Heuristic Hamiltonian for fermions interacting via meson fields*

M. Bolsterli
Theoretical Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545
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The splitting of the Dirac field operator into two two-component parts corresponding to particle and antiparticle field operators is used to make a no-pair, low-momentum-transfer approximation to the fermion current densities that appear in the Hamiltonians that characterize the interaction of the Dirac field with scalar, vector, and pseudoscalar fields. The resulting approximate Hamiltonians can only be regarded as heuristic, but they describe interesting quantum field theories without ultraviolet divergences. The properties of the scalar and vector theories are described in some detail; in particular, the energy per particle for infinite uniform matter is given in the Fermi-sea approximation.

I. INTRODUCTION AND RESULTS

In general, the interaction of a meson field with a Dirac fermion field is described by an interaction Hamiltonian density that is the product of the meson field operator with a current density operator for the fermion. For example, in the case of a neutral scalar-meson field interacting with a Dirac field, the standard Yukawa interaction Hamiltonian is

$$\begin{aligned} H_{I,\mathrm{NS}} &= -g\int \phi(\vec{r})J_\beta(\vec{r})d\vec{r}, \\ J_\beta(\vec{r}) &= :\psi^\dagger(\vec{r})\beta\psi(\vec{r}):, \end{aligned} \tag{1.1}$$

where $J_\beta(\vec{r})$ is the scalar current density operator; the colons indicate that the creation and annihilation parts of ψ and $\psi^\dagger$ are normal-ordered. The pseudosclar current density is[1]

$$J_{\rho_2}(\vec{r}) = :\psi^\dagger(\vec{r})\rho_2\psi(\vec{r}):, \tag{1.2}$$

and the four-vector current densities are

$$\begin{aligned} J_1(\vec{r}) &= :\psi^\dagger(\vec{r})\psi(\vec{r}):, \\ J_{\vec{\alpha}}(\vec{r}) &= :\psi^\dagger(\vec{r})\vec{\alpha}\psi(\vec{r}):. \end{aligned} \tag{1.3}$$

In order to describe the approximations to be considered, the following Fourier transforms are needed:

$$\begin{aligned} \psi(\vec{r}) &= (2\pi)^{-3/2}\int e^{i\vec{p}\cdot\vec{r}}\tilde{\psi}(\vec{p})d\vec{p}, \\ J_Y(\vec{r}) &= (2\pi)^{-3}\int e^{-i(\vec{p}-\vec{q})\cdot\vec{r}}\tilde{J}_Y(\vec{p},\vec{q})d\vec{p}\,d\vec{q}, \\ \tilde{J}_Y(\vec{p},\vec{q}) &= :\tilde{\psi}^\dagger(\vec{p})Y\tilde{\psi}(\vec{q}):. \end{aligned} \tag{1.4}$$

In the following considerations of approximations to $\tilde{J}_Y(\vec{p},\vec{q})$, the vectors $\vec{k}$ and $\vec{K}$ are used; throughout they take the values

$$\begin{aligned} \vec{k} &= \vec{p}-\vec{q}, \\ \vec{K} &= \tfrac{1}{2}(\vec{p}+\vec{q}). \end{aligned} \tag{1.5}$$

The usual nonrelativistic approximation to $\tilde{J}_Y(\vec{p},\vec{q})$ is obtained by taking both $|\vec{p}|$ and $|\vec{q}|$ much less than M, the fermion mass, *and* neglecting pair terms (sometimes this is phrased in terms of $|\vec{p}|$ and $|\vec{q}|$ being small compared to the meson mass; as will be seen in the following, it is really the fermion mass that enters). In this paper, a weaker approximation scheme for $\tilde{J}_Y(\vec{p},\vec{q})$ is presented, namely, a low-momentum-transfer approximation useful when $|\vec{k}| = |\vec{p}-\vec{q}| \ll M$ *and* pair terms are neglected. Naturally, the usual nonrelativistic approximation is obtained when the additional approximation $|\vec{K}| = |\tfrac{1}{2}(\vec{p}+\vec{q})| \ll M$ is made.

The neglect of pair terms is a traditional approximation for which it seems impossible to provide a logical justification. Simply, it is necessary in order to obtain a tractable strong-coupling theory. For this reason, the approximations described in the following can only be regarded as heuristic. The Hamiltonians that will be presented have only a tenuous relation to relativistic quantum field theory, but they are significant in that they lead to interesting noncovariant quantum field theories.

For example, the leading low-momentum-transfer approximation, without pair terms, to the scalar current density $\tilde{J}_\beta(\vec{p},\vec{q})$ is

$$\begin{aligned} \tilde{J}_{\beta,\mathrm{appr}}(\vec{p},\vec{q}) &= \frac{M}{\epsilon(\vec{K})}[\tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{q}) + \tilde{\chi}_-^\dagger(-\vec{q})\tilde{\chi}_-(-\vec{p})], \\ \epsilon(\vec{p}) &= (p^2+M^2)^{1/2}, \end{aligned} \tag{1.6}$$

where M is the fermion mass, and $\tilde{\chi}_+(\vec{p})$ and $\tilde{\chi}_-(\vec{p})$ are the two-component annihilation operators for fermion and antifermion, respectively, with momentum $\vec{p}$. The corresponding interaction Hamiltonian

$$H_{I,S,\mathrm{appr}} = -\frac{g}{(2\pi)^3}\int e^{-i(\vec{p}-\vec{q})\cdot\vec{r}}\tilde{J}_{\beta,\mathrm{appr}}(\vec{p},\vec{q}) \times\phi(\vec{r})d\vec{r}\,d\vec{p}\,d\vec{q}, \tag{1.7}$$

