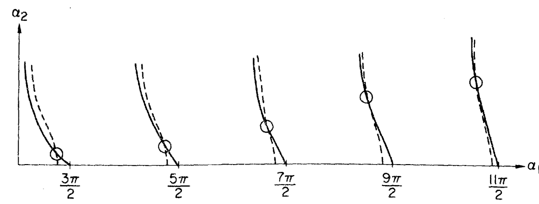

FIG. 6. The values of α_1 and α_2 that solve (7.18) and correspond to second-sheet poles in the S matrix (7.7). The dashed line is a plot of (7.18a); the solid line is a plot of (7.18b).

Once α_1 and α_2 are found, the corresponding values of β_1 and β_2 given by (7.15) are

$$\beta_1 = \alpha_2\tan\alpha_1\left(\frac{\csc^2\alpha_1 - \mathrm{csch}^2\alpha_2}{\cot^2\alpha_1 + \coth^2\alpha_2}\right), \quad \beta_2 = \alpha_1\cot\alpha_1\left(\frac{\csc^2\alpha_1 + \mathrm{csch}^2\alpha_2}{\cot^2\alpha_1 + \coth^2\alpha_2}\right). \qquad (7.19)$$

Figure 6 shows that the only allowed values of positive α_1 lie in the strips

$$n\pi < \alpha_1 < (n+\tfrac{1}{2})\pi.$$

Because of (7.19) this means that all the poles have $\beta_2 > 0$ and hence all lie on the second sheet as claimed. [Of course, α_2 must also be positive but its sign is lost in going from (7.15) to (7.16).]

Let the real and imaginary parts of the energy at the pole be given by

$$\tilde{E} = E_P + \frac{i}{2}\gamma_P. \qquad (7.20)$$

Then

$$E_P = \frac{\alpha_1^2 - \alpha_2^2}{2mR^2} - V = \frac{\beta_1^2 - \beta_2^2}{2mR^2}, \qquad (7.21)$$

$$\gamma_P = \frac{2\alpha_1\alpha_2}{mR^2} = \frac{2\beta_1\beta_2}{mR^2} \qquad (7.22)$$

because of the definitions (7.14). Equations (7.18) have the property that if $\alpha_1 + i\alpha_2$ is a solution, then so is $-\alpha_1 + i\alpha_2$. Therefore from (7.19) both $\beta_1 + i\beta_2$ and $-\beta_1 + i\beta_2$ are solutions. Thus in (7.22) there are automatically poles at $E_P + \frac{1}{2}i\gamma_P$ and $E_P - \frac{1}{2}\gamma_P$ as claimed.

The existence of the second-sheet poles is thus proved but it would be nice to determine their locations more accurately. Fortunately, approximate solutions to the coupled transcendental equations (7.18) may be obtained for α_1 large and α_2

small. In this limit Fig. 6 shows that α_1 is a little less than $(n+\frac{1}{2})\pi$ for n a large positive integer. Therefore let

$$\alpha_1 = (n+\tfrac{1}{2})\pi - \frac{\lambda}{(n+\frac{1}{2})\pi} \tag{7.23}$$

and determine λ. Equation (7.18b) gives

$$\alpha_2 = \frac{\lambda[b^2+\lambda(\lambda-2)]^{1/2}}{a}\left[1+O\left(\frac{\lambda^2}{a^2}\right)\right], \tag{7.24}$$

where

$$a \equiv (n+\tfrac{1}{2})\pi,$$

$$b \equiv [(n+\tfrac{1}{2})^2\pi^2 - 2mVR^2]^{1/2}.$$

For α_2 to be small it is necessary that

$$b \ll a .$$

This is in accordance with (7.13). Substituting (7.23) and (7.24) into (7.18a) determines λ via the equation

$$\frac{1}{\lambda}+\frac{1}{a^2}\left(1-\frac{\lambda}{3}\right) = 1 - \frac{1}{3}\frac{\lambda^2[b^2+\lambda(\lambda-2)]}{a^2} + O\left(\frac{1}{a^4}\right).$$

Because a is large, λ must be near 1. In fact

$$\lambda = 1 + \frac{b^2+1}{3a^2}.$$

The solution is then

$$\alpha_1 = a - \frac{1}{a} - \frac{b^2+1}{3a^3} + \cdots,$$

$$\alpha_2 = \frac{(b^2-1)^{1/2}}{a}\left(1+\frac{b^2+1}{3a^2}+\cdots\right). \tag{7.25a}$$

The corresponding values of β are

$$\beta_1 = (b^2-1)^{1/2}\left(1-\frac{1}{2a^2}+\cdots\right),$$

$$\beta_2 = 1 + \frac{2b^2-1}{6a^2} + \cdots . \tag{7.25b}$$

These solutions give for the real energy (7.21)

$$E_P = \frac{1}{2mR^2}\left(b^2 - 2 + \frac{4-5b^2}{3a^2}+\cdots\right).$$

This is in good agreement with the energy at which the bump on the real axis occurs, which according to (7.11a) is

$$E_B = \frac{b^2}{2mR^2}.$$

The width which follows from (7.22) is

$$\gamma_P = \frac{2(b^2-1)^{1/2}}{mR^2}\left(1+\frac{b^2-2}{3a^2}+\cdots\right), \tag{7.26a}$$

whereas the width of the real axis bump is

$$\gamma_B = \frac{2b}{mR^2} \tag{7.26b}$$

according to (7.11b).

C. Schrödinger wave functions

The wave functions for an unstable state is the analytic continuation in E of $\Phi(E,r)$ to the point $\tilde{E}$ on the second sheet. At this value of energy, $\eta = \infty$ and $D = 0$ but the product $D\eta$ is finite. To calculate norms and inner products with such states it is useful to introduce the wave function

$$\psi(E,r) = \begin{cases} \dfrac{C(E)}{r}\sin Kr, & r \leq R \\[2mm] \dfrac{D(E)\eta(E)}{r}e^{ikr}, & r \geq R \end{cases} \tag{7.27}$$

where C, D, and η are the same functions of E as in (7.2). Note that whereas Φ is an energy eigenstate for any energy E because it satisfies the boundary conditions for any E, ψ certainly is not. However, ψ is an energy eigenstate at energy $\tilde{E}$ on the second sheet because it coincides with Φ there:

$$\psi_{\text{II}}(\tilde{E},r) = \Phi_{\text{II}}(\tilde{E},r).$$

On the second sheet ψ is given by

$$\psi_{\text{II}}(E,r) = \begin{cases} \dfrac{-C_{\text{II}}(E)}{r}\sin Kr, & r \leq R \\[2mm] \dfrac{D_{\text{II}}(E)\eta_{\text{II}}(E)}{r}e^{-ikr}, & r \geq R \end{cases} \tag{7.28}$$

where

$$\text{Im}\,k \geq 0, \quad \text{Im}\,K \geq 0$$

as always. The continuity of ψ_{II} and its derivative at $\tilde{E}$ imply

$$-C\sin\alpha = (D\eta)e^{-i\beta},$$

$$C\cos\alpha = i\frac{\beta}{\alpha}(D\eta)e^{-i\beta}, \tag{7.29}$$

where α and β are given by (7.14) and C, D, η are the values at $\tilde{E}$ on the second sheet. [Note that the boundary conditions (7.29) contain the pole constraints (7.15) on α and β.]

The norm of ψ_{II} obviously cannot be obtained by integrating the absolute square of the wave function (7.28) because it diverges exponentially in r. This divergence in r results from continuing to the second sheet; ψ decreases exponentially on the first sheet, as (7.27) shows. The norm of ψ_{II} is obtained by calculating the norm on the first sheet and then analytically continuing the answer to the second sheet (see Sec. III). The first-sheet norm is simply

$$\langle\psi(E)|\psi(E)\rangle = \int d^3r\,|\psi(E,r)|^2$$
$$= 2\pi\,|C(E)|^2\left[\frac{\sin(K^*-K)R}{K^*-K} - \frac{\sin(K^*+K)R}{K^*+K}\right] + 4\pi\,|D(E)\eta(E)|^2\,\frac{e^{i(k-k^*)R}}{i(k-k^*)}\,. \tag{7.30}$$

Analytically continuing (7.30) to the second sheet gives

$$\langle\psi_{\rm II}(\tilde E)|\psi_{\rm II}(\tilde E)\rangle = 2\pi R\,|C|^2\left(\frac{\sinh 2\alpha_2}{2\alpha_2} - \frac{\sin 2\alpha_1}{2\alpha_1}\right) - 2\pi R\,|D\eta|^2\,\frac{e^{2\beta_2}}{\beta_2}\,. \tag{7.31}$$

Because of the two boundary conditions (7.29) this whole quantity vanishes. To see this, multiply the first boundary condition by the complex conjugate of the second and vice versa to get

$$|C|^2\sin\alpha\cos\alpha^* = i\frac{\beta^*}{\alpha^*}|D\eta|^2e^{2\beta_2}\,,$$
$$|C|^2\sin\alpha^*\cos\alpha = -i\frac{\beta}{\alpha}|D\eta|^2e^{2\beta_2}\,.$$

