Renormalization of the energy-momentum tensor in ϕ^4 theory*

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The problem of the improvement term of the energy-momentum tensor $\theta_{\mu\nu}$ in ϕ^4 theory is reconsidered. Renormalization-group methods (due to 't Hooft) with dimensional regularization are used. A unique finite improvement coefficient, depending only on the regulator parameter, is shown to renormalize $\theta_{\mu\nu}$. This $\theta_{\mu\nu}$ has a soft trace at a fixed point. It coincides with the $\theta_{\mu\nu}$ suggested by conformal ideas and by Callan, Coleman, and Jackiw (CCJ), if summation of the perturbation theory divergences is allowed. But order by order, the CCJ $\theta_{\mu\nu}$ is finite only up to the three-loop level, and not beyond, even if it is correctly dimensionally regularized.

I. INTRODUCTION

This paper aims at a definitive treatment of the renormalization of the energy-momentum tensor in ϕ^4 theory in four dimensions. In view of the literature (e.g. Refs. 1–6) on the subject, a further paper might appear superfluous. However, as we will see, this is not so.

In this paper we only investigate ϕ^4 theory. It is the simplest theory to have problems[1] with the "improvement term"; these are the problems of interest. Further, as Freedman and Weinberg[4] explain, the treatment of these problems is essentially the same in more complicated theories.

The bare Lagrangian is, as usual,

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi_0)^2 - \tfrac{1}{2}m_0^2\phi_0^2 - g_0\phi_0^4/4! \,. \tag{1.1}$$

Standard manipulations[7] derive a canonical energy-momentum tensor $T_{\mu\nu}$ as the current for translations in space-time:

$$T_{\mu\nu} = \partial_\mu\phi_0\partial_\nu\phi_0 - g_{\mu\nu}[\tfrac{1}{2}(\partial\phi_0)^2 - \tfrac{1}{2}m_0^2\phi_0^2 - g_0\phi_0^4/4!\,] \,. \tag{1.2}$$

One may add to this any quantity whose divergence is zero and which does not contribute to the Ward identities. The only such improvement term we need to consider is proportional to $(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\phi^2$. Then we have an improved energy-momentum tensor

$$\theta_{\mu\nu} \equiv T_{\mu\nu} - H_0(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\,\phi_0^2 \,. \tag{1.3}$$

We wish to find a renormalized energy-momentum tensor; that is, we wish to make finite the Green's functions of the renormalized field $\phi \equiv Z^{-1/2}\phi_0$ with an insertion of this tensor. Coupling and mass renormalizations in the Lagrangian are carried over into $T_{\mu\nu}$ of course—see (1.2).

Green's functions of $T_{\mu\nu}$ diverge, and a nonzero improvement term is necessary[1] to make them finite. Moreover, the Ward identities and power counting show[1-4] that a suitable choice of H_0 is sufficient[8] to give a renormalized energy-momentum tensor. Thus renormalization has preserved the concept of an energy-momentum tensor, for it is always allowed to add to $T_{\mu\nu}$ a term whose divergence is identically zero.

The reasons for wanting to renormalize $T_{\mu\nu}$ are as follows: There is of course the idea that the energy-momentum tensor is physical and hence finite. More important is that through this tensor gravity couples to matter.[9] So to require that gravitational interactions be finite requires the tensor to be renormalized. Now Green's functions coupling several gravitons to matter are finite if those for one graviton are finite.[10] So we only need to deal with the case of one insertion of $\theta_{\mu\nu}$. Then renormalizability of $\theta_{\mu\nu}$ gives renormalizability of the combined theory of gravity and matter, with gravity treated to lowest order and the self-interactions of matter to all orders. Renormalization of graviton loops is a separate and very much harder problem.[11]

In general relativity, the energy-momentum tensor is[9] $2(-g)^{-1/2}\delta\mathcal{S}/\delta g^{\mu\nu}$, where $\mathcal{S}$ is the action and g the determinant of $g_{\mu\nu}$. For ϕ^4 theory

$$\mathcal{S} = \int d^4x\sqrt{-g}\,(\mathcal{L} - \tfrac{1}{2}H_0R\phi_0^2) \,. \tag{1.4}$$

The apparently nonminimal term $-\tfrac{1}{2}H_0R\phi_0^2$ is zero in flat space, but its functional derivative by $g_{\mu\nu}$ is not. In fact[1-4] it gives the improvement term in (1.3). Note that the term $-\tfrac{1}{2}H_0R\phi_0^2$ is not necessarily nonminimal. One might say[12] that the minimal way to go from flat to curved space is not for the kinetic energy term to be $\tfrac{1}{2}(\partial\phi_0)^2$ but for it to be the conformally invariant $\tfrac{1}{2}(\partial\phi_0)^2 - \tfrac{1}{12}R\phi_0^2$. However, for the present, we leave H_0 undetermined. We treat it as a coupling constant for one of the interactions of gravity and matter.

Requiring renormalizability of the graviton-matter coupling has forced the introduction of an extra counterterm not necessarily present in one's original conception of the theory. In such a situation one normally argues that there is an arbitrariness in the renormalization prescription. To be able to compensate this, a renormalized parameter corresponding to the counterterm is needed. (The standard example is the quantum electrodynamics of a scalar field, where renormalizability necessitates a four-point coupling for the scalar.) These arguments apply to the case in hand. We write

$$H_0 Z = G + h_R Z_4 , \tag{1.5}$$

where Z is the wave-function renormalization constant, G is some counterterm that leads to a finite energy-momentum tensor, Z_4 is the renormalization factor for insertions of ϕ^2, and h_R is an arbitrary finite parameter, termed the finite improvement coefficient. The presence of the factor Z in Eq. (1.5) comes from expressing the theory in terms of the renormalized field ϕ instead of the bare field ϕ_0.