together with a Dirac Hamiltonian for the fermion field ψ,

$$H_D(\psi) = \int :\tilde{\psi}^\dagger(\vec{p})(\vec{\alpha}\cdot\vec{p}+\beta M)\tilde{\psi}(\vec{p}): d\vec{p}$$
$$= \int \epsilon(\vec{p})[\tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{p}) + \tilde{\chi}_-^\dagger(\vec{p})\tilde{\chi}_-(\vec{p})]d\vec{p}, \quad (1.8)$$

and the usual Hamiltonian for the scalar field,

$$H_S(\phi) = \tfrac{1}{2}\int :[\pi^2 + (\nabla\phi)^2 + m^2\phi^2]: d\vec{r}\ , \quad (1.9)$$

gives a noncovariant quantum field theory without divergent integrals. In the usual nonrelativistic approximation, the factor $M/\epsilon(\vec{K})$ is replaced by unity; that theory then contains ultraviolet divergences.[2]

As a second sample, the pseudoscalar current density has the following leading term in the momentum transfer (without pairs):

$$\tilde{J}_{\rho 2,\mathrm{appr}}(\vec{p},\vec{q}) = \tilde{\chi}_+^\dagger(\vec{p})Y_P(\vec{p},\vec{q})\tilde{\chi}_+(\vec{q}) + \tilde{\chi}_-^\dagger(-\vec{q})Y_P(\vec{p},\vec{q})\tilde{\chi}_-(-\vec{p}), \quad (1.10)$$
$$Y_P(\vec{p},\vec{q}) = \frac{i}{2\epsilon(\vec{K})}\vec{\sigma}\cdot\left\{\vec{k} - \frac{(\vec{k}\cdot\vec{K})\vec{K}}{\epsilon(\vec{K})[\epsilon(\vec{K})+M]}\right\}.$$

The operator $Y_P(\vec{p},\vec{q})$ is the pion absorption operator in the low-momentum-transfer approximation. Of course, for $|\vec{K}| \ll M$, Y_P reduces to the usual $\vec{\sigma}\cdot\nabla/2M$ form. The interaction Hamiltonian corresponding to Eq. (1.10), together with the Dirac Hamiltonian (1.8) and the pseudoscalar field Hamiltonian [also given by (1.9)], gives a noncovariant quantum field theory without ultraviolet divergences.

Recently there has been considerable discussion of whether or not the pion absorption operator should be invariant under Galilean transformations.[3-6] The operator Y_P given here is not Galilean invariant. For example, consider the case that $|\vec{k}|$ is very small and that $p^2/2M - q^2/2M \approx m$, where m is the pion mass, that is, the fermions are "on-shell" but the meson is not. Then $\vec{k}\cdot\vec{K} \approx Mm$ and

$$Y_P \approx \frac{i}{2M}\vec{\sigma}\cdot\left(\vec{k} - \frac{m}{2M}\vec{K}\right)\ , \quad (1.11)$$

which is not a Galilean-invariant form. The ambiguity in the pion absorption operator arising from interaction of the fermion field with an external scalar or vector potential[3-5] takes a simpler form here. The scalar potential interaction is

$$\int \frac{M}{\epsilon(\vec{K})}[\tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{q}) + \tilde{\chi}_-^\dagger(-\vec{q})\tilde{\chi}_-(-\vec{p})]\tilde{V}_S(\vec{p}-\vec{q})d\vec{p}\,d\vec{q}, \quad (1.12)$$

while the vector potential gives

$$\int [\tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{q}) - \tilde{\chi}_-^\dagger(-\vec{q})\tilde{\chi}_-(-\vec{p})]\tilde{V}_V(\vec{p}-\vec{q})d\vec{p}\,d\vec{q}\ . \quad (1.13)$$

There is thus no ambiguity of the type discussed in Ref. 3 other than the usual decrease in effective strength of a scalar interaction by the factor $M/\epsilon(\vec{K})$.

The approximate vector current density operators are

$$\tilde{J}_{1,\mathrm{appr}}(\vec{p},\vec{q}) = \tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{q}) - \tilde{\chi}_-^\dagger(-\vec{q})\tilde{\chi}_-(-\vec{p}), \quad (1.14)$$
$$\tilde{J}_{\vec{\alpha},\mathrm{appr}}(\vec{p},\vec{q}) = \frac{\vec{K}}{\epsilon(\vec{K})}[\tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{q}) + \tilde{\chi}_-^\dagger(-\vec{q})\tilde{\chi}_-(-\vec{p})].$$

For a transverse vector field interacting with a Dirac field, this approximation applied to the transverse current density leads to a theory without divergent integrals. This treatment can be combined with a previous treatment of the longitudinal and scalar parts of the vector field[7] to give a heuristic Hamiltonian for a vector field theory without divergent integrals.

Section II gives the details of the low-momentum-transfer approximation scheme, for which the starting point is the introduction of two-component particle and antiparticle field operators. The results of Sec. II are used to formulate a heuristic Hamiltonian for the interaction of a Dirac field with scalar, vector, and pseudoscalar fields in Sec. III. In the next two sections, the particular cases of interaction via scalar and transverse vector fields, respectively, are considered in more detail. Finally, Sec. VI gives the energy per particle in infinite uniform fermion matter in the Fermi-sea approximation for an interaction mediated by scalar and vector fields.