Add and subtract these,

$$|C|^2\sin 2\alpha_1 = i\left(\frac{\beta^*}{\alpha^*} - \frac{\beta}{\alpha}\right)|D\eta|^2e^{2\beta_2}\,,$$
$$|C|^2\sinh 2\alpha_2 = \left(\frac{\beta^*}{\alpha^*} + \frac{\beta}{\alpha}\right)|D\eta|^2e^{2\beta_2}\,.$$

Substitute this into (7.31) to get

$$\langle\psi_{\rm II}(\tilde E)|\psi_{\rm II}(\tilde E)\rangle = 2\pi R|D\eta|^2e^{2\beta_2}\left[\frac{1}{2\alpha_2}\left(\frac{\beta^*}{\alpha^*}+\frac{\beta}{\alpha}\right) - \frac{i}{2\alpha_1}\left(\frac{\beta^*}{\alpha^*}-\frac{\beta}{\alpha}\right) - \frac{1}{\beta_2}\right] = 2\pi R\,|D\eta|^2e^{2\beta_2}\left(\frac{\beta_1}{\alpha_1\alpha_2} - \frac{1}{\beta_2}\right).$$

Now from the definition of α and β

$$\alpha^2 - \beta^2 = 2mVR\,.$$

Separating real and imaginary parts gives

$$\alpha_1\alpha_2 - \beta_1\beta_2 = 0\,.$$

Thus

$$\langle\psi_{\rm II}(\tilde E)|\psi_{\rm II}(\tilde E)\rangle = 0\,. \tag{7.32}$$

Note that the vanishing of this norm depends essentially on the reality of V, i.e., on the Hermiticity of the Hamiltonian.

As indicated in Sec. III the effective norm of an unstable state is the inner product with the state of energy $\tilde E^*$. This will be explicitly calculated now and used in several later subsections. It was shown subsequent to (7.22) that such poles do exist. Because of time-reversal invariance

$$\Phi_{E^*}(\vec r) = \Phi_E(\vec r)^*\,.$$

Therefore (7.27) satisfies

$$\psi(E^*,r) = \psi(E,r)^*\,. \tag{7.33}$$

This means that evaluating

$$\langle\psi_{\rm II}(\tilde E^*)|\psi_{\rm II}(\tilde E)\rangle$$

first requires calculating

$$\langle\psi(E^*)|\psi(E)\rangle = \int d^3x\,\psi(E^*,r)^*\psi(E,r) = \int d^3x\,[\psi(E,r)]^2$$

and then continuing the result to E. Performing the integration gives

$$\langle\psi(E^*)|\psi(E)\rangle = 2\pi\,[C(E)]^2\left(R - \frac{\sin 2KR}{2K}\right) - 4\pi\,[D(E)\eta(E)]^2\,\frac{e^{2ikR}}{2ik}\,.$$

The continuation of this to the second sheet is

$$\langle\psi_{\rm II}(\tilde E^*)|\psi_{\rm II}(\tilde E)\rangle = 2\pi RC^2\left[1 - \frac{\sin 2\alpha}{2\alpha}\right] + 2\pi R\,(D\eta)^2\,\frac{e^{-2i\beta}}{i\beta}\,. \tag{7.34}$$

The boundary conditions (7.29) may be manipulated into another useful form by multiplying them together and by adding the squares:

$$-C^2\sin\alpha\cos\alpha = i\,\frac{\beta}{\alpha}(D\eta)^2e^{-2i\beta}\,,$$
$$C^2 = \left(1 - \frac{\beta^2}{\alpha^2}\right)(D\eta)^2e^{-2i\beta}\,.$$

Using these in (7.34) gives

$$\langle\psi_{\rm II}(\tilde E^*)|\psi_{\rm II}(\tilde E)\rangle = 2\pi R\,(D\eta)^2 \times e^{-2i\beta}\left(1 - \frac{\beta^2}{\alpha^2}\right)\left(1 + \frac{1}{i\beta}\right).$$

For subsequent purposes it is useful to define normalized states

$$|\Psi_{\text{II}}(\tilde{E})\rangle = |\psi_{\text{II}}(\tilde{E})\rangle \frac{1}{\sqrt{N}},$$
$$|\Psi_{\text{II}}(\tilde{E}^*)\rangle = |\psi_{\text{II}}(\tilde{E}^*)\rangle \frac{1}{\sqrt{N^*}}, \tag{7.35}$$

where

$$N = 2\pi R (D\eta)^2 e^{-2i\beta}\left(1 - \frac{\beta^2}{\alpha^2}\right)\left(1 + \frac{1}{i\beta}\right)$$
$$= 2\pi R C^2 \left(1 + \frac{1}{i\beta}\right). \tag{7.36}$$

D. Calculation of γ_F

One simple application of these curious wave functions is to calculate the decay width γ_F given by Fermi's golden rule and see how it compares with γ_B and γ_P. This requires evaluating the matrix element

$$\langle \vec{k}^* | V | \Psi_{\text{II}}(\tilde{E})\rangle, \tag{7.37}$$

where

$$\vec{k}^2 = 2m\tilde{E}.$$

To calculate the decay amplitude use the first-sheet wave functions (7.27) to get

$$\langle \vec{k}^* | V | \psi(E)\rangle = -\int d^3x\, e^{-i\vec{k}\cdot\vec{x}} V\theta(R-r)\frac{C(E)}{r}\sin Kr$$
$$= -C(E)\frac{2\pi V}{k}\left[\frac{\sin(K-k)R}{K-k} - \frac{\sin(K+k)R}{K+k}\right].$$

Continuing this to the second sheet and using the definitions of α and β gives

$$\langle \vec{k}^* | V | \psi_{\text{II}}(\tilde{E})\rangle = C\frac{2\pi VR^2}{\beta}\left[\frac{\sin(\alpha-\beta)}{\alpha-\beta} - \frac{\sin(\alpha+\beta)}{\alpha+\beta}\right].$$

Using (7.15) simplifies this to

$$\langle \vec{k}^* | V | \psi_{\text{II}}(\tilde{E})\rangle = \frac{2\pi C}{m}\left[\sin\alpha\cos\beta - \frac{\alpha}{\beta}\cos\alpha\sin\beta\right].$$

The boundary condition (7.29) can be used to eliminate α and get

$$\langle \vec{k}^* | V | \psi_{\text{II}}(\tilde{E})\rangle = \frac{-2\pi}{m} D\eta.$$

The normalized amplitude (7.37) is obtained by dividing this by N in (7.36):

$$\langle \vec{k}^* | V | \Psi_{\text{II}}(\tilde{E})\rangle = -\frac{2\pi}{m}\frac{e^{i\beta}}{\left[2\pi R\left(1-\frac{\beta^2}{\alpha^2}\right)\left(1+\frac{1}{i\beta}\right)\right]^{1/2}}. \tag{7.38}$$

Denote the continuation of this amplitude to real energy by T,

$$T = \lim_{\tilde{E}\to\text{real}}\left[\langle \vec{k}^* | V | \Psi_{\text{II}}(\tilde{E})\rangle\right],$$
$$= -\frac{2\pi}{m}\frac{e^{i\beta_1}}{\left[2\pi R\left(1-\frac{\beta_1}{\alpha_1}\right)^2\left(1+\frac{1}{i\beta_1}\right)\right]^{1/2}}. \tag{7.39}$$

The real part of $\tilde{E}$ is defined as E_P in (7.20). The decay rate is then

$$\gamma_F = \int \frac{d^3k}{(2\pi)^3}\, 2\pi\delta\left(\frac{k^2}{2m} - E_P\right)|T|^2.$$

The phase-space integration gives

$$\gamma_F = \frac{m(2mE_P)^{1/2}}{\pi}|T|^2$$
$$= \frac{m\beta_1}{\pi R}|T|^2.$$

By (7.39) this is

$$\gamma_F = \left(\frac{2\beta_1}{mR^2}\right)\frac{1}{\left[1-\left(\frac{\beta_1}{\alpha_1}\right)^2\right]\left[1+\left(\frac{1}{\beta_1}\right)^2\right]^{1/2}}.$$

Using the explicit values of α_1 and β_1 in (7.25) gives

$$\gamma_F = \frac{2(b^2-1)^{1/2}}{mR^2}\left[1 + \frac{b^2-1}{a^2} + \frac{1}{2(b^2-1)} + \cdots\right]. \tag{7.40}$$

This value of the width is to be compared with (7.26) for γ_B and γ_P.