Since ϕ^2 is multiplicatively renormalizable, the only necessary dependence on h_R is as in (1.5) even though h_R is dimensionless. (This of course could change in the presence of graviton loops, but we ignore these.)

It is generally hoped that, given a theory of matter in Minkowski space, it can be combined with general relativity with no new parameters. Introduction of h_R is therefore considered undesirable. (But one might cite it as an example of how constrained the structure of relativistic quantum theories is.) So we look for some natural criterion for fixing H_0 or h_R. There seem to exist four such criteria. One of them was formulated[12] independently of considerations of renormalizability. We prove that they all give the same $\theta_{\mu\nu}$ and force it to be finite.[13] A fifth criterion applies only at a fixed point; it will be satisfied by the $\theta_{\mu\nu}$ given by the other criteria.

The criteria are as follows:

(1) The kinetic energy in (1.4) must be conformal invariant.[12] In n-dimensional space-time this gives $H_0 = \frac{1}{4}(n-2)/(n-1)$.

(2) θ^μ_μ must be a soft operator,[1,2] *assuming, incorrectly if $n=4$, the validity of canonical field theory.* Equivalently scale invariance must be softly broken. This also gives $H_0 = \frac{1}{4}(n-2)/(n-1)$.

(3) The finite improvement program.[1-4] Criteria (1) and (2) do not result in a finite $\theta_{\mu\nu}$, in perturbation theory.[15] So choose, if possible, H_0 as a function, finite at $n=4$, of n and the renormalized coupling, g_R, and mass, m_R, so that $\theta_{\mu\nu}$ is finite. Since Z_4/Z is divergent in perturbation theory, the freedom to arbitrarily change the renormalized improvement coefficient, h_R, is irrelevant here.

(4) The renormalization-group (RG) covariant $\theta_{\mu\nu}$. Choose h_R such that the renormalization-group equations for $\theta_{\mu\nu}$ have no $\partial/\partial h_R$ term. Thus changes of the renormalization mass μ do not need a change in h_R to compensate. This criterion is new in the context of $\theta_{\mu\nu}$. But note the following: (a) a zero-mass theory can be defined[16] as one that satisfies a homogeneous Callan-Symanzik equation; (b) Jegerlehner[17] has used a similar proposal to renormalize $(\phi^6)_4$ theory.

(5) At a fixed point of the renormalization group, require[6] *θ^μ_μ to be soft.* Only[6] at a fixed point can such a requirement be imposed.

As will be proved in the next sections the first four criteria are equivalent at $n=4$ and give a finite $\theta_{\mu\nu}$, contrary to previous[3,4] expectations. Further, the $\theta_{\mu\nu}$ so defined will be shown to have a soft trace at a fixed point. The first two criteria only work if perturbation theory is summed.[13] We show that if $H_0 = \text{constant} + O(n-4)$, then the variations of H_0 at $n \neq 4$ are irrelevant, and the constant must be $\frac{1}{6}$, in agreement with criteria (1) and (2). Despite the problems,[13] the proof can be regarded as good heuristic support for saying that the $\theta_{\mu\nu}$ we define is the correct minimal one.

Unfortunately, the results do not agree if[15] $n \neq 4$. This is not a disaster since criteria (3) and (4) are formulated to deal with the divergences at $n=4$; there is no reason to apply them if $n \neq 4$.

This paper is organized as follows: Section II sets up the renormalization group for $\theta_{\mu\nu}$, and defines the RG-covariant $\theta_{\mu\nu}$, using 't Hooft's[14] methods. In Sec. III equivalence to criteria (1) and (2) is proved. In Sec. IV the finite improvement program is shown to succeed, to be unique, and to agree with the RG-covariant definition. The resulting H_0 is independent of g_R and m_R, and equals $\frac{1}{6} + O(n-4)$. But in Sec. V we will see that this value is *not* $\frac{1}{4}(n-2)/(n-1)$, which up to the three-loop level appears to work.

In Sec. VI, $\theta_{\mu\nu}$, as defined in Secs. II–V, is shown to have a soft trace at a fixed point of the renormalization group. Thus it agrees with Schroer's[6] definition. Finally, Sec. VII contains some concluding remarks.

Heavy use is made throughout of dimensional renormalization, that is, dimensional regularization[18] with counterterms defined[14] to be sums of pure poles at $n=4$.

II. RG-COVARIANT $\theta_{\mu\nu}$

Consider the energy-momentum tensor

$$\theta_{\mu\nu} = Z\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}[\tfrac{1}{2}Z(\partial\phi)^2 - \tfrac{1}{2}m_B{}^2\phi^2 - g_B\phi^4/4!\,] - H_B(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\phi^2 , \tag{2.1}$$

now written in terms of the renormalized field $\phi \equiv Z^{-1/2}\phi_0$. The counterterms in Z, m_B^2, and g_B are defined[14] as sums of pure poles at $n=4$. In the case of g_B there is also a factor μ^{4-n} to soak up the n dependence of its mass dimension, where μ is the unit of mass.[14] Also we choose to write

$$H_B = G(g_R, n) + [k(g_R, m_R^2/\mu^2) + h_R] Z_m(g_R, n) . \tag{2.2}$$

Here, G is a sum of pure poles with no constant term, h_R is the renormalized improvement coefficient, $Z_m = m_B^2/m_R^2$, and k is a *finite* function to be chosen. Finiteness of $Z_m\phi^2$ [and hence of $\theta_{\mu\nu}$ as defined by Eqs. (2.1) and (2.2) for any k] follows from the arguments of Ref. 19. So also does the fact that g_B, Z, Z_m, and G do not depend on m_R.