II. LOW-MOMENTUM-TRANSFER APPROXIMATION TO FERMION CURRENT DENSITIES

The essential formalism for reducing the Dirac field from a single four-component field to two two-component ones has been given by Foldy and Wouthuysen[8] and Bethe and Salpeter.[9] This section restates these results in the form needed here. The correct treatment of normal ordering for antiparticle operators requires that the charge conjugation operator σ_y be used.[1]

The Dirac field Hamiltonian is

$$H_D = \int :\tilde{\psi}^\dagger(\vec{p})(\vec{\alpha}\cdot\vec{p}+\beta M)\tilde{\psi}(\vec{p}): d\vec{p}\ , \quad (2.1)$$

where ψ is a four-component Dirac field operator; the corresponding momentum and angular momentum are

$$\vec{P}=\int :\tilde{\psi}^\dagger(\vec{p})\vec{p}\tilde{\psi}(\vec{p}):d\vec{p}\ ,$$

$$\vec{J}=\int :\tilde{\psi}^\dagger(\vec{p})(\vec{I}+\tfrac{1}{2}\vec{\Sigma})\,\tilde{\psi}(\vec{p}):d\vec{p}\ , \qquad (2.2)$$

$$\vec{I}=i\nabla_{\vec{p}}\times\vec{p}\ .$$

The unitary matrix

$$U(\vec{p})=\left[\frac{\epsilon(\vec{p})+M}{2\epsilon(\vec{p})}\right]^{1/2}\begin{pmatrix}1 & -\dfrac{\vec{\sigma}\cdot\vec{p}}{\epsilon(\vec{p})+M}\sigma_y\\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\epsilon(\vec{p})+M} & \sigma_y\end{pmatrix} \qquad (2.3)$$

diagonalizes $\vec{\alpha}\cdot\vec{p}+\beta M$,

$$U^\dagger(\vec{p})(\vec{\alpha}\cdot\vec{p}+\beta M)U(\vec{p})=\begin{pmatrix}\epsilon(\vec{p}) & 0\\ 0 & -\epsilon(\vec{p})\end{pmatrix}, \qquad (2.4)$$

leaves $\vec{p}$ invariant,

$$U^\dagger(\vec{p})\vec{p}U(\vec{p})=\begin{pmatrix}\vec{p} & 0\\ 0 & \vec{p}\end{pmatrix}, \qquad (2.5)$$

and changes $\vec{I}+\frac{1}{2}\vec{\Sigma}$,

$$U^\dagger(\vec{p})(\vec{I}+\tfrac{1}{2}\vec{\Sigma})U(\vec{p})=\begin{pmatrix}\vec{I}+\frac{1}{2}\vec{\sigma} & 0\\ 0 & \vec{I}-\frac{1}{2}\vec{\sigma}^T\end{pmatrix}, \qquad (2.6)$$

where use is made of

$$\sigma_y\vec{\sigma}\sigma_y=-\vec{\sigma}^T\ , \qquad (2.7)$$

with T indicating the transpose. Let the two-component field operators $\tilde{\chi}_+(\vec{p})$ and $\tilde{\chi}_-(\vec{p})$ be defined by

$$U^\dagger(\vec{p})\tilde{\psi}(\vec{p})=\begin{pmatrix}\tilde{\chi}_+(\vec{p})\\ \tilde{\chi}_-^*(-\vec{p})\end{pmatrix}=\tilde{\chi}_4(\vec{p})\ . \qquad (2.8)$$

The normal ordering in Eqs. (2.1) and (2.2) is defined by letting $\tilde{\chi}_+(\vec{p})$ and $\tilde{\chi}_-(\vec{p})$ be annihilation operators. Now

$$\begin{aligned}H_D&=\int\tilde{\chi}_+^\dagger(\vec{p})\epsilon(\vec{p})\tilde{\chi}_+(\vec{p})d\vec{p}\\ &\quad-\int :\tilde{\chi}_-^T(-\vec{p})\epsilon(\vec{p})\tilde{\chi}_-^*(-\vec{p}):d\vec{p} \qquad (2.9)\\ &=\int\epsilon(\vec{p})[\tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{p})+\tilde{\chi}_-^\dagger(\vec{p})\tilde{\chi}_-(\vec{p})]d\vec{p}\ ,\end{aligned}$$

which is just Eq. (1.8). Similarly, since

$$\vec{I}^T=\vec{I}^{*\dagger}=-\vec{I} \qquad (2.10)$$

it follows that

$$\vec{P}=\int\vec{p}[\tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{p})+\tilde{\chi}_-^\dagger(\vec{p})\tilde{\chi}_-(\vec{p})]d\vec{p}$$

$$\vec{J}=\int[\tilde{\chi}_+^\dagger(\vec{p})(\vec{I}+\tfrac{1}{2}\vec{\sigma})\tilde{\chi}_+(\vec{p})+\tilde{\chi}_-^\dagger(\vec{p})(\vec{I}+\tfrac{1}{2}\vec{\sigma})\tilde{\chi}_-(\vec{p})]d\vec{p}\ , \qquad (2.11)$$

so that $\tilde{\chi}_+(\vec{p})$ and $\tilde{\chi}_-(\vec{p})$ are two-component field operators for the fermion and antifermion, respectively. The nonrelativistic approximation to H_D is

$$H_{D,\mathrm{NR}}=\int\left(M+\frac{p^2}{2M}\right)[\tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{p})+\tilde{\chi}_-^\dagger(\vec{p})\tilde{\chi}_-(\vec{p})]d\vec{p}, \qquad (2.12)$$

so that in this limit $\tilde{\chi}_+$ and $\tilde{\chi}_-$ are the Schrödinger field operators for the particle and the antiparticle.