E. An exact test

The idea of introducing a wave function for an unstable particle may be checked exactly. The general expression for the S matrix

$$\langle \vec{k}_2 | S | \vec{k}_1\rangle = \langle \vec{k}_2 | \vec{k}_1\rangle$$
$$- 2\pi i\delta(E_2 - E_1)\left\langle \vec{k}_2\right|\left(V + V\frac{1}{E-H}V\right)\left|\vec{k}_1\right\rangle \tag{7.41}$$

must have a pole on the second sheet at $\tilde{E}$. According to Sec. II this pole is produced by the resolvent in (7.41) and has a residue determined by the wave function of the unstable state. The scattering amplitude at the pole is therefore predicted to be

$$\langle \vec{k}_2 | S | \vec{k}_1\rangle \underset{E\to\tilde{E}}{\sim} -2\pi i\delta(E_2 - E_1)\frac{C}{E-\tilde{E}},$$

where $E = E_1 = E_2$ and the residue is

$$C \equiv \langle \vec{k}^* | V | \Psi_{\text{II}}(\tilde{E})\rangle\langle \Psi_{\text{II}}(\tilde{E}^*) | V | \vec{k}\rangle.$$

Because of the time-reversal property (7.33)

$$\langle \Psi_{\text{II}}(\tilde{E}^*) | V | \vec{k}\rangle = \langle \vec{k}^* | V | \Psi_{\text{II}}(\tilde{E})\rangle.$$

This is the matrix element already calculated in (7.38). Hence the residue C is predicted to be

$$C=\frac{2\pi}{m^2R}\ \frac{e^{2i\beta}}{(1-\beta^2/\alpha^2)(1+1/i\beta)}\ . \tag{7.42}$$

The real test of this calculation is the knowledge of the exact S matrix for $l=0$. If the calculations that have been done with the exponentially diverging wave functions for unstable states are valid, then (7.42) must be an exact result. According to (7.4) for s-wave scattering

$$\begin{aligned}\langle\vec{p}_2|S|\vec{p}_1\rangle &= (2\pi)^3\frac{\delta(p_2-p_1)}{4\pi p_2 p_1}\,\eta(E)\\ &= -2\pi i\delta(E_2-E_1)\left[\frac{i\pi}{mp}\,\eta(E)\right].\end{aligned} \tag{7.43}$$

On the second sheet

$$\eta_{\rm II}(E)=\frac{f(E)+E}{f(E)-E}\,e^{2ikR}\ ,$$

where $f(E)$ is given by (7.12). At the pole

$$\eta_{\rm II}(E)\underset{E\to\tilde{E}}{\sim}\frac{2\tilde{E}}{(E-\tilde{E})[f'(\tilde{E})-1]}\,e^{2i\beta}\ .$$

This gives for (7.43)

$$\langle\vec{p}_2|S|\vec{p}_1\rangle\underset{E\to\tilde{E}}{\sim}-2\pi i\delta(E_2-E_1)\frac{C'}{E-\tilde{E}}\ ,$$

with

$$C'\equiv\frac{-i\pi}{m\tilde{k}}\ \frac{2\tilde{E}}{f'(\tilde{E})-1}\,e^{2i\beta}. \tag{7.44}$$

Calculating the required derivative gives

$$f'(\tilde{E})-1=\frac{-i\beta}{2}\left(1-\frac{\beta^2}{\alpha^2}\right)\left(1+\frac{1}{i\beta}\right).$$

Putting this into (7.44) gives

$$C'=\frac{2\pi}{m^2R}\ \frac{e^{2i\beta}}{(1-\beta^2/\alpha^2)(1+1/i\beta)}\ . \tag{7.45}$$

Hence $C'=C$ and the wave-function calculation is exact.

APPENDIX A: EXISTENCE OF $|\phi(\vec{k})j,j_3\rangle$

The resonance pole will occur in many scattering amplitudes. To demonstrate the existence of the state $|\phi\rangle$ used in Sec. II consider, in particular, an elastic scattering amplitude in which the pole occurs in total center-of-mass energy, i.e., in s. This scattering amplitude is obtained from the Fourier transform of a Green's function with an equal number, n, of incoming and outgoing lines. Denote this Green's function by

$$\mathcal{G}(x',x)=\langle 0|T[\mathcal{Q}^\dagger(x')\mathcal{Q}(x)]|0\rangle, \tag{A1}$$

where

$$\begin{aligned}x&=(x_1,x_2,\ldots,x_n),\\ \mathcal{Q}(x)&=T[\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\cdots\psi_{\alpha_n}(x_n)],\end{aligned} \tag{A2}$$

and the $\psi(x)$ are renormalized Heisenberg fields. Let k^μ be the total energy-momentum four vector of the corresponding elastic scattering amplitude (e.g., $s=k^2$). Introduce average and relative coordinates by

$$x_i=X+r_i,\quad \sum_1^n r_i=0.$$

Then X and k are conjugate variables.

By hypothesis, analytic continuation of the on-shell Green's function in the variable k^0 leads to a pole on the second sheet. To construct the state $|\phi\rangle$ it is necessary that the same pole occur in the off-shell function

$$G(k',k)\equiv\int d^4X'd^4X\,e^{ik'\cdot X'}\mathcal{G}(x',x)e^{-ik\cdot X} \tag{A3}$$

independently of the $2n-2$ relative coordinates r and r'. This was proved quite rigorously for the four-point function by Bros,[28] who examined the inhomogeneous Bethe-Salpeter equation for the four-point function. From Fredholm theory the solution is the quotient of two functions: a numerator that depends on the relative coordinates and a denominator that does not. Bros proved that both the numerator and denominator are holomorphic on the sheet reached by continuing through the two-particle cut. Therefore poles in the Green's function can only come from zeros of the denominator. Because of the simplicity of the Fredholm denominator, the location of such a pole in the off-shell function is automatically independent of the relative coordinates and is the same as the pole in the scattering amplitude. It will be assumed here that the same holds true for the general case (A3) as well.

Among the $(2n)!$ time orderings in (A1), isolate the $(n!)^2$ terms for which $(x_i')^0>(x_j)^0$, for all i and j, by separating

$$\begin{aligned}\mathcal{G}(x',x)=&\langle 0|\mathcal{Q}^\dagger(x')\,\theta(t'_{\rm min}-t_{\rm max})\mathcal{Q}(x)|0\rangle\\ &+\Delta\mathcal{G}(x',x).\end{aligned}$$

When there are only two fields in $\mathcal{Q}$,

$$\begin{aligned}t_{\rm max}&=\tfrac{1}{2}(t_1+t_2)+\tfrac{1}{2}|t_1-t_2|\,,\\ t_{\rm min}&=\tfrac{1}{2}(t_1+t_2)-\tfrac{1}{2}|t_1-t_2|\,.\end{aligned}$$

Generally for n fields

$$\begin{aligned}t_{\rm max}&=X^0+\alpha(r_1^0,r_2^0,\ldots,r_n^0),\\ t_{\rm min}&=X^0+\beta(r_1^0,r_2^0,\ldots,r_n^0).\end{aligned}$$

This separation and the fact that

$$\mathcal{A}(X,r)=e^{iX^0H}\mathcal{A}(\vec{X},r)e^{-iX^0H} \tag{A4}$$

allows the Fourier transform (A3) to be performed. The result is

$$G(k',k) = \left\langle\phi(\vec{k}')\left|e^{-i\beta(k'^0-H)}\frac{2\pi i\delta(k'^0-k^0)}{k^0-H}e^{i\alpha(k^0-H)}\right|\phi(\vec{k})\right\rangle + \Delta G(k',k), \tag{A5}$$

where

$$\begin{aligned}&\mathrm{Im}k^0>0,\\&\alpha=\alpha(r_i^0),\\&\beta=\beta(r_i'^0),\\&|\phi(\vec{k})\rangle\equiv\int d^3X\,e^{i\vec{k}\cdot\vec{X}}\mathcal{A}(\vec{X},r)|0\rangle.\end{aligned} \tag{A6}$$

Because the resonance pole in the scattering amplitude is on the second sheet, it can only come from terms in (A5) that have a branch point in k^0. The displayed term has branch points along the positive real k^0 axis coming from thresholds in H. Of the terms in ΔG, $(n!)^2$ of them have cuts only along the negative real k^0 axis coming from $(k^0+H)^{-1}$. None of the other terms in ΔG have branch points at all because they come from finite ranges of integration of the variable X'^0-X^0. Thus the unstable particle pole can only come from the analytic continuation of the first term of (A5). Since this term has a pole then so does the analytic continuation of

$$\left\langle\phi(\vec{k}')\left|\frac{1}{k^0-H}\right|\phi(\vec{k})\right\rangle. \tag{A7}$$

Because the spatial dependence of the Heisenberg fields is

$$\mathcal{A}(\vec{X},r)=e^{-i\vec{X}\cdot\vec{\mathcal{P}}}\mathcal{A}(0,r)e^{i\vec{X}\cdot\vec{\mathcal{P}}},$$

where $\vec{\mathcal{P}}$ is the full momentum operator, it follows that $|\phi\rangle$ in (A6) is an eigenstate of momentum,

$$\vec{\mathcal{P}}|\phi(\vec{k})\rangle=\vec{k}|\phi(\vec{k})\rangle. \tag{A8}$$