We will consider the Green's functions

$$G_N(p_i) = \left\langle T \prod_{i=1}^{N} \tilde{\phi}(p_i) \right\rangle , \tag{2.3a}$$

$$G_{N\mu\nu}(p_i; q) = \left\langle T \tilde{\theta}_{\mu\nu}(q) \prod \tilde{\phi}(p_i) \right\rangle , \tag{2.3b}$$

where the tildes imply a Fourier transform into momentum space.

The derivation of renormalization-group equations (RGE's) follows the usual steps.[20] For G_N we have the standard RGE

$$0 = \left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g_R} - \gamma_m m_R^2 \frac{\partial}{\partial m_R^2} + \tfrac{1}{2} N \gamma \right) \times G_N(g_R, m_R^2, \mu, n) . \tag{2.4}$$

Here, dependence on momenta has been suppressed, and

$$\beta = (n-4) g_B / (g_B' - 2Z'Z^{-1} g_B) , \tag{2.5}$$

$$\gamma = \beta Z' Z^{-1} , \tag{2.6}$$

$$\gamma_m = \beta Z_m' Z_m^{-1} - \gamma , \tag{2.7}$$

where the prime indicates differentiation with respect to g_R.

Now β, γ, and γ_m are finite and g_B, Z, and Z_m are sums of poles. So Eqs. (2.5) to (2.7) imply[14] that γ and γ_m are independent of n, while

$$\beta = (n-4) g_R + \bar{\beta}(g_R) . \tag{2.8}$$

These equations also imply[14] relations among the various pole terms. Moreover, these equations can be treated as differential equations determining the renormalization "constants" g_B, Z, and Z_m in terms of the RG functions β, γ, and γ_m. If the perturbation series for these quantities are assumed summable, then[14] Eqs. (2.5) to (2.7) provide considerable information about the renormalization constants when $n \sim 4$. This technique will be crucial in Sec. III.

The derivation of the RGE for $G_{N\mu\nu}$ is exactly analogous to that of the RGE for G_N. We get

$$0 = \left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g_R} - \gamma_m m_R^2 \frac{\partial}{\partial m_R^2} + \tfrac{1}{2} N \gamma - \delta \frac{\partial}{\partial h_R} \right) \times G_{N\mu\nu}(g_R, m_R^2, h_R, \mu, n) , \tag{2.9}$$

where

$$\delta(g_R, h_R, m_R^2/\mu^2, n) = Z_m^{-1} \left[\beta \frac{\partial}{\partial g_R} - (2+\gamma_m) m_R^2 \frac{\partial}{\partial m_R^2} - \gamma \right] \times [G + (h_R + k) Z_m] . \tag{2.10}$$

Both G_N and δ are to be regarded as functionals of k, and the $\partial/\partial g_R$, $\partial/\partial m_R^2$, and $\partial/\partial \mu$ in Eqs. (2.9) and (2.10) act on the dependence on these variables of k. The h_R and k dependence of δ can be exhibited:

$$\delta = \zeta(g_R) + \gamma_m h_R + \left[\beta \frac{\partial}{\partial g_R} - (2+\gamma_m) m_R^2 \frac{\partial}{\partial m_R^2} + \gamma_m \right] k , \tag{2.11}$$

where

$$\zeta = \beta Z_m^{-1} G' - \gamma Z_m^{-1} G . \tag{2.12}$$

Note that, as G is a sum of poles with no constant term, ζ is independent of n.

Now the RGE (2.9) for $G_{N\mu\nu}$ is just like the RGE (2.4) for G_N, except for the term $\delta\partial/\partial h_R$. If we *choose* k to satisfy

$$\left[\bar{\beta} \frac{\partial}{\partial g_R} - (2+\gamma_m) m_R^2 \frac{\partial}{\partial m_R^2} + \gamma_m \right] k = -\zeta , \tag{2.13}$$

then $\delta = \gamma_m h_R + O(n-4)$. So setting $h_R = 0$ will then make (2.9) just like (2.4) at $n=4$. One can say that, with $h_R = 0$, $\theta_{\mu\nu}$ transforms covariantly under the same renormalization group as G_N. The extra nonminimal improvement got by setting $h_R \neq 0$ transforms into itself exactly as an insertion of ϕ^2 does.

On the other hand, if k were not chosen according to Eq. (2.13), then a change in μ would be equivalent to the changes implied by the RGE (2.4) with, in addition, a change in h_R.

To obtain a unique solution of Eq. (2.13) boundary conditions need to be specified; the equation only determines what happens along individual characteristics. The obvious conditions, which we choose, are that k be a power series in g_R and that k not diverge as $m_R \to 0$. Now the lowest-order terms in $\bar{\beta}$ and γ_m are[21]

$$\bar{\beta} = 3g_R^2/(16\pi^2) + O(g_R^3) , \tag{2.14a}$$

$$\gamma_m = -g_R/(16\pi^2) + O(g_R^2) . \tag{2.14b}$$

Moreover, a one-loop calculation gives

$$G=\frac{g_R}{96\pi^2(n-4)}+O(g_R{}^2)\,, \tag{2.15}$$

so that by Eq. (2.12)

$$\zeta=g_R/(96\pi^2)+O(g_R{}^2)\,. \tag{2.16}$$

Then (2.13) has a unique solution satisfying the boundary conditions. It is independent of m_R—otherwise the first m_R-dependent term in k diverges like $\ln m_R$ when $m_R\to 0$.

In a more general theory, with more than one coupling constant, we will need an extra boundary condition for each extra coupling.[22] But in such a theory, it would be sensible to have a separate unit of mass for each coupling. For each there would be an RGE. This point probably merits investigation. Note that the usual application of the RG is to scaling properties of Green's functions. There one only needs the effect of a single, common scaling of each unit of mass. Here, however, we are interested in the arbitrariness of the renormalization procedure.