Now the current density $J_Y(\vec{p},\vec{q})$ can obviously be written

$$\begin{aligned}J_Y(\vec{p},\vec{q})&=\tilde{\chi}_4^\dagger(\vec{p})U^\dagger(\vec{p})YU(\vec{q})\tilde{\chi}_4(\vec{q})\\ &=\tilde{\chi}_+^\dagger(\vec{p})Y_{++}(\vec{p},\vec{q})\tilde{\chi}_+(\vec{p})+\tilde{\chi}_-^\dagger(-\vec{q})Y_{--}(\vec{p},\vec{q})\tilde{\chi}_-(-\vec{p})\\ &\quad+\tilde{\chi}_+^\dagger(\vec{p})Y_{+-}(\vec{p},\vec{q})\sigma_y\tilde{\chi}_-^*(-\vec{q})\\ &\quad+\tilde{\chi}_-^T(-\vec{p})\sigma_yY_{-+}(\vec{p},\vec{q})\tilde{\chi}_+(\vec{q}), \qquad (2.13)\\ Y_{++}(\vec{p},\vec{q})&=[U^\dagger(\vec{p})YU(\vec{q})]_{++}\ ,\\ Y_{--}(\vec{p},\vec{q})&=-[U^\dagger(\vec{p})YU(\vec{q})]_{--}^T\ ,\\ Y_{+-}(\vec{p},\vec{q})&=[U^\dagger(\vec{p})YU(\vec{q})]_{+-}\sigma_y\ ,\\ Y_{-+}(\vec{p},\vec{q})&=\sigma_y[U^\dagger(\vec{p})YU(\vec{q})]_{-+}\ .\end{aligned}$$

The ++ and −− terms are the particle and antiparticle parts of J, respectively; the other two terms are the pair terms.

If all terms in J are kept, then Eq. (2.13) is just a transcription of the original relativistic current density into a very awkward form. However, it is now possible to approximate the various Y_{ij}, in particular, by keeping just the leading term in an expansion in powers of the momentum transfer $\vec{p}-\vec{q}=\vec{k}$. With the reduced vectors

$$\begin{aligned}\vec{k}_{\mathrm{red}}&=\vec{k}-\frac{\vec{k}\cdot\vec{K}}{\epsilon(\vec{K})[\epsilon(\vec{K})+M]}\vec{K}\ ,\\ \vec{\sigma}_{\mathrm{red}}&=\vec{\sigma}-\frac{\vec{\sigma}\cdot\vec{K}}{\epsilon(\vec{K})[\epsilon(\vec{K})+M]}\vec{K}\ ,\end{aligned} \qquad (2.14)$$

the results are shown in Table I.

III. HEURISTIC HAMILTONIAN

From Table I and Ref. 7, the form of the heuristic Hamiltonian for the interaction of a Dirac field with scalar, vector, and pseudoscalar fields is

$$\begin{aligned}H&=H_D\{\chi_+,\chi_-\}+H_{S,P,V}\{a\}+H_{I,S,\mathrm{appr}}\{\chi_+,\chi_-,a_S\}\\ &\quad+H_{I,V,\mathrm{appr}}\{\chi_+,\chi_-,a_L,a_T\}+H_{I,P,\mathrm{appr}}\{\chi_+,\chi_-,a_P\},\end{aligned} \qquad (3.1)$$

TABLE I. Leading terms in the momentum transfer k.

	Y	$\begin{pmatrix} Y_{++} & Y_{+-} \\ Y_{-+} & Y_{--} \end{pmatrix}$
Scalar	β	$\frac{1}{\epsilon(\vec{K})}\begin{pmatrix} M & -\vec{\sigma}\cdot\vec{K} \\ -\vec{\sigma}\cdot\vec{K} & M \end{pmatrix}$
Pseudoscalar	ρ_2	$\frac{i}{2\epsilon(\vec{K})}\begin{pmatrix} \vec{\sigma}\cdot\vec{k}_{\text{red}} & -2\epsilon(\vec{K}) \\ 2\epsilon(\vec{K}) & \vec{\sigma}\cdot\vec{k}_{\text{red}} \end{pmatrix}$
Vector	1	$\frac{1}{2\epsilon(\vec{K})}\begin{pmatrix} 2\epsilon(\vec{K}) & \vec{\sigma}\cdot\vec{k}_{\text{red}} \\ -\vec{\sigma}\cdot\vec{k}_{\text{red}} & -2\epsilon(\vec{K}) \end{pmatrix}$
	$\vec{\alpha}$	$\frac{1}{\epsilon(\vec{K})}\begin{pmatrix} \vec{K} & \epsilon(\vec{K})\vec{\sigma}_{\text{red}} \\ \epsilon(\vec{K})\sigma_{\text{red}} & \vec{K} \end{pmatrix}$
Pseudovector	ρ_1	$\frac{1}{\epsilon(\vec{K})}\begin{pmatrix} \vec{\sigma}\cdot\vec{K} & M \\ M & -\vec{\sigma}\cdot\vec{K} \end{pmatrix}$
	$\rho_1\vec{\alpha}$	$\frac{1}{\epsilon(\vec{K})}\begin{pmatrix} M\vec{\sigma}+\epsilon(\vec{K})(\vec{\sigma}-\vec{\sigma}_{\text{red}}) & i\vec{\sigma}\times\vec{K} \\ -i\vec{\sigma}\times\vec{K} & M\vec{\sigma}+\epsilon(\vec{K})(\vec{\sigma}-\vec{\sigma}_{\text{red}}) \end{pmatrix}$