As yet $|\phi\rangle$ has n discrete field indices $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ that are suppressed in (A6). These indices may be coupled in pairs with ordinary Clebsch-Gordan coefficients leading to an n-fold Clebsch-Gordan coefficient that satisfies

$$\sum_{\alpha'}\mathfrak{D}^{l_1}_{\alpha_1,\alpha_1'}(R)\cdots\mathfrak{D}^{l_n}_{\alpha_n,\alpha_n'}(R)\begin{pmatrix}l_1\cdots l_n;j\\\alpha_1'\cdots\alpha_n';j_3\end{pmatrix} = \sum_{j_3'}\begin{pmatrix}l_1\cdots l_n;j\\\alpha_1\cdots\alpha_n;j_3'\end{pmatrix}D^j_{j_3',j_3}(R),$$

where $\mathfrak{D}^l$ are the rotations appropriate to the field representation [usually $(l,0)\oplus(0,l)$]. The final form for $|\phi\rangle$ is then

$$|\phi(\vec{k})j,j_3\rangle=\sum_\alpha\int(d^4x)^n\delta(X^0)e^{i\vec{k}\cdot X}\left[T[\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)]|0\rangle\begin{pmatrix}l_1\cdots l_n;j\\\alpha_1\cdots\alpha_n;j_3\end{pmatrix}f(r_1,\ldots,r_n)\right] \tag{A9}$$

Here f is a convenient smearing function of Lorentz invariants r^2 and $r_i\cdot r_j$. If $|\phi\rangle$ in (A6) gives a pole in (A7), then at least one of the (j,j_3) combinations in (A9) must also because of the completeness of the Clebsch-Gordan coefficients. Under rotations (A9) transforms as

$$U[R]|\phi(\vec{k})j,j_3\rangle=\sum_{j_3'}|\phi(R\vec{k})j,j_3'\rangle\mathfrak{D}_{j_3',j_3}(R).$$

Also,

$$\vec{J}^2|\phi(\vec{k})j,j_3\rangle=j(j+1)|\phi(\vec{k})j,j_3\rangle,$$

$$J_3|\phi(\vec{k})j,j_3\rangle=j_3|\phi(\vec{k})j,j_3\rangle.$$

The function f may be chosen to guarantee the normalization

$$\langle j_3',j'\phi(\vec{k}')|\phi(\vec{k})j,j_3\rangle=\delta^3(\vec{k}'-\vec{k})\delta_{j',j}\delta_{j_3',j_3}.$$

APPENDIX B: RESOLVENT IDENTITIES

The identities (2.8) and (2.12) follow very easily from the definitions

$$R(k^0)=\frac{1}{k^0-H},$$

$$r(k^0)=\frac{1}{k^0-H'}.$$

The full resolvent satisfies the three integral equations

$$R(k^0)=r(k^0)+R(k^0)H''r(k^0),$$

$$R(k^0)=r(k^0)+r(k^0)H''R(k^0),$$

$$R(k^0)=r(k^0)+r(k^0)H''r(k^0)+r(k^0)H''R(k^0)H''r(k^0).$$

Use the fact that $r(k^0)$ commutes with A and B to calculate the projections

$$AR(k^0)B=[AR(k^0)A]H''r(k^0),$$

$$BR(k^0)A=r(k^0)H''[AR(k^0)A],$$

$$BR(k^0)B=Br(k^0)B+r(k^0)H''[AR(k^0)A]H''r(k^0).$$

Sum these three projections together with $AR(k^0)A$ itself to get

$$R(k^0) = Br(k^0)B + [1 + r(k^0)H'']AR(k^0)A[H''r(k^0) + 1] \tag{B1}$$

as claimed in (2.8)

Now to simplify $AR(k^0)A$ use the integral equation

$$R(k^0) = r(k^0) + r(k^0)H''r(k^0) + r(k^0)H''r(k^0)H''R(k^0).$$

The diagonal projection of this is

$$AR(k^0)A = \frac{A}{k^0 - E_\phi} + \frac{1}{k^0 - E_\phi} H''r(k^0)H''[AR(k^0)A], \tag{B2}$$

where E_ϕ is the expectation value

$$\langle\phi(\vec{k}_1)|H'|\phi(\vec{k}_2)\rangle = E_\phi \delta^3(\vec{k}_1 - \vec{k}_2).$$

Rewrite (B2) as

$$[k^0 - E_\phi - H''r(k^0)H'']AR(k^0)A = A.$$

The expectation value of this is

$$\int d^3k_2 \langle\phi(\vec{k}_1)|k^0 - H' - H''r(k^0)H''|\phi(\vec{k}_2)\rangle \times \left\langle\phi(\vec{k}_2)\left|\frac{1}{k^0 - H}\right|\phi(\vec{k}_3)\right\rangle = \delta^3(\vec{k}_1 - \vec{k}_3). \tag{B3}$$

The definition of $D(k)$ in (2.7) is

$$\left\langle\phi(\vec{k}_2)\left|\frac{1}{k^0 - H}\right|\phi(\vec{k}_3)\right\rangle \equiv \frac{\delta^3(\vec{k}_2 - \vec{k}_3)}{D(k_2)}.$$

Therefore (B3) shows that an equivalent formula for $D(k)$ is

$$\langle\phi(\vec{k}_1)|k^0 - H' - H''r(k^0)H''|\phi(\vec{k}_2)\rangle = \delta^3(\vec{k}_1 - \vec{k}_2)D(k),$$

as claimed in (2.12).

APPENDIX C: THE TOPOLOGY OF $\mathfrak{h}\uparrow$

Because the unstable particles have zero norm a new, positive-definite-norm operator must be found to give meaning to (3.15). This norm should be positive-definite both in the unstable-particle sector of $\mathfrak{h}\uparrow$ as well as in the physical Hilbert space $\mathfrak{h} \subset \mathfrak{h}\uparrow$. The difficulty is that certain states like $|\phi\rangle$ are in the Hilbert space $\mathfrak{h}$ and yet are not orthogonal to the unstable states. In particular,

$$\langle\phi(\vec{k}')|\psi_n(\vec{k})\rangle = \delta^3(\vec{k}' - \vec{k}). \tag{C1}$$

Essentially the problem is that the three states $|\psi_n\rangle$, $|\psi_{\bar{n}}\rangle$, and $|\phi\rangle$ are linearly independent but not orthogonal. To construct an orthogonal basis is straightforward. First let

$$I_{1u} = \int d^3k[\,|\Psi(k)\rangle\langle\Psi(k^*)| + |\Psi(k^*)\rangle\langle\Psi(k)|\,]. \tag{C2}$$

This is the identity operator in the one unstable-particle sector:

$$I_{1u}|\Psi(k)\rangle = |\Psi(k)\rangle, \tag{C3a}$$

$$I_{1u}|\Psi(k^*)\rangle = |\Psi(k^*)\rangle. \tag{C3b}$$

The projection of $|\phi\rangle$ along the unstable sector is

$$I_{1u}|\phi(\vec{k})\rangle = |\Psi(k)\rangle\frac{1}{\sqrt{N}} + |\Psi(k^*)\rangle\frac{1}{\sqrt{N^*}},$$

with N taken from (3.12). The part of $|\phi\rangle$ orthogonal to the unstable-particle sector is just

$$I_{1s}|\phi(\vec{k})\rangle = |\phi(\vec{k})\rangle - |\Psi(k)\rangle\frac{1}{\sqrt{N}} - |\Psi(k^*)\rangle\frac{1}{\sqrt{N^*}}. \tag{C3c}$$

An orthogonal basis is therefore provided by the three states (C3a), (C3b), and (C3c).

The identity operator (C2) for the unstable-particle sector may be extended in a Hermitian manner to include many unstable particles or mixtures of stable particles and at least one unstable particle. With the extension I_u so defined let

$$I_s|F\rangle \equiv \begin{cases} |F\rangle - I_u|F\rangle & \text{if } |F\rangle \in \mathfrak{h}, \\ 0 & \text{if } |F\rangle \in \mathfrak{h}. \end{cases} \tag{C4}$$

Thus I_s projects out what may be called the very stable part of the Hilbert space $\mathfrak{h}$. The identity operator in the full space $\mathfrak{h}\uparrow$ is then

$$I = I_s + I_u, \tag{C5}$$

where $I_sI_u = I_uI_s = 0$. The important point is that the first term of (C5) is I_s and not the usual identity in $\mathfrak{h}$. Note, too, that (C5) assumes that the eigenstates of H span $\mathfrak{h}\uparrow$.