Now, given that $\partial k/\partial m_R=0$, the solution of Eq. (2.13) is

$$k(g_R)=\exp\left[-\int_c^{g_R}\frac{\gamma_m(g)}{\bar\beta(g)}dg\right]\left\{\text{const}-\int_d^{g_R}dg\,\frac{\zeta(g)}{\bar\beta(g)}\exp\left[\int_c^{g}\frac{\gamma_m(g')}{\bar\beta(g')}dg'\right]\right\}, \tag{2.17}$$

where c and d are arbitrary. Write

$$\begin{aligned}\bar\beta(g_R)&=g_R{}^2\tilde\beta(g_R)\,,\\ \gamma_m(g_R)&=g_R\tilde\gamma_m(g_R)\,,\\ \zeta(g_R)&=g_R\tilde\zeta(g_R)\,,\end{aligned} \tag{2.18}$$

so that $\tilde\beta(0)$, $\tilde\gamma_m(0)$, and $\tilde\zeta(0)$ are all nonzero. Then

$$k=\tfrac{1}{6}f(g_R)^{-1}-f(g_R)^{-1}g_R{}^{1/3}\times\int_0^{g_R}\frac{dg}{g^{4/3}}\left[\frac{\tilde\zeta(g)}{\tilde\beta(g)}f(g)-\tfrac{1}{18}\right], \tag{2.19}$$

where the boundary conditions have been used and

$$f(g_R)=\exp\int_0^{g_R}\frac{dg}{g}\left[\frac{\tilde\gamma_m(g)}{\tilde\beta(g)}-\frac{\tilde\gamma_m(0)}{\tilde\beta(0)}\right]. \tag{2.20}$$

Note that in lowest order $k=\frac{1}{6}$, agreeing with the original guess[1] for the improvement coefficient.

III. EQUALITY WITH CCJ

In this section we will prove that the RG-covariant $\theta_{\mu\nu}$ defined in Sec. II agrees at $n=4$ with that obtained by taking $H_0=\frac{1}{6}+o((n-4)^{1/3})$, independently of the $o((n-4)^{1/3})$ terms. Thus it agrees with the CCJ[1] definition. Moreover, replacing "$\frac{1}{6}$" by any other finite value gives a divergent $\theta_{\mu\nu}$. These results are *not*[3,4] true order by order in perturbation theory—for more light on this see Secs. IV and V.

Our proof uses 't Hooft's method[14] to sum the divergences. This of course assumes that the perturbation theory account of the divergences is sufficiently close to the truth. So, at the present time, our results must be regarded as heuristic, although very suggestive.

Note that assuming summability of the divergences and of the RG coefficients is not necessarily wrong, for in those (superrenormalizable) theories whose existence is proved the divergences are precisely those indicated by perturbation theory.[23] Of course in such theories there are only finitely many divergences, which greatly simplifies matters. However, at least it is known that summability of the perturbation series (for the divergences) is possible.

In any case the results here can be taken to mean that $\theta_{\mu\nu}$ as defined by RG covariance or by the finite improvement program (Sec. IV) is the embodiment of the CCJ idea. [In Sec. IV one particular choice of $H_0(n)=\frac{1}{6}+O(n-4)$ will be found to renormalize $\theta_{\mu\nu}$ in perturbation theory.]

The difference between the RG-covariant $\theta_{\mu\nu}$ and the CCJ $\theta_{\mu\nu}$ with improvement coefficient H_0 is

$$\begin{aligned}\theta_{\mu\nu}-{}^{\text{CCJ}}\theta_{\mu\nu}&\equiv(H_0Z-G-kZ_m)(\partial_\mu\partial_\nu-g_{\mu\nu}\Box)\phi^2\\ &=(H_0ZZ_m{}^{-1}-GZ_m{}^{-1}-k)(\partial_\mu\partial_\nu-g_{\mu\nu}\Box)N[\phi^2]\,.\end{aligned} \tag{3.1}$$

Here the normal product $N[\phi^2]$ is defined by the conventions of Ref. 24. Since $N[\phi^2]$ is finite, we only need consider

$$\Delta\equiv H_0ZZ_m{}^{-1}-GZ_m{}^{-1}-k\,. \tag{3.2}$$

Now Eqs. (2.6), (2.7), and (2.12) imply that

$$\beta(ZZ_m{}^{-1})'+\gamma_mZZ_m{}^{-1}=0\,, \tag{3.3}$$

$$\beta(GZ_m{}^{-1})'+\gamma_mGZ_m{}^{-1}=\zeta\,. \tag{3.4}$$

Hence

$$ZZ_m{}^{-1}=\exp\left[-\int_0^{g_R}\frac{\tilde\gamma_m(g)}{n-4+g\tilde\beta(g)}dg\right], \tag{3.5}$$

$$GZ_m{}^{-1}=ZZ_m{}^{-1}\int_0^{g_R}\frac{\tilde\zeta(g)}{n-4+g\tilde\beta(g)}Z^{-1}Z_m dg\,, \tag{3.6}$$

where the boundary conditions that $ZZ_m^{-1}=1$ and $GZ_m^{-1}=0$ when $g_R=0$ and $n\neq 4$ have been used.

Next we examine the behavior of ZZ_m^{-1} and GZ_m^{-1} as $n\to 4$ with g_R fixed. This behavior is singular since the integrals in Eqs. (3.5) and (3.6) diverge if $n=4$.