with H_D given by Eq. (2.9),

$$\begin{aligned} H_{S,P,V}\{a\} &= \int \omega_S(\vec{p})a_S^\dagger(\vec{p})a_S(\vec{p})d\vec{p} + \int \omega_P(\vec{p})a_P^\dagger(\vec{p})a_P(\vec{p})d\vec{p} \\ &\quad + \int \omega_V(\vec{p})\left[a_L^\dagger(\vec{p})a_L(\vec{p}) + \sum_{i=1}^{2} a_{Ti}^\dagger(\vec{p})a_{Ti}(\vec{p})\right]d\vec{p}\,, \\ H_{I,S,\text{appr}} &= -g_S \int J_{\beta,\text{appr}}(\vec{r})\phi_S(\vec{r})d\vec{r}, \\ H_{I,P,\text{appr}} &= -g_P \int J_{\rho 2,\text{appr}}(\vec{r})\phi_P(\vec{r})d\vec{r}\,, \\ H_{I,V,\text{appr}} &= g_V\int J_{1,\text{appr}}(\vec{r})\frac{1}{\nabla^2}[\nabla\cdot\vec{\Pi}_L(\vec{r}) - \tfrac{1}{2}g_V J_{1,\text{appr}}(\vec{r})]d\vec{r} \\ &\quad - g_V \int \vec{J}_{\vec{\alpha},\text{appr}}(\vec{r})\cdot\vec{V}_T(\vec{r})d\vec{r}\,. \end{aligned} \tag{3.2}$$

The relation of the fields to the creation and annihilation operators is

$$\begin{aligned} \phi_{S\text{ or }P}(\vec{r}) &= \int \frac{e^{i\vec{k}\cdot\vec{r}}}{[16\pi^3\omega_{S\text{ or }P}(\vec{k})]^{1/2}} \\ &\quad \times[a_{S\text{ or }P}(\vec{k}) + a^\dagger_{S\text{ or }P}(-\vec{k})]d\vec{k}\,, \\ \vec{V}_T(\vec{r}) &= \sum_{i=1}^{2}\int \frac{e^{i\vec{k}\cdot\vec{r}}}{[16\pi^3\omega_V(\vec{k})]^{1/2}}\,\hat{\eta}_i(\vec{k}) \\ &\quad \times[a_{Ti}(\vec{k}) + a^\dagger_{Ti}(-\vec{k})]d\vec{k}\,, \\ \frac{1}{\nabla^2}\nabla\cdot\vec{\Pi}_L(\vec{r}) &= -\int \frac{m\,e^{i\vec{k}\cdot\vec{r}}}{k[16\pi^3\omega_V(\vec{k})]^{1/2}}[a_L(\vec{k}) + a_L^\dagger(-\vec{k})]\,, \end{aligned} \tag{3.3}$$

where

$$\omega_I(\vec{k}) = (k^2 + m_I{}^2)^{1/2}\,, \tag{3.4}$$

and the $\hat{\eta}_i(k)$ are orthogonal transverse polarization unit vectors that satisfy

$$\hat{\eta}_i^*(\vec{k}) = \hat{\eta}_i(-\vec{k}). \tag{3.5}$$

The Hamiltonian of Eq. (3.1) has no ultraviolet divergences and can be treated in strong coupling by using variational methods; Ref. 7 gives the techniques for the case of the longitudinal vector field only.

The next two sections of this paper are devoted to the cases of interaction with a scalar field and a transverse vector field, respectively. The heuristic Hamiltonian for an interaction with a pseudoscalar field will be discussed in a future paper.

IV. SCALAR FIELD INTERACTION

The heuristic Hamiltonian for interacting Dirac and scalar fields is

$$H = H_D\{\chi_+,\chi_-\} + H_S\{a\} + H_{I,S,\text{appr}}\{\chi_+,\chi_-,a\}\,, \tag{4.1}$$

where H_D is given in Eq. (2.9),

$$\begin{aligned} H_S\{a\} &= \int \omega(\vec{k})a^\dagger(\vec{k})a(\vec{k})d\vec{k}\,, \\ \omega(\vec{k}) &= (k^2 + m^2)^{1/2}, \end{aligned} \tag{4.2}$$

and $H_{I,S,\text{appr}}$ is given by Eqs. (3.2) and (3.3) and is equal to

$$H_{I,S,\mathrm{appr}} = -\frac{\gamma^{1/2}}{2\pi}\int \frac{\delta(\vec{k}-\vec{p}+\vec{q})}{\omega^{1/2}(\vec{k})}\frac{M}{\epsilon(\vec{K})} \times[\tilde{\chi}_+^\dagger(\vec{p})\tilde{\chi}_+(\vec{q})+\tilde{\chi}_-^\dagger(-\vec{q})\tilde{\chi}_-(-\vec{p})] \times[a(\vec{k})+a^\dagger(-\vec{k})]d\vec{k}\,d\vec{p}\,d\vec{q}\,, \tag{4.3}$$

where

$$\gamma = g^2/4\pi\;. \tag{4.4}$$

The energy and effective mass of the single fermion can be found by using the localized-state methods of Refs. 10 and 7. The single-particle localized state is

$$|\vec{x};\phi,f\rangle = \int \tilde{\chi}_+^\dagger(\vec{r})f(\vec{r}-\vec{x})d\vec{r}\,W_{\vec{x}}^\dagger\{\phi\}|\Omega\rangle,$$
$$\tilde{\chi}_+(\vec{r})|\Omega\rangle = a(\vec{k})|\Omega\rangle = 0, \tag{4.5}$$
$$W_{\vec{x}}^\dagger\{\phi\} = \exp\left\{\int[\phi(\vec{k})a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}} - \tfrac{1}{2}|\phi(\vec{k})|^2]d\vec{k}\right\}.$$