Before constructing a positive-definite-norm operator [see (C13)], it is worthwhile to investigate in more detail why the inner product (C1) does not vanish. To do this, express $|\phi\rangle$ as a superposition of energy eigenstates. By assumption the asymptotic scattering states, either in or out, span $\mathfrak{h}$. Thus put

$$|\phi(\vec{k})\rangle = \int dE\,|E,\vec{k}\rangle^{\text{in}}\varphi(E). \tag{C6}$$

The state $|E,\vec{k}\rangle^{\text{in}}$ is a direct product of asymptotic stable-particle states $(e, P, \gamma, \nu_e, \nu_\mu)$ with total energy E and total momentum $\vec{k}$. (The relative energies and momenta are suppressed.) The state $|\phi\rangle$ was specially chosen in Sec. II and Appendix A so that a pole would occur in the matrix elements

$$\frac{\delta^3(\vec{k}' - \vec{k})}{D(k)} = \left\langle\phi(\vec{k}')\left|\frac{1}{k^0 - H}\right|\phi(\vec{k})\right\rangle.$$

Because of (C6)

$$\frac{1}{D(k)} = \int dE \frac{\varphi(E)^*\varphi(E)}{k^0 - E} . \qquad \text{(C7)}$$

Analytically continuing the k^0 dependence of (C7) clockwise into the lower half plane merely distorts the dE contour ahead of it. A pole can occur in (C7) at $k^0 = \bar{k}^0$ only if the contour is trapped against a fixed pole of the integrand at $E = \bar{k}^0$ in the lower half plane. Thus

$$\varphi^*(E)\varphi(E) \to \frac{c}{E - \bar{k}^0} , \qquad \text{(C8)}$$

where c is finite but nonvanishing. The pole in (C8) cannot be of higher order because the pole in (C7) would not be first order then. Furthermore, $\varphi(E)$ cannot have *ad hoc* poles and cuts because they would produce spurious singularities in (C6) and hence in the S matrix. Hence it is the pole in the weighting function (C8) that allows certain states in the physical Hilbert space $\mathfrak{h}$ to have a nonzero overlap with the unstable-particle states.

All this is in preparation for discussing the norm. For a general state

$$|G\rangle = \int d^3k\, |\Psi(k)\rangle\, G_A(\vec{k}) + \int d^3k\, |\Psi(k^*)\rangle G_B(\vec{k}), \qquad \text{(C9)}$$

the inner-product norm is

$$\langle G|G\rangle = \int d^3k (G_A^* G_B + G_A G_B^*). \qquad \text{(C10)}$$

This norm is, of course, real but not necessarily positive. Clearly (C10) is just the expectation value of (C2):

$$\langle G|G\rangle = \langle G|I_{1u}|G\rangle .$$

This suggests defining a diagonal operator

$$\Omega_{1u} \equiv \int d^3k\, [\,|\Psi(k)\rangle\langle\Psi(k)| + |\Psi(k^*)\rangle\langle\Psi(k^*)|\,]. \qquad \text{(C11)}$$

The expectation value

$$\langle G|\Omega_{1u}|G\rangle = \int d^3k (|G_A|^2 + |G_B|^2) \qquad \text{(C12)}$$

is then positive-definite. Again (C11) can be generalized to the entire unstable sector. Therefore define the full norm in $\mathfrak{h}\uparrow$ by

$$\Omega = I_s + \Omega_u$$
$$\|F\|^2 \equiv \langle F|\Omega|F\rangle \quad \text{for } |F\rangle \in \mathfrak{h}\uparrow . \qquad \text{(C13)}$$

To see how (C13) acts, calculate the norm of $|\phi\rangle$. From (C11)

$$\langle\phi(\vec{k})|\Omega_u|\phi(\vec{k}')\rangle = \left(\frac{1}{N} + \frac{1}{N^*}\right)\delta^3(\vec{k} - \vec{k}') .$$

This, of course, is not generally positive. To calculate the I_s contribution to the norm use

$$\langle\phi(\vec{k})|I_s|\phi(\vec{k}')\rangle = [\langle\phi(\vec{k})|I_s][I_s|\phi(\vec{k})\rangle] .$$

Substituting (C3c) gives

$$\langle\phi(\vec{k})|I_s|\phi(\vec{k}')\rangle = \left(1 - \frac{1}{N} - \frac{1}{N^*}\right)\delta^3(\vec{k} - \vec{k}')$$

so that

$$\langle\phi(\vec{k})|\Omega|\phi(\vec{k}')\rangle = \delta^3(\vec{k} - \vec{k}') .$$

This shows that for states in $\mathfrak{h}$ the Ω norm agrees with the usual norm in $\mathfrak{h}$:

$$\langle F|\Omega|F\rangle = \langle F|F\rangle \quad \text{for } |F\rangle \in \mathfrak{h} .$$

Thus $\|F\|$ is a genuine norm. It satisfies

(a) $\|F\| \geq 0,\ \|F\| = 0$ iff $|F\rangle = 0$;

(b) $\|\alpha F\| = |\alpha|\ \|F\|$;

(c) $\|F_1 + F_2\| \leq \|F_1\| + \|F_2\|$.

A further bonus is that $\mathfrak{h}\uparrow$ is Cauchy complete in this norm. More precisely, any sequence of states $|F_n\rangle$ of the form (C9) that is Cauchy complete, i.e., for which

$$\|\,|F_n\rangle - |F_m\rangle\| \to 0 \quad \text{as } n, m \to \infty ,$$

must necessarily have a limit $|F\rangle \in \mathfrak{h}\uparrow$. This is a trivial consequence of (C12) and the fact that square-integrable functions are Cauchy complete. The more substantive version of completeness is why only states of the form (C10) should be considered. This is just the question, previously mentioned, of whether the eigenstates of H span $\mathfrak{h}\uparrow$.

The definition of the norm operator in (C13) is not unique. For example, define linear combinations

$$|\pi_1(k)\rangle \equiv |\Psi(k)\rangle\alpha + |\Psi(k^*)\rangle\beta^* ,$$
$$|\pi_2(k)\rangle \equiv |\Psi(k)\rangle\frac{|\alpha\beta|}{\beta} - |\Psi(k^*)\rangle\frac{|\alpha\beta|}{\alpha^*} ,$$

where α and β are complex numbers satisfying

$$\alpha\beta + \alpha^*\beta^* = 1 .$$

The $|\pi\rangle$ have inner products

$$\langle\pi_1(k')|\pi_1(k)\rangle = \delta^3(\vec{k}' - \vec{k}) ,$$
$$\langle\pi_2(k')|\pi_2(k)\rangle = -\delta^3(\vec{k}' - \vec{k}) ,$$
$$\langle\pi_1(k')|\pi_2(k)\rangle = 0 .$$

The identity operator (C2) written in terms of these new states is

$$I_{1u} = \int d^3k [\,|\pi_1(k)\rangle\langle\pi_1(k)| - |\pi_2(k)\rangle\langle\pi_2(k)|\,] . \qquad \text{(C14)}$$

The norm operator Ω in (C11) is diagonal in the

$|\Psi\rangle$ basis but not in the $|\pi\rangle$ basis. The minus sign in (C14) suggests defining a new positive-definite norm operator that is diagonal in $|\pi\rangle$:

$$\Omega' \equiv \int d^3k[|\pi_1(k)\rangle\langle\pi_1(k)| + |\pi_2(k)\rangle\langle\pi_2(k)|] .$$

Obviously $|\pi_2\rangle$ here is formally analogous to the timelike photons of the Gupta-Bleuler quantization of QED and Ω' is analogous to their η norm.

Sets which are open in the Ω norm are also open in the Ω' norm because the two are connected by the transformation

$$T \equiv \int d^3k[|\pi_1(k)\rangle\langle\Psi(k)| + |\pi_2(k)\rangle\langle\Psi(k^*)|] ,$$

in that

$$\Omega = T^\dagger\Omega' T .$$

Because of (C2)

$$T^\dagger T = I_{1u} .$$

Because of (C14)

$$TT^\dagger = I_{1u} .$$

Thus T is unitary. The topology of $\mathfrak{h}\uparrow$ is therefore not changed by such a change in the definition of norm. In particular, if

$$\langle\Delta F|\Omega|\Delta F\rangle = 0 ,$$

then

$$\langle\Delta F|\Omega'|\Delta F\rangle = 0 .$$

Therefore

$$|F_1\rangle = |F_2\rangle$$

has an invariant meaning.

APPENDIX D: MULTIPLE POLES AND MULTIPOLE GHOSTS

The analysis of Sec. III applied to any first-order pole in the scattering amplitude. It is well known that unitarity and the asymptotic condition forbid multiple poles on the physical sheet.[29] No such prohibition applies to unphysical sheets.[13] An analysis similar to Sec. III will show that whenever there are multiple poles the state space $\mathfrak{h}\uparrow$ necessarily contains some curious new states.

The term "multipole ghosts" originated in Heisenberg's indefinite-metric quantization of the point-source Lee model.[16] In such a quantization the usual prohibition does not apply and, indeed, the scattering amplitude has a second-order pole on the real axis (see Sec. III B). Associated with the double pole are two states: an energy eigenstate and a ghost of that state. Nakanishi has discussed the possibility of multiple-ghost states generally in an indefinite-metric quantization and has shown that if there are such states the scattering amplitude has a multiple pole on the physical sheet.[30]

This discussion borrows the name "multipole ghosts" but differs in two respects. First, the existence of the states is deduced from the existence of the multiple poles (rather than conversely). Second, the quantization is conventional in that the multiple poles are only on unphysical sheets.