Separating out the leading behavior at $g\sim 0$ of the integrand in Eq. (3.5) gives

$$ZZ_m^{-1}=\left[\frac{g_R\tilde\beta(0)+n-4}{n-4}\right]^{1/3}\exp\left\{\int_0^{g_R}dg\left[\frac{\tilde\gamma_m(0)}{n-4+g\tilde\beta(0)}-\frac{\tilde\gamma_m(g)}{n-4+g\tilde\beta(g)}\right]\right\}, \tag{3.7}$$

where Eqs. (2.14) have been used. Hence

$$\begin{aligned}\tfrac{1}{6}ZZ_m^{-1}-GZ_m^{-1}&=ZZ_m^{-1}\left(\frac{1}{6}-\int_0^{g_R}dg\frac{\tilde\xi(g)(n-4)^{1/3}}{[n-4+g\tilde\beta(g)][n-4+g\tilde\beta(0)]^{1/3}}\exp\left\{\int_0^{g}dg'\left[\frac{\tilde\gamma_m(g')}{n-4+g'\tilde\beta(g')}-\frac{\tilde\gamma_m(0)}{n-4+g'\tilde\beta(0)}\right]\right\}\right)\\
&=ZZ_m^{-1}\left[\frac{1}{6}\left[\frac{n-4}{n-4+g_R\tilde\beta(0)}\right]^{1/3}-(n-4)^{1/3}\int_0^{g_R}dg\left(\frac{\tilde\xi(g)}{[n-4+g\tilde\beta(g)][n-4+g\tilde\beta(0)]^{1/3}}\right.\right.\\
&\qquad\times\exp\left\{\int_0^{g}dg'\left[\frac{\tilde\gamma_m(g')}{n-4+g'\tilde\beta(g')}-\frac{\tilde\gamma_m(0)}{n-4+g'\tilde\beta(0)}\right]\right\}\\
&\qquad\left.\left.-\frac{\tilde\xi(0)}{[n-4+g\tilde\beta(0)]^{4/3}}\right)\right].\end{aligned} \tag{3.8}$$

So using Eqs. (2.19), (2.20), and (3.7) we get

$$\tfrac{1}{6}ZZ_m^{-1}-GZ_m^{-1}-k\to 0 \text{ as } n\to 4\,. \tag{3.9}$$

Hence Δ in Eq. (3.2) is zero at $n=4$ if $H_0=\frac{1}{6}$. Also, if a term that vanishes faster than $(n-4)^{1/3}$ is added to H_0, then Δ still vanishes. This is because by Eq. (3.7) ZZ_m^{-1} is of order $(n-4)^{-1/3}$. Thus, in particular, we can take $H_0=\frac{1}{4}(n-2)/(n-1)$, as suggested by the arguments of Sec. I.

This completes the proof that ${}^{\rm CCJ}\theta_{\mu\nu}$, with the original improvement coefficient, is finite and equal to the RG-covariant $\theta_{\mu\nu}$. The question of whether this is true order by order is taken up in the next two sections.

IV. PERTURBATIVE FINITE IMPROVEMENT

In Sec. III we "proved" that the CCJ[1] tensor is finite after summing perturbation theory. It does not follow that this tensor is finite order by order. In fact, as we will see, it is not.

The task of this section is to prove existence and uniqueness of a finite $H_0=H_0(g_R,m_R,\mu,n)$ which renormalizes $\theta_{\mu\nu}$ in perturbation theory. We will see that this H_0 depends in fact on n only, and the one-loop calculation in Ref. 1 implies that $H_0=\frac{1}{6}+O(n-4)$. Then in Sec. V we will show that $H_0=\frac{1}{4}(n-2)/(n-1)+O((n-4)^3)$, where the correction terms do not vanish. So CCJ's ansatz, even when dimensionally regularized correctly, does not perturbatively renormalize $\theta_{\mu\nu}$. However, the $\theta_{\mu\nu}$ we do construct will be proved to coincide at $n=4$ with the RG-covariant one.

The result that a finite improvement works perturbatively is in apparent contradiction with Ref. 4. However, there it was assumed that $H_0=\frac{1}{6}+F(g_R)$. But the results about to be proved show that no such ansatz can work, given[4] that higher-order corrections to $H_0=\frac{1}{6}$ are needed.

We consider the energy-momentum tensor

$$\theta_{\mu\nu}(H_0)=T_{\mu\nu}-ZH_0(\partial_\mu\partial_\nu-g_{\mu\nu}\Box)\phi^2\,, \tag{4.1}$$

where H_0 is to be determined, but Z, $m_B{}^2$, and g_B are as in Sec. II. Comparison with Eqs. (2.1) and (2.2) shows that

$$\theta_{\mu\nu}(H_0)=\text{finite}-(ZH_0-G)Z_m^{-1}(\partial_\mu\partial_\nu-g_{\mu\nu}\Box)N[\phi^2]\,. \tag{4.2}$$

So $\theta_{\mu\nu}$ is finite if and only if $Z_m^{-1}(ZH_0-G)$ is finite. We will now prove the following:

(a) H_0 can be chosen as a power series in g_R and $n-4$, such that, perturbatively, $Z_m^{-1}(ZH_0-G)$ is finite at $n=4$. The coefficients in the power series may depend on m_R and μ. Such an H_0 is unique and in fact depends only on n.

(b) $Z_m^{-1}(ZH_0-G)-k$ is of order $n-4$ near $n=4$, where k is defined by Eq. (2.13).

Statement (a) is of the success of the finite improvement program; statement (b) is that it is equivalent at $n=4$ to using the RG-covariant $\theta_{\mu\nu}$. Thus all the possibilities for a "natural" $\theta_{\mu\nu}$ agree.

First recall the RGE's (3.3) and (3.4) for ZZ_m^{-1} and GZ_m^{-1} and the defining equation (2.13) of k:

$$\left(\beta\frac{\partial}{\partial g_R}+\gamma_m\right)Z_m^{-1}Z=0\,, \tag{4.3}$$

$$\left(\beta\frac{\partial}{\partial g_R}+\gamma_m\right)Z_m^{-1}G=\zeta\,, \tag{4.4}$$

$$\left(\bar\beta\frac{\partial}{\partial g_R}+\gamma_m\right)k=-\zeta\,, \tag{4.5}$$

where $\partial k/\partial m_R^2 = 0$ has been used.