Then the normalization and Hamiltonian matrix elements are

$$D(\vec{x}) = \langle\vec{x};\phi,f|\vec{0};\phi,f\rangle = D_F(\vec{x})D_B(\vec{x}),$$
$$A(\vec{x}) = \langle\vec{x};\phi,f|H|\vec{0};\phi,f\rangle,$$
$$D_F(\vec{x}) = \int e^{i\vec{p}\cdot\vec{x}}|\tilde{f}(\vec{p})|^2 d\vec{p},$$
$$D_B(\vec{x}) = \exp\left[-\int|\phi(\vec{k})|^2(1-e^{i\vec{k}\cdot\vec{x}})d\vec{k}\right], \tag{4.6}$$
$$A(\vec{x}) = D(\vec{x})\int\omega(\vec{k})|\phi(\vec{k})|^2 e^{i\vec{k}\cdot\vec{x}}d\vec{k} + D_B(\vec{x}) \times\left\{\int\epsilon(\vec{p})|\tilde{f}(\vec{p})|^2 e^{i\vec{p}\cdot\vec{x}}d\vec{p} - \frac{\gamma^{1/2}}{2\pi}\int\frac{\delta(\vec{k}-\vec{p}+\vec{q})}{\omega^{1/2}(\vec{k})}\frac{M}{\epsilon(\vec{K})}\tilde{f}^\dagger(\vec{p})\tilde{f}(\vec{q})[\phi(\vec{k})e^{i\vec{p}\cdot\vec{x}}+\phi^*(-\vec{k})e^{i\vec{q}\cdot\vec{x}}]d\vec{p}\,d\vec{q}\,d\vec{k}\right\}.$$

The strong-coupling localized-state approximation is obtained by choosing ϕ and f to minimize the localized-state energy functional F_{LS},

$$F_{\mathrm{LS}}\{\phi,f\} = A(\vec{0})/D(\vec{0}). \tag{4.7}$$

Variation of ϕ gives

$$\phi_{\mathrm{LS}}(\vec{k}) = \frac{\gamma^{1/2}}{2\pi\omega^{3/2}(\vec{k})}\tilde{\rho}_{\mathrm{LS}}(\vec{k}),$$
$$\tilde{\rho}_{\mathrm{LS}}(\vec{k}) = \int\delta(\vec{k}-\vec{p}+\vec{q})\frac{M}{\epsilon((\vec{p}+\vec{q})/2)}\tilde{f}^\dagger(\vec{p})\tilde{f}(\vec{q})\,d\vec{p}\,d\vec{q}, \tag{4.8}$$

and elimination of ϕ results in

$$F_{\mathrm{LS}}\{f\} = \int\epsilon(\vec{p})|\tilde{f}(\vec{p})|^2 d\vec{p} - \frac{\gamma}{2}\int\frac{e^{-m|\vec{r}-\vec{s}|}}{|\vec{r}-\vec{s}|}d(\vec{x})d(\vec{y})f^\dagger(\vec{r}+\tfrac{1}{2}\vec{x})f(\vec{r}-\tfrac{1}{2}\vec{x})f^\dagger(\vec{s}-\tfrac{1}{2}\vec{y})f(\vec{s}+\tfrac{1}{2}\vec{y})d\vec{r}\,d\vec{s}\,d\vec{x}\,d\vec{y},$$
$$d(\vec{x}) = (2\pi)^{-3}\int e^{i\vec{p}\cdot\vec{x}}\frac{M}{\epsilon(\vec{p})}d\vec{p}\;. \tag{4.9}$$

If the momenta that occur in $\tilde{f}(\vec{p})$ are small compared to M, then $d(\vec{x})$ can be replaced by $\delta(\vec{x})$ and F_{LS} becomes

$$F_{\mathrm{LS,NR}}\{f\} = \int\left(M+\frac{p^2}{2M}\right)|\tilde{f}(\vec{p})|^2 d\vec{p} - \frac{\gamma}{2}\int|\tilde{f}(\vec{r})|^2\frac{e^{-m|\vec{r}-\vec{s}|}}{|\vec{r}-\vec{s}|}|\tilde{f}(\vec{s})|^2 d\vec{r}\,d\vec{s}\;. \tag{4.10}$$

Some numerical results for this energy functional are given in Ref. 11. The effective mass in this approximation is computed as in Ref. 7. The result is

$$M^*_{\mathrm{LS}} = M + \frac{\gamma}{6\pi^2}\int\frac{k^2|\tilde{\rho}_{\mathrm{LS}}(\vec{k})|^2}{\omega^4(\vec{k})}d\vec{k}\;. \tag{4.11}$$

In the nonrelativistic approximation,

$$\rho_{\text{LS,NR}}(\vec{k}) = \int e^{-i\vec{k}\cdot\vec{r}} |\tilde{f}(\vec{r})|^2 d\vec{r} , \tag{4.12}$$

and then $M^*_{\text{LS,NR}}$ is given in Ref. 11.

Another approximation suitable for weak coupling is obtained by using the TLS energy functional F_{TLS},

$$F_{\text{TLS}} = \int A(\vec{x})d\vec{x} \Big/ \int D(\vec{x})d\vec{x} , \tag{4.13}$$

and assuming that the range of $f(\vec{r})$ is very short compared to that of $D_B(\vec{r})$. Then $f(\vec{r})$ can be replaced by a δ function and $D_B(\vec{r})$ by $D_B(0) = 1$ in most places. The interaction term in $\int A$ becomes