Suppose then that the scattering amplitude contains a pole of order L on the second sheet at $k^0 = \tilde{k}^0$. This sheet is reached just as in (3.3) by continuing clockwise around a particular n-particle branch point. At the pole

$$D_n(\tilde{k}) = 0 .$$

Furthermore, because the zero is of order L

$$\left.\frac{dD_n(k)}{dk^0}\right|_{\tilde{k}} = 0$$

and generally

$$\left.\frac{d^{l-1}D_n(k)}{d(k^0)^{l-1}}\right|_{\tilde{k}} = 0, \quad l = 1, 2, \ldots, L .$$

Using expression (2.12) for $D(k)$, these three equations are

$$\langle\phi(\vec{k}')[\tilde{k}^0 - H' - H''r_n(\tilde{k}^0)H'']|\phi(\vec{k})\rangle = 0 , \qquad \text{(D1a)}$$

$$\langle\phi(\vec{k}')|\{1 + H''[r_n(\tilde{k}^0)]^2H''\}|\phi(\vec{k})\rangle = 0 , \qquad \text{(D1b)}$$

$$\langle\phi(\vec{k}')|H''[r_n(\tilde{k}^0)]^l H''|\phi(\vec{k})\rangle = 0, \quad l = 3, 4, \ldots, L . \qquad \text{(D1c)}$$

The Lth derivative of $D(k)$ does not vanish. Let it be

$$\langle\phi(\vec{k}')|H''[r_n(k^0)]^{L+1}H''|\phi(\vec{k})\rangle \equiv C\delta(\vec{k}' - \vec{k}) , \qquad \text{(D2)}$$

where C is a finite but nonvanishing constant analogous to N in (3.12).

The analytic continuation of (2.13) is still

$$(k^0 - H)|\psi_n(k)\rangle = |\phi(\vec{k})\rangle D_n(k) .$$

Thus

$$(\tilde{k}^0 - H)|\psi_n(\tilde{k})\rangle = 0$$

as before. Now, however, the equation

$$(k^0 - H)^2|\psi_n(k)\rangle = (k^0 - H)|\phi(\vec{k})\rangle D_n(k)$$

contains new information. Differentiate this with respect to k^0 and then set $k^0 = \tilde{k}^0$:

$$(\tilde{k}^0 - H)^2\left[\left.\frac{d|\psi_n(k)\rangle}{dk^0}\right|_{\tilde{k}}\right] = 0 .$$

Generally, if the equation

$$(k^0 - H)^l|\psi_n(k)\rangle = (k^0 - H)^{l-1}|\phi(\vec{k})\rangle D_n(k)$$

is differentiated $l-1$ times the result is

$$(\tilde{k}^0 - H)^l \left[\frac{d^{l-1}|\psi_n(k)\rangle}{d(k^0)^{l-1}} \bigg|_{\tilde{k}} \right] = 0, \quad l = 1, 2, 3, \ldots, L.$$

These equations may be summarized by

$$(\tilde{k}^0 - H)^l |\psi_n^l(\tilde{k})\rangle = 0, \tag{D3a}$$

where

$$|\psi_n^l(\tilde{k})\rangle \equiv (-)^{l-1} \frac{d^{l-1}|\psi_n(k)\rangle}{d(k^0)^{l-1}} \bigg|_{\tilde{k}}, \quad l = 1, 2, \ldots, L. \tag{D3b}$$

These equations are useless unless $|\psi_n(k)\rangle$ can actually be differentiated. The explicit form (2.10) shows that it can. Furthermore, because the k^0 dependence of (2.10) is so simple the results are just

$$|\psi_n^1(\tilde{k})\rangle = [1 + r_n(\tilde{k}^0) H''] |\phi(\vec{k})\rangle, \tag{D4a}$$

$$|\psi_n^l(\tilde{k})\rangle = [r_n(\tilde{k}^0)]^l H'' |\phi(\vec{k})\rangle, \quad l = 2, 3, \ldots, L. \tag{D4b}$$

Now, of course,

$$(\tilde{k}^0 - H)|\psi_n^1(\tilde{k})\rangle = 0 \tag{D5}$$

as always. Next try to evaluate

$$(\tilde{k}^0 - H)|\psi_n^2(\tilde{k})\rangle.$$

Because

$$(\tilde{k}^0 - H')r_n(\tilde{k}^0) = 1$$

it is clear from (D4b) that

$$(\tilde{k}^0 - H')|\psi_n^2(\tilde{k})\rangle = r_n(\tilde{k}^0) H'' |\phi(\vec{k})\rangle. \tag{D6}$$

Furthermore,

$$H''|\psi_n^2(\tilde{k})\rangle = H''[r_n(\tilde{k}^0)]^2 H'' |\phi(\vec{k})\rangle.$$

From the definitions of H'' and the reduced resolvent in Sec. II the right-hand side of this equation is proportional to $|\phi\rangle$. Thus

$$H''|\psi_n^2(\tilde{k})\rangle = |\phi(\tilde{k})\rangle \lambda, \tag{D7}$$

where λ is given by

$$\langle \phi(\vec{k}')| H''[r_n(\tilde{k}^0)]^2 H'' |\phi(\vec{k})\rangle = \delta^3(\vec{k}' - \vec{k})\lambda.$$

Referring to (D1b) shows that $\lambda = -1$. Subtracting (D7) from (D6) gives

$$(\tilde{k}^0 - H)|\psi_n^2(\tilde{k})\rangle = [1 + r_n(\tilde{k}^0) H''] |\phi(\tilde{k})\rangle.$$

Hence

$$(\tilde{k}^0 - H)|\psi_n^2(\tilde{k})\rangle = |\psi_n^1(\tilde{k})\rangle. \tag{D8}$$

Now do the same thing for the higher-order states:

$$(\tilde{k}^0 - H)|\psi_n^l(\tilde{k})\rangle = (\tilde{k}^0 - H')|\psi_n^l(\tilde{k})\rangle - H''|\psi_n^l(\tilde{k})\rangle.$$

For $l \geq 3$ the second term vanishes because of (D1c). The first term is just

$$(\tilde{k}^0 - H')|\psi_n^l(\tilde{k})\rangle = [r_n(\tilde{k}^0)]^{l-1} H'' |\phi(\vec{k})\rangle, \quad l = 3, 4, \ldots, L.$$

Therefore the general result is

$$(\tilde{k}^0 - H)|\psi_n^l(\tilde{k})\rangle = |\psi_n^{l-1}(\tilde{k})\rangle, \quad l = 2, 3, \ldots, L \tag{D9}$$

even though the $l = 2$ case required special treatment. Because of (D9) the fact that

$$(\tilde{k}^0 - H)^l |\psi_n^l(\tilde{k})\rangle = 0$$

merely reflects the fact that

$$(\tilde{k}^0 - H)|\psi_n^1(\tilde{k})\rangle = 0.$$

The states $|\psi_n^l\rangle$ are sometimes called multipole ghost states. It is easy to show that

$$\langle \psi_n^l(\tilde{k})|\psi_n^{l'}(\tilde{k})\rangle = 0 \tag{D10}$$

because the Hamiltonian is Hermitian as in Sec. III. Again Hermitian analyticity requires that $D(k)$ have a zero of order L when continued counterclockwise to $k^0 = \tilde{k}^{0*}$ in the upper half plane. There are thus L additional states $|\psi_{\bar{n}}^l(\tilde{k}^*)\rangle$ analogous to (D4). The inner product of conjugate partners is

$$\langle \psi_{\bar{n}}^l(\tilde{k}^*)|\psi_n^{l'}(\tilde{k}')\rangle = \begin{cases} C\delta^3(\vec{k} - \vec{k}') & \text{if } l + l' = L + 1, \\ 0 & \text{otherwise}. \end{cases} \tag{D11}$$

The constant C is given in (D2).

APPENDIX E: OTHER ONE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS

The states constructed in Sec. III by analytic continuation automatically have $q^2 > 0$. Unfortunately they are not irreducible representations of the Poincaré group. To construct irreducible representations requires introducing complex boosts because of (4.34). Once these boosts are allowed, however, a state with any momentum $k = p + iq$ satisfying $k^2 = \mathfrak{M}^2$ can be generated. Beltrametti and Luzzatto[23] observed that there are three classes of momentum vectors that are distinguished by whether the plane containing p and q (1) intersects the interior of the light cone, (2) is tangent to it, or (3) is completely spacelike. The quantity $(p \cdot q)^2 - p^2 q^2$ is correspondingly positive, zero, or negative. A suitable real Lorentz transformation will bring any k into one of the three standard forms as follows:

Type 1. $\Delta > 0$:

$$\bar{k} = (\alpha, 0, 0, \beta), \quad \mathfrak{M}^2 = \alpha^2 - \beta^2. \tag{E1}$$

Type 2. $\Delta = 0$:

$$\bar{k} = (\alpha, 0, i\mathfrak{M}, \alpha). \tag{E2}$$

Type 3. $\Delta < 0$:

$$\bar{k} = (0, 0, \alpha, \beta), \quad \mathfrak{M}^2 = -\alpha^2 - \beta^2. \tag{E3}$$

The subgroup of the real Lorentz group that leaves $\bar{k}$ invariant is one dimensional in each case. Denote the little-group generator by g_i. Then the three classes have, respectively,

$$g_1 = J_3\,,\quad g_2 = J_2 + K_1\,,\quad g_3 = K_1\,. \tag{E4}$$

[Of course for any of the momentum classes the subgroup of the complex Lorentz group that leaves a particular $\bar{k}$ invariant has not one but three generators, viz. the three $S_i(\bar{k})$ given by (4.27). In each case, however, only one of these three generators is proportional to a Hermitian generator g_i.]