We need the following series expansions:

$$Z_m^{-1}Z = \sum_{N=0}^{\infty} g_R^N \Theta_N(n) = 1 + \sum_{N=1}^{\infty}\sum_{M=1}^{N} \frac{g_R^N \theta_{MN}}{(n-4)^M}, \tag{4.6}$$

$$Z_m^{-1}G = \sum_{N=1}^{\infty} g_R^N \Omega_N(n) = \sum_{N=1}^{\infty}\sum_{M=1}^{N} \frac{g_R^N \omega_{MN}}{(n-4)^M}, \tag{4.7}$$

$$k = \sum_{N=0}^{\infty} k_N g_R^N, \tag{4.8}$$

$$\beta = (n-4)g_R + \bar\beta(g_R) = (n-4)g_R + \sum_{N=2}^{\infty} \beta_N g_R^N, \tag{4.9}$$

$$\gamma_m = \sum_{N=1}^{\infty} \gamma_{mN} g_R^N, \tag{4.10}$$

$$\zeta = \sum_{N=1}^{\infty} \zeta_N g_R^N. \tag{4.11}$$

It is convenient to present the proofs in a series of lemmas and theorems. First comes a technical result whose point will be apparent when it is used in lemma 2.

Lemma 1. Suppose α is not an integer. Define for each non-negative integer M an $(L+1)$-dimensional vector $\vec{x}^M$ with components

$$x_N^M = \frac{\Gamma(\alpha+1)}{\Gamma(M+N+1)\Gamma(\alpha-N-M+1)}, \quad \text{for } 0 \le N \le L. \tag{4.12}$$

Then the $\vec{x}^M$ span the space of $(L+1)$-dimensional vectors.

Proof. Define vectors $\vec{y}^{M,m}$ by

$$y_N^{M,m} = N(N-1)\cdots(N-m+1) \times \frac{\Gamma(M+2m+1)\Gamma(\alpha-m-M+1)}{\Gamma(N+M+m+1)\Gamma(\alpha-N-M+1)} = \frac{\Gamma(N+1)\Gamma(M+2m+1)\Gamma(\alpha-m-M+1)}{\Gamma(N-m+1)\Gamma(N+M+m+1)\Gamma(\alpha-N-M+1)}. \tag{4.13}$$

Then we have the following:

(i) The first m components of $\vec{y}^{M,m}$ are zero, while the rest are nonzero. Hence the $\vec{y}$'s span the space.

(ii) $\vec{y}^{M,0} = \vec{x}^M \Gamma(M+1)\Gamma(\alpha-M+1)/\Gamma(\alpha+1)$.

(iii) $\vec{y}^{M,m+1} = (M+2m+1)(M+2m+2)/(\alpha+m+1) \times (\vec{y}^{M,m} - \vec{y}^{M+1,m})$. Since $\alpha+m+1$ is never zero, this implies that the $\vec{x}$'s span the space.

Lemma 2. Suppose that $X(n, g_R, m_R)$ is a power series in $(n-4)$ and g_R, hence finite at $n=4$. If $XZ_m^{-1}Z$ is finite at $n=4$, then $X=0$.

Proof. Consider the leading terms in the expansion of $XZ_m^{-1}Z$ in powers $g_R^a(n-4)^b$. The leading terms are those for which $a+b$ is least.

Suppose that $X \neq 0$; then let the leading terms of X be

$$\sum_{m=0}^{l} (n-4)^m g_R^{l-m} a_m, \tag{4.14}$$

where at least one a_m is nonzero. The leading terms of $Z_m^{-1}Z$ are obtained from Eq. (4.3) using the first nontrivial terms of β and γ_m [see Eqs. (2.14)], and we have

$$Z_m^{-1}Z = \left[1 + \frac{3g_R}{16\pi^2(n-4)}\right]^{1/3} + \text{nonleading terms} = \sum_{N=0}^{\infty} \left[\frac{3g_R}{16\pi^2(n-4)}\right]^N \frac{\Gamma(\frac{4}{3})}{\Gamma(N+1)\Gamma(\frac{4}{3}-N)}. \tag{4.15}$$

Since $XZ_m^{-1}Z$ is finite, in particular its leading pole terms are zero. These are contained in the product of (4.14) and (4.15) and hence

$$\sum_{m=0}^{l} \left(\frac{3}{16\pi^2}\right)^m \frac{a_m \Gamma(\frac{4}{3})}{\Gamma(m+N+1)\Gamma(\frac{4}{3}-N-m)} = 0, \tag{4.16}$$

for every positive integer N. So, by lemma 1, all the a_m's vanish, contrary to hypothesis. Hence $X=0$. Note that the proof would fail if the exponent $\frac{1}{3}$ in (4.15) were replaced by an integer.

Theorem 1. If $H_0(n, g_R, m_R, \mu)$ is finite and if

$$Z_m^{-1}(ZH_0 - G) \tag{4.17}$$

is finite, then H_0 is unique and is independent of g_R, m_R, and μ.

Proof. If there is also an $\bar{H}_0$, say satisfying the conditions of the theorem, then $(H_0 - \bar{H}_0)ZZ_m^{-1}$ is finite, and lemma 2 gives $H_0 = \bar{H}_0$, so that H_0 is unique.

Now Z_m, Z, and G are independent of m_R, so differentiating (4.17) with respect to m_R shows that $Z_m^{-1}Z\partial H_0/\partial m_R$ is finite and hence that $\partial H_0/\partial m_R = 0$. Similarly, $\partial H_0/\partial \mu = 0$.

Applying $(\beta\, \partial/\partial g_R + \gamma_m)$ to (4.17) and using Eqs. (4.3) and (4.4) shows that $Z_m^{-1}Z\beta\partial H_0/\partial g_R$ is finite, and hence that $\partial H_0/\partial g_R = 0$.