$$-\frac{\gamma^{1/2}}{2\pi} \int \frac{M}{\omega^{1/2}(\vec{k})} \left[\frac{\delta(\vec{k}+\vec{q})}{\epsilon(\vec{q}/2)} \phi(\vec{k}) + \frac{\delta(\vec{k}-\vec{q})}{\epsilon(\vec{q}/2)} \phi^*(-\vec{k})\right] d\vec{k}\, d\vec{q} = -\frac{\gamma^{1/2}}{2\pi} \int \frac{M}{\omega^{1/2}(\vec{k})\epsilon(\vec{k}/2)} [\phi(\vec{k}) + \phi^*(-\vec{k})] d\vec{k} , \tag{4.14}$$

and the corresponding energy functional is

$$F_\delta\{\phi\} = (2\pi)^{-3} \int \epsilon(\vec{p}) e^{i\vec{p}\cdot\vec{x}} D_B(\vec{x})\, d\vec{p}\, d\vec{x} + \int \omega(\vec{k}) |\phi(\vec{k})|^2 d\vec{k} - \frac{\gamma^{1/2}}{2\pi} M \int \frac{d\vec{k}}{\omega^{1/2}(\vec{k})\epsilon(\vec{k}/2)} [\phi(\vec{k}) + \phi^*(-\vec{k})] . \tag{4.15}$$

In general, F_δ can be minimized numerically to obtain ϕ and the energy. For γ small, it is clear that ϕ is of order $\gamma^{1/2}$, so that to get the term of order γ in F_δ, D_B can be expanded and

$$\begin{aligned}(2\pi)^{-3} \int e^{i\vec{p}\cdot\vec{x}} D_B(\vec{x}) d\vec{x} &\approx (2\pi)^{-3} \int e^{i\vec{p}\cdot\vec{x}} \left[1 - \int |\phi(\vec{k})|^2 (1 - e^{i\vec{k}\cdot\vec{x}}) d\vec{k}\right] d\vec{x} \\ &= \delta(\vec{p}) \left(1 - \int |\phi(\vec{k})|^2 d\vec{k}\right) + \int \delta(\vec{p}+\vec{k}) |\phi(\vec{k})|^2 d\vec{k} .\end{aligned} \tag{4.16}$$

Then for weak coupling the functional becomes

$$F_{\text{WC}}\{\phi\} \approx M + \int [\omega(\vec{k}) + \epsilon(-\vec{k}) - M] |\phi(\vec{k})|^2 d\vec{k} - \frac{\gamma^{1/2} M}{2\pi} \int \frac{d\vec{k}}{\omega^{1/2}(\vec{k})\epsilon(\vec{k}/2)} [\phi(\vec{k}) + \phi(-\vec{k})]\, d\vec{k} , \tag{4.17}$$

and hence

$$\phi_{\text{WC}}(\vec{k}) = \frac{\gamma^{1/2} M}{2\pi\omega^{1/2}(\vec{k})\epsilon(\vec{k}/2)[\omega(\vec{k}) + \epsilon(-\vec{k}) - M]} , \tag{4.18}$$

$$E_{\text{WC}} = M - \frac{\gamma M^2}{4\pi^2} \int \frac{d\vec{k}}{\omega(\vec{k})\epsilon^2(\vec{k}/2)[\omega(\vec{k}) + \epsilon(-\vec{k}) - M]} .$$

The effective mass is

$$M^*_{\text{WC}} = M + \frac{\gamma}{6\pi^2} \int \frac{k^2 d\vec{k}}{\omega(\vec{k})\epsilon^2(\vec{k}/2)[\omega(\vec{k}) + \epsilon(-\vec{k}) - M]^3} . \tag{4.19}$$

For infinite fermion matter, with momentum states filled up to the Fermi momentum Q, the energy per particle is easily obtained for the heuristic Hamiltonian. The result is

$$\frac{E}{N} = \epsilon_F(Q) - \frac{g^2}{2m^2} \frac{\rho_S^2(Q)}{\rho(Q)} , \tag{4.20}$$

where

$$\begin{aligned}\epsilon_F(Q) &= \int \epsilon(\vec{p}) d_F\vec{p} \Big/ \int d_F\vec{p} , \\ \rho(Q) &= (2\pi)^{-3} 2n \int d_F\vec{p} , \\ \rho_S(Q) &= (2\pi)^{-3} 2n \int \frac{M}{\epsilon(\vec{p})} d_F\vec{p} .\end{aligned} \tag{4.21}$$

Here n is the number of kinds of fermions; $\int d_F\vec{p}$ is an integration over the fermi sea. The result (4.20) is like the one obtained by Walecka[12]; it differs in that M in $\epsilon(\vec{p})$ is not the same as M^*. Also, the binding energy per particle is

$$E/N - E_1 , \tag{4.22}$$

where E_1 is the single-particle energy, which is approximated in Eqs. (4.10), (4.15), and (4.18). The constant E_1 is independent of the Fermi momentum (or density), so that the behavior of the binding energy as a function of density can be obtained without a value of E_1. As was noted in Ref. 12 the function in (4.20) or (4.22) starts with positive slope at $Q = 0$, then turns down, if g^2/m^2 is not too small, and finally turns back upward for large values of Q and becomes asymptotically equal to $\epsilon_F(Q)$; that is, the energy per particle has a minimum for some value of Q.