Regardless of the type, an irreducible representation of the full Lorentz group may be defined as in (5.15) by

$$|k, j, \sigma\rangle \equiv U[L(k, \bar{k})L^c(\bar{k}, \mathfrak{M})]\,|\Psi(\mathfrak{M})j, j_3\rangle(2\mathfrak{M})^{1/2}\,|_{j_3=\sigma}\,, \tag{E5}$$

where the meaning of σ may differ from one class to another. For type-1 momenta (E5) actually yielded the states of Sec. V C on which the one-dimensional little group was represented irreducibly. It will now be shown that states of type 3 can also have their one-dimensional little group represented irreducibly but states of type 2 do not have such a representation. To see this, apply a real Lorentz transformation Λ to (E5) to get

$$U[\Lambda]|k, j, \sigma\rangle = U[L(\Lambda k, \bar{k})\,L^c(\bar{k}, \mathfrak{M})]\,U[Z_i]|\mathfrak{M}, j, \sigma\rangle\,, \tag{E6}$$

where

$$Z_i \equiv L^c(\bar{k}, \mathfrak{M})^{-1}\{L(\Lambda k, \bar{k})^{-1}\Lambda\,L(k, \bar{k})\}\,L^c(\bar{k}, \mathfrak{M})\,. \tag{E7}$$

The quantity in curly brackets in each case is a real Lorentz transformation that leaves $\bar{k}$ invariant. It is therefore generated by the corresponding g_i so that

$$U[L(\Lambda k, \bar{k})^{-1}\Lambda\,L(k, \bar{k})] = e^{i\lambda_W g_i}\,, \tag{E8}$$

where λ_W is a real Wigner parameter. (For type 1, λ_W is the real Wigner rotation angle around the z axis; for type 3, λ_W gives the real boost velocity along the x axis; and for type 2, λ_W is a mixture of rotation angle around y and boost along x.)

States with momenta of type 1 are the helicity states discussed in Sec. V C. Consider now states of type 3. Choose the complex boost in (E5) to be

$$U[L^c(\bar{k}, \mathfrak{M})] = e^{i\theta J_1}\,e^{-\pi K_2/2}\,. \tag{E9}$$

(The velocity boost in the y direction is purely imaginary.) The corresponding Lorentz matrix is

$$L^c(\bar{k}, \mathfrak{M})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 1 & 0 & 0 \\ i\cos\theta & 0 & 0 & -\sin\theta \\ i\sin\theta & 0 & 0 & \cos\theta \end{pmatrix}. \tag{E10}$$

Obviously

$$L^c(\bar{k}, \mathfrak{M})^{\mu}{}_{\nu}\,\mathfrak{M}^{\nu} = \begin{pmatrix} 0 \\ 0 \\ i\mathfrak{M}\cos\theta \\ i\mathfrak{M}\sin\theta \end{pmatrix}, \tag{E11}$$

in accordance with the $\bar{k}$ of (E3). To calculate Z use (E8) and

$$[J_1, K_1] = 0$$

to get

$$U[Z] = e^{\pi K_2/2}\,e^{i\lambda_W K_1}\,e^{-\pi K_2/2}\,. \tag{E12}$$

Because

$$[K_1, K_2] = -iJ_3\,,$$

this becomes

$$U[Z] = e^{-\lambda_W J_3}\,. \tag{E13}$$

The transformation law (E6) for type-3 states is therefore one dimensional as claimed:

$$U[\Lambda]|k, j, \sigma\rangle = |\Lambda k, j, \sigma\rangle\,e^{-\lambda_W\sigma}\,. \tag{E14}$$

Because λ_W and σ are real this representation is not unitary. More importantly, it is not even unitary when k is continued to real values.

For states with momentum of type 2 there is no choice of the complex boost $L^c(\bar{k}, \mathfrak{M})$ that will represent the one-dimensional little group irreducibly. Suppose there were such a choice. The Z in (E7) could only be a rotation around the z axis if

$$L^c(\bar{k}, \mathfrak{M})^{-1}(J_2 + K_1)L^c(\bar{k}, \mathfrak{M}) = \zeta J_3\,, \tag{E15}$$

for some constant ζ. This would then ensure that

$$U[Z] = e^{i\lambda_W\zeta J_3}$$

and (E6) would then represent the little group irreducibly. Unfortunately, there is no L^c that satisfies (E15). To see this, write (E15) as

$$(J_2 + K_1)L^c(\bar{k}, \mathfrak{M}) = \zeta L^c(\bar{k}, \mathfrak{M})J_3\,. \tag{E16}$$

The generators may be represented by the 4×4 matrices

$$J_2+K_1=\begin{bmatrix}0&-i&0&0\\-i&0&0&i\\0&0&0&0\\0&-i&0&0\end{bmatrix},$$

$$J_3=\begin{bmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{bmatrix}.$$

Explicit multiplication shows that the only matrices L^c that satisfy (E16) have all zeros in the second row. Such matrices have zero determinant and thus cannot represent Lorentz transformations. This shows that there is no state like (E5) which represents the little group of type-2 momenta irreducibly.

APPENDIX F: HOW $q\to 0$ IN HELICITY AMPLITUDES

It is crucial that q goes to zero in the plane $\vec{p}\parallel\vec{q}$. Because of this, the helicity states of Sec. V C are actually produced in (6.10) and yet their observed transformation law is (6.18). To see how $q\to 0$, begin with the transformation law

$$\langle\Lambda k^*,j,\sigma'|T[\psi_\alpha(\Lambda r)\bar\psi_\beta(-\Lambda r)]|0\rangle$$
$$=e^{-i\theta_W\sigma'}\langle k^*,j,\sigma'|T[\psi_{\alpha'}(r)\bar\psi_{\beta'}(-r)]|0\rangle \times\mathfrak{D}_{\alpha',\alpha}(\Lambda)\mathfrak{D}_{\beta',\beta}(\Lambda)$$

appropriate to the residue in (6.5). Let $q\to 0$ in some unspecified manner. Then project out the two-particle irreducible representation (6.4):

$$\langle\Lambda p,j,\sigma'|\Lambda p,j,\sigma;1+2\rangle =e^{-i\theta'\sigma'}\sum_{\sigma''}\langle p,j,\sigma'|p,j,\sigma'';1+2\rangle\mathfrak{D}_{\sigma'',\sigma}(\tilde R), \tag{F1}$$

where

$$\theta'\equiv\lim_{q\to 0}\theta_W \tag{F2}$$

and $\tilde R$ is given by (6.4).

Equation (F1) may be used to investigate how $q\to 0$. Recall from (6.4) that

$$\tilde R=L(W,\tilde p')R(\tilde p',p')\Lambda R(p,\tilde p)L(\tilde p,W).$$

Choose

$$p=(p^0,0,0,|\vec{p}|)$$
$$\Lambda=e^{i\theta J_3}.$$

Then $p=\tilde p=p'=\tilde p'$ so that

$$\tilde R=e^{i\theta J_3}.$$

This greatly simplifies (F1) because now

$$D_{\sigma'',\sigma}(\tilde R)=\delta_{\sigma'',\sigma}e^{i\theta\sigma}$$

and (F1) reduces to

$$\langle p,j,\sigma'|p,j,\sigma;1+2\rangle =e^{-i\theta'\sigma'}\langle p,j,\sigma'|p,j,\sigma;1+2\rangle e^{i\theta\sigma}. \tag{F3}$$

The content of (F3) is that the amplitude vanishes unless

$$\theta'\sigma'=\theta\sigma.$$

The angle θ' will depend on whether or not $\vec{q}\parallel\vec{p}$. In fact, it will be shown that

$$\theta'=\begin{cases}\theta & \text{if } \vec{q}\parallel\vec{p},\\ 0 & \text{otherwise}.\end{cases} \tag{F4}$$

This has two consequences: If $\vec{q}\parallel\vec{p}$ then the amplitude vanishes unless $\sigma'=\sigma$. If $\vec{q}$ and $\vec{p}$ are not parallel, the amplitude vanishes regardless of the value of σ'.