Theorem 2. Let $H_0 = \sum_{N=0}^{\infty} \eta_N (n-4)^N$, and define the η_N's such that the terms of order $(n-4)^{-1}$ in $Z_m^{-1}(ZH_0 - G)$ vanish, i.e., define recursively

$$\eta_N = \left(\omega_{1,N+1} - \sum_{M=0}^{N-1} \eta_M \theta_{M+1,N+1}\right) \Big/ \theta_{N+1,N+1}. \tag{4.18}$$

Then $H_0 Z_m^{-1} Z - G Z_m^{-1}$ is finite.

Proof. Note that $\theta_{NN} \neq 0$, by Eq. (4.15). So Eq. (4.18) does define η_N.

Now picking out the order g_R^N terms in Eq. (4.3) gives (if $N>0$)

$$\Theta_N=\frac{-1}{N(n-4)}\sum_{M=0}^{N-1}(\gamma_{m,N-M}+M\beta_{N+1-M})\Theta_M\,. \tag{4.19}$$

Similarly, Eq. (4.4) gives

$$\Omega_N=\frac{1}{N(n-4)}\left[\zeta_N-\sum_{M=0}^{N-1}(\gamma_{m,N-M}+M\beta_{N+1-M})\Omega_M\right]. \tag{4.20}$$

Hence

$$H_0\Theta_N-\Omega_N=\frac{-1}{N(n-4)}\left[\zeta_N+\sum_{M=0}^{N-1}(\gamma_{m,N-M}+M\beta_{N+1-M})\times(H_0\Theta_M-\Omega_M)\right]. \tag{4.21}$$

Let us prove by induction on N that $H_0\Theta_N-\Omega_N$ is finite. Certainly this is true for $N=0$. So suppose $H_0\Theta_M-\Omega_M$ is finite for $0\le M\le N-1$. Then the factor in square brackets in Eq. (4.21) is finite. So $H_0\Theta_N-\Omega_N$ has at worst a single pole, whose vanishing is precisely the definition (4.18) of H_0. Hence $H_0\Theta_N-\Omega_N$ is finite, and by induction this is true for all N. The theorem is thus proved.

Theorem 3.

$$Z_m^{-1}(ZH_0-G)-k=O(n-4)\text{ as }n\to 4\,. \tag{4.22}$$

Proof. By Eqs. (2.8), (4.3), and (4.4)

$$\left\{\left[(n-4)g_R+\bar\beta\right]\frac{\partial}{\partial g_R}+\gamma_m\right\}(Z_m^{-1}ZH_0-Z_m^{-1}G)=-\zeta\,. \tag{4.23}$$

Now $Z_m^{-1}(ZH_0-G)$ is finite at $n=4$. Thus Eq. (4.23) shows that $Z_m^{-1}(ZH_0-G)$ evaluated at $n=4$ satisfies the same equation as k. Since it also satisfies the boundary condition of being a power series in g_R, it must equal k. Equation (4.22) follows.

Note the following:

(1) Since η_0 is determined by the one-loop divergences, H_0 is $\frac{1}{6}$ at $n=4$, and this is the CCJ[1] value.

(2) The power series for H_0 is to be regarded as formal, like the ordinary perturbation series. It is effectively a loop expansion: η_N only contributes at $n=4$ when ZZ_m^{-1} has a pole of order N, i.e., at N-loop order.

(3) Although η_N contributes to the finite part of $\theta_{\mu\nu}$ at N loops, it is determined by the divergences in G at $N+1$ loops.

(4) An apparent paradox is that in the proof of finiteness (theorem 2), the Ω_N's enter only as spectators and their divergences seem to be irrelevant. However, these are determined by the ζ_N's, which do enter nontrivially.

(5) In trying to make $Z_m^{-1}(ZH_0-G)$ finite, we have precisely enough conditions on H_0 to be able to eliminate the single-pole terms. Then the essential part of the proof is that the homogeneous parts of the RGE's (4.3) and (4.4) are the same. Consequently the vanishing of the single-pole terms propagates to all the poles.

(6) When $\theta_{\mu\nu}$ is expressed in terms of the bare field, its only dependence on H_0 is that indicated explicitly in Eq. (1.3). Thus the uniqueness of the H_0 satisfying the finite improvement program extends to all renormalization prescriptions, if dimensional regularization is used.

V. FAILURE OF $H_0=\frac{1}{4}(n-2)/(n-1)$

The observation that prompted this paper was that to take $H_0=\frac{1}{4}(n-2)/(n-1)$ would ensure success of the finite improvement program at the three-loop level.[25] As far as the leading divergences are concerned, mere examination of Eqs. (1.25) and (1.26) of Ref. 4 suffices. Complete cancellation of the divergences in $\theta_{\mu\nu}$ requires that Green's functions with an insertion of $(n-4)g_B\phi^4$ at momentum q be finite. A fairly straightforward calculation[26] proves this at the three-loop level.

Cancellation of divergences between three-loop diagrams appears to be highly nonaccidental. So it is necessary to understand it. What will emerge is that the cancellation is in fact accidental. An examination of the renormalization group for θ^μ_μ will show that the divergence at a particular order is really a divergence at one order less. Then the cancellation at three loops turns out to be a consequence of the *topology* of one- and two-loop self-energy graphs. The appropriate three-loop calculation will prove that $H_0=\frac{1}{4}(n-2)/(n-1)$ gives a divergent $\theta_{\mu\nu}$ at four-loops; a correction of order $(n-4)^3$ is needed to H_0.

Note that $H_0=\frac{1}{4}(n-2)/(n-1)$ gives an incorrect, though finite, $\theta_{\mu\nu}$ at three loops. The correct η_3, which contributes to the finite part to this order, is, in the method of Sec. IV, used to cancel a divergence in four-loop diagrams. What we are about to prove will show that this divergence is implied by certain properties of three-loop diagrams.