V. TRANSVERSE VECTOR FIELD INTERACTION

From Table I, the heuristic Hamiltonian for interacting Dirac and transverse vector fields is

$$H=H_D\{\chi_+,\chi_-\}+H_{\mathrm{TV}}\{a\}+H_{I,\mathrm{TV,appr}}\{a,\chi_+,\chi_-\},$$
$$H_{\mathrm{TV}}\{a\}=\int\omega(\vec{k})\sum_{i=1}^{2}a_i^\dagger(\vec{k})a_i(\vec{k})d\vec{k},\tag{5.1}$$
$$H_{I,\mathrm{TV,appr}}\{a,\chi_+,\chi_-\}=-\frac{\gamma^{1/2}}{2\pi}\int\frac{\delta(\vec{k}-\vec{p}+\vec{q})}{\omega^{1/2}(\vec{k})}\sum_{i=1}^{2}\frac{\vec{K}\cdot\hat{\eta}_i(\vec{k})}{\epsilon(\vec{K})}[\chi_+^\dagger(\vec{p})\chi_+(\vec{q})+\chi_-^\dagger(-\vec{q})\chi_-(-\vec{p})][a_i(\vec{k})+a_i^\dagger(-\vec{k})]\,d\vec{k}\,d\vec{p}\,d\vec{q}\,,$$

where the $\hat{\eta}_i(\vec{k})$ are transverse polarization vectors:

$$\vec{k}\cdot\hat{\eta}_i(\vec{k})=0\,,$$
$$\hat{\eta}_i^*(\vec{k})=\hat{\eta}_i(-\vec{k})\,,\tag{5.2}$$
$$\hat{\eta}_i(\vec{k})\cdot\hat{\eta}_j(\vec{k})=\delta_{i,j}\,.$$

Again

$$\gamma=g^2/4\pi\,.\tag{5.3}$$

Now in the localized-state approximation,

$$\phi_{i,\mathrm{LS}}(\vec{k})=\frac{\gamma^{1/2}}{2\pi\omega^{3/2}(\vec{k})}\tilde{\rho}_{i,\mathrm{LS}}(\vec{k})\,,\tag{5.4}$$
$$\tilde{\rho}_{i,\mathrm{LS}}(\vec{k})=\hat{\eta}_i(\vec{k})\cdot\int\delta(\vec{k}-\vec{p}+\vec{q})\frac{\vec{p}+\vec{q}}{2\epsilon((\vec{p}+\vec{q})/2)}\times\tilde{f}^\dagger(\vec{p})\tilde{f}(\vec{q})d\vec{p}\,d\vec{q}\,.$$

If f is an S wave, then $\tilde{\rho}_{i,\mathrm{LS}}(\vec{k})$ must be zero, since the result of the integration can only be a vector along $\vec{k}$. Thus, the transverse vector field makes no contribution to the single-particle energy or effective mass in the LS approximation.

In the TLS approximation with the range of $f(\vec{r})$ taken to be short compared to that of $D_B(\vec{r})$, the transverse vector field again makes no contribution to the single-particle energy or effective mass. Similarly, for infinite fermion matter, there is no contribution to the energy per particle in the Fermi-sea approximation.

VI. INFINITE MATTER WITH SCALAR AND VECTOR FIELDS

Sections IV and V can be combined with the result of Ref. 7 to give the following expression for the energy per particle for fermions interacting via scalar and vector fields in the Fermi-sea approximation:

$$\frac{E}{N}-E_1=\epsilon_F(Q)-E_1-\frac{g_S^{\,2}}{2m_S^{\,2}}\frac{\rho_S^{\,2}(Q)}{\rho(Q)}+\frac{g_V^{\,2}}{2m_V^{\,2}}\rho(Q)-\frac{3g_V^{\,2}}{32\pi^2}Q\,,\tag{6.1}$$

where

$$\rho(Q)=\frac{n}{3\pi^2}Q^3,$$
$$\rho_S(Q)=\frac{n}{\pi^2}\int_0^Q\frac{M}{(p^2+M^2)^{1/2}}p^2dp\,,\tag{6.2}$$
$$\epsilon_F(Q)=\frac{3}{Q^3}\int_0^Q(p^2+M^2)^{1/2}p^2dp\,,$$

and n is the number of kinds of fermions present; the matter is assumed to be symmetric, that is, equal densities of the n spin-$\frac{1}{2}$ constituents. Equation (6.1) differs from the E/N given by Walecka[12] in two respects: the subtraction of E_1 and the last term proportional to K, of which the latter is much more important. Without the term in Q, saturation occurs when ρ_S/ρ starts to become small *or* because $g_S^{\,2}/m_S^{\,2}$ and $g_V^{\,2}/m_V^{\,2}$ are both large and nearly balance. The first alternative would require Q to be a significant fraction of M and is therefore excluded for nuclear matter. The very delicate balance between $g_S^{\,2}/m_S^{\,2}$ and $S_V^{\,2}/m_V^{\,2}$ required by the second alternative is always possible, but makes the computation of higher-order effects imperative. With the term in Q, it is possible to neglect relativistic effects and set

$$\rho_S\approx\rho,$$
$$\epsilon_F(Q)\approx\frac{3Q^2}{10M}\,.\tag{6.3}$$

Then

$$\frac{E}{N}-E_1\approx\frac{3Q^2}{10M}-E_1+\left(\frac{g_V^{\,2}}{m_V^{\,2}}-\frac{g_S^{\,2}}{m_S^{\,2}}\right)\frac{nQ^3}{6\pi^2}-\frac{3g_V^{\,2}}{32\pi^2}Q\,.\tag{6.4}$$

As long as $g_V^{\,2}/m_V^{\,2}>g_S^{\,2}/m_S^{\,2}$, this expression always exhibits saturation, provided, of course, that the minimum of $E/N-E_1$ occurs for a negative value of $E/N-E_1$.

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[1]The notation is that of Bjorken and Drell with the modification that the pseudoscalar operator is written ρ_2. All Dirac matrices are taken only in the standard representation:

$$\beta=\rho_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \vec{\alpha}=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix},\quad \rho_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},$$

$$\gamma_5=\rho_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \vec{\Sigma}=\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}=\rho_1\vec{\alpha}.$$

The generalization of the results to other representations is straightforward.

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