To demonstrate (F4) take

$$q=(q^0,|\vec{q}|\sin\eta,0,|\vec{q}|\cos\eta).$$

Since Λ is a z rotation,

$$q'=(q^0,|\vec{q}|\sin\eta\cos\theta,|\vec{q}|\sin\eta\sin\theta,|\vec{q}|\cos\eta).$$

Thus η and θ are the polar and azimuthal angles of q' and $\vec{q}\parallel\vec{p}$ corresponds to $\eta=0$. To calculate θ' requires first calculating θ_W from

$$e^{i\theta_W J_3}=L(\bar k,k')e^{i\theta J_3}L(k,\bar k). \tag{F5}$$

A specific form for the real transformation $L(k,\bar k)$ is necessary. Consider the case $q^2>0$ as in (6.20). Then

$$\bar p=\left(\frac{p\cdot q}{(q^2)^{1/2}},0,0,\left[\frac{(p\cdot q)^2-p^2q^2}{q^2}\right]^{1/2}\right),$$
$$\bar q=((q^2)^{1/2},0,0,0)$$

from (5.12a). A real transformation $L(\bar k,k)$ from k to $\bar k$ may always be accomplished by boosting q to the rest form $\bar q$ and then rotating into the z axis:

$$L(\bar k,k)=R(\hat z,B(\bar q,q)p)B(\bar q,q).$$

Thus

$$e^{i\theta_W J_3}=R(\hat z,B(\bar q,q')p)B(\bar q,q')e^{i\theta J_3}B(q,\bar q) \times R(B(\bar q,q)p,\hat z), \tag{F6}$$

where $p'=p$ has been used in the argument of the leftmost rotation. The two boosts are given by

$$B(q,\bar q)=e^{i\mu\hat n\cdot\vec{K}},$$
$$B(q',\bar q)=e^{i\mu\hat n'\cdot\vec{K}},$$

where

$$\hat{n} = (\sin\eta, 0, \cos\eta),$$

$$\hat{n}' = (\sin\eta\cos\theta, \sin\eta\sin\theta, \cos\eta),$$

$$\tanh\mu = |\vec{q}|/q^0 .$$

Because $\hat{n}'$ is just $\hat{n}$ rotated by θ around $\hat{z}$,

$$B(\bar{q}, q')e^{i\theta J_3} = e^{i\theta J_3}B(\bar{q}, q).$$

Hence (F6) is just

$$e^{i\theta_W J_3} = R(\hat{z}, B(\bar{q}, q')p)e^{i\theta J_3}R(B(\bar{q}, q)p, \hat{z}). \quad \text{(F7)}$$

To calculate θ_W these rotations must be explicitly displayed. The rightmost rotation above rotates a vector with momentum along $\hat{z}$ until it is parallel to

$$B(\bar{q}, q)p = \begin{bmatrix} p^0\cosh\mu - |\vec{p}|n_3\sinh\mu \\ -p^0 n_1\sinh\mu + |\vec{p}|n_1 n_3(\cosh\mu - 1) \\ 0 \\ -p^0 n_3\sinh\mu + |\vec{p}|[1 + n_3 n_3(\cosh\mu - 1)] \end{bmatrix}, \quad \text{(F8)}$$

where the vector components are written col(t, x, y, z). Thus the rotation is around the y axis and is given by

$$R(B(q, \bar{q})p, \hat{z}) = \begin{pmatrix} \cos\omega & 0 & \sin\omega \\ 0 & 1 & 0 \\ -\sin\omega & 0 & \cos\omega \end{pmatrix}, \quad \text{(F9)}$$

where

$$\tan\omega = \frac{-p^0 n_1\sinh\mu + |\vec{p}|n_1 n_3(\cosh\mu - 1)}{-p^0 n_3\sinh\mu + |\vec{p}|[1 + n_3 n_3(\cosh\mu - 1)]}. \quad \text{(F10)}$$

This gives for (F7)

$$e^{i\theta_W J_3} = R(\hat{z}, B(\bar{q}, q')p)\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \times\begin{pmatrix} \cos\omega & 0 & \sin\omega \\ 0 & 1 & 0 \\ -\sin\omega & 0 & \cos\omega \end{pmatrix}. \quad \text{(F11)}$$

The remaining rotation to be calculated rotates a vector with momentum along $\hat{z}$ until it is parallel to

$$B(\bar{q}, q')p = \begin{bmatrix} p^0\cosh\mu - |\vec{p}|n_3'\sinh\mu \\ -p^0 n_1'\sinh\mu + |\vec{p}|n_1' n_3'(\cosh\mu - 1) \\ -p^0 n_2'\sinh\mu + |\vec{p}|n_2' n_3'(\cosh\mu - 1) \\ -p^0 n_3'\sinh\mu + |\vec{p}|[1 + n_3' n_3'(\cosh\mu - 1)] \end{bmatrix}. \quad \text{(F12)}$$

Comparing $\hat{n}'$ with $\hat{n}$ shows that this vector is just (F8) rotated around the z axis by θ. Thus

$$R(B(q', \bar{q})p, \hat{z}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \times\begin{pmatrix} \cos\omega & 0 & \sin\omega \\ 0 & 1 & 0 \\ -\sin\omega & 0 & \cos\omega \end{pmatrix}. \quad \text{(F13)}$$

This rather simplifies (F11), which becomes

$$e^{i\theta_W J_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{(F14)}$$

This means that $\theta_W = 0$ and hence $\theta' = 0$ as claimed for $\vec{q}$ not parallel to $\vec{p}$.

It appears, in fact, that $\theta_W = 0$ regardless of the direction of $\vec{q}$. This is not correct. Suppose that $\vec{q} \parallel \vec{p}$. Then $\hat{n}$ is along the z axis. Thus the vector in (F8) has only t and z components so that

$$R(B(\bar{q}, q)p, \hat{z}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{(F15a)}$$

In other words $\omega = 0$. The vector (F12) in this case also has only t and z components so that

$$R(B(\bar{q}, q')p, \hat{z}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(F15b)}$$

and hence (F7) becomes

$$e^{i\theta_W J_3} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(F16)}$$

as claimed in (F4). Generally, of course, the rotation that takes the vector (0, 0, 1) into itself is not just the identity. It can be any fixed rotation around the z axis. This fixed rotation must appear in both (F15a) and (F15b). It then cancels in (F16) so that $\theta_W = \theta$ regardless. This is the reason that (F15b) is not just the $\omega = 0$ limit of (F13).

*Work supported by the U. S. Energy Research and Development Administration.

[1]E. P. Wigner, Ann. Math. 40, 149 (1939).

[2]P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford Univ. Press, London, 1958), 4th edition, p. 201.

[3]R. Peierls and P. L. Kapur, Proc. R. Soc. London A166, 277 (1938).

[4]E. P. Wigner and L. Eisenbud, Phys. Rev. 72, 29 (1947).

[5]A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30, 257 (1958); G. Breit, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1959), Vol. XLI, Part I.

[6]H. Feshbach, Ann. Phys. (N.Y.) 19, 287 (1962).

[7]L. Fonda and R. G. Newton, Ann. Phys. (N.Y.) 10, 490 (1960).

[8]C. Møller, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 22, No. 19 (1946).

[9]R. Peierls, *Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics* (Pergamon, New York, 1955), p. 296.

[10]H. P. Stapp, Nuovo Cimento 32, 103 (1964).

[11]P. Roman, *Advanced Quantum Theory* (Addison-Wesley, Reading, Mass., 1965), pp. 396 and 491.

[12]R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge Univ. Press, New York, 1966), p. 250.

[13]M. L. Goldberger and K. M. Watson, Phys. Rev. 136, B1472 (1964).

[14]T. D. Lee, Phys. Rev. 95, 1329 (1954).

[15]G. Källén and W. Pauli, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 30, No. 7 (1955).

[16]W. Heisenberg, Nucl. Phys. 4, 532 (1957).

[17]K. L. Nagy, Nuovo Cimento Suppl. 17, 92 (1970).

[18]V. Glaser and G. Källén, Nucl. Phys. 2, 706 (1956).

[19]S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), p. 65.

[20]S. Weinberg, Phys. Rev. 134, B882 (1964).

[21]J. D. Talman, *Special Functions* (Benjamin, New York, 1968), p. 66.

[22]D. Zwanziger, Phys. Rev. 131, 2818 (1963).

[23]E. G. Beltrametti and G. Luzzatto, Nuovo Cimento 36, 1217 (1965).

[24]H. J. Bremermann and L. J. Durand, J. Math. Phys. 2, 240 (1961).

[25]M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).

[26]K. Huang and H. A. Weldon, Phys. Rev. D 11, 257 (1975).

[27]H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 1425 (1955).

[28]J. Bros, in *Analytic Methods in Mathematical Physics*, edited by R. P. Gilbert and R. G. Newton (Gordon and Breach, New York, 1970), p. 121. I am indebted to H. P. Stapp for this reference.

[29]R. J. Eden and P. V. Landshoff, Phys. Rev. 136, B1817 (1964).

[30]N. Nakanishi, Prog. Theor. Phys. Suppl. 51, 1 (1972).