To prove all this we need to consider the renormalization of scalar operators of dimension 4 and less, viz., ϕ^2, ϕ^4, $(\partial\phi)^2$, and $\Box\phi^2$. By power counting these form a closed set under renormalization. At momentum $q=0$ their renormalization is determined in terms of g_B, m_B^2, and Z by use of the action principle and the equation of motion. When $q\neq 0$, extra terms proportional to $\Box\phi^2$ are needed.

Defining the counterterms to be pure pole series, we write

$$m_R^2 N[\phi^2] = m_B^2 \phi^2 , \tag{5.1}$$

$$\mu^{4-n} \frac{g_R}{4!} N[\phi^4] = \frac{g_R}{4!} \frac{\partial g_B}{\partial g_R} \phi^4 - \tfrac{1}{2} g_R \frac{\partial Z}{\partial g_R} (\partial \phi)^2 + \tfrac{1}{2} g_R \frac{\partial m_B^2}{\partial g_R} \phi^2 + \tfrac{1}{2} A \Box \phi^2 , \tag{5.2}$$

$$\tfrac{1}{2} N[(\partial \phi)^2] = \frac{1}{12}\left(g_R \frac{\partial g_B}{\partial g_R} - g_B \right) \phi^4 + \left(\tfrac{1}{2} Z - g_R \frac{\partial Z}{\partial g_R}\right) (\partial \phi)^2 + g_R \frac{\partial m_B^2}{\partial g_R} \phi^2 + \tfrac{1}{2} B \Box \phi^2 , \tag{5.3}$$

$$\Box N[\phi^2] = Z_m \Box \phi^2 . \tag{5.4}$$

Equation (5.4) is of course implied by Eq. (5.1). Vacuum expectation values are for us irrelevant and so are subtracted out implicitly. The normal products are defined (in n dimensions) by Ref. 24, and A and B depend only on g_R and n, by Ref. 19.

Now we have the equations of motion[24]

$$\tfrac{1}{6} \mu^{4-n} g_R N[\phi^4] + m_R^2 N[\phi^2] + N[\phi \Box \phi] = 0 , \tag{5.5}$$

$$\tfrac{1}{6} g_B \phi^4 + m_B^2 \phi^2 + Z \phi \Box \phi = 0 . \tag{5.6}$$

Hence, because $\Box N[\phi^2] = 2N[(\partial\phi)^2] + 2N[\phi\Box\phi]$, we have

$$B = 2A - \tfrac{1}{2}(Z - Z_m) . \tag{5.7}$$

Finiteness of $\theta_{\mu\nu}$ is equivalent[1-4] to finiteness of its trace, and from Eq. (4.1) we have

$$\begin{aligned}\theta^\mu_\mu &= (\tfrac{1}{2}n - 1) Z \phi \Box \phi + \tfrac{1}{2} n m_B^2 \phi^2 + \tfrac{1}{24} n g_B \phi^4 + [\tfrac{1}{2} - \tfrac{1}{4} n - H_0 (1-n)] Z \Box \phi^2 \\ &= m_B^2 \phi^2 + (4-n) g_B \phi^4 / 4! + [\tfrac{1}{2} - \tfrac{1}{4} n - H_0(1-n)] Z \Box \phi^2 ,\end{aligned} \tag{5.8}$$

where the equation of motion (5.6) has been used.

From Eqs. (5.1) to (5.4) it follows with the aid of definitions (2.5) to (2.7) that

$$\frac{g_B \phi^4}{4!} = \frac{(\beta - 2 g_R \gamma)}{n-4} \frac{\mu^{4-n}}{4!} N[\phi^4] + \frac{\tfrac{1}{2}\gamma}{n-4} N[(\partial\phi)^2] - \frac{(\gamma + \gamma_m)}{(n-4)} \frac{m_R^2}{2} N[\phi^2] + \frac{Z_m^{-1}}{n-4}\left[\tfrac{1}{4}\gamma (Z - Z_m) - \frac{\beta A}{2 g_R}\right] \Box N[\phi^2] . \tag{5.9}$$

So

$$\begin{aligned}\theta^\mu_\mu &= \tfrac{1}{24} \mu^{4-n} (2 \gamma g_R - \beta) N[\phi^4] - \tfrac{1}{2}\gamma N[(\partial\phi)^2] + \tfrac{1}{2} m_R^2 (2 + \gamma + \gamma_m) N[\phi^2] \\ &\quad + \{[\tfrac{1}{2} - \tfrac{1}{4} n - (1-n) H_0] Z + \tfrac{1}{2} \beta A / g_R + \tfrac{1}{4}\gamma (Z_m - Z)\} \times Z_m^{-1} \Box N[\phi^2] \\ &= \text{finite} + (D Z_m^{-1} Z + \tfrac{1}{2} E) \Box N[\phi^2] ,\end{aligned} \tag{5.10}$$

where

$$D = (n-1) H_0 - \tfrac{1}{4}(n-2) , \tag{5.11}$$

$$E = Z_m^{-1} \beta A / g_R - \tfrac{1}{2} Z_m^{-1} Z \gamma . \tag{5.12}$$

Then finiteness of $\theta_{\mu\nu}$ is equivalent to finiteness of $Z_m^{-1} Z D + \tfrac{1}{2} E$. The results of Sec. IV show that there is a unique finite choice of D in perturbation theory that makes $\theta_{\mu\nu}$ finite. The question is now whether this choice is $D = 0$.

Now the only one- and two-loop diagrams contributing to A and Z have the topology of Fig. 1. So it is easy to see that, to $O(g_R)$, $A/g_R = \tfrac{1}{2} Z'$ $(= \tfrac{1}{2} Z\gamma/\beta)$. This implies that $D = O((n-4)^2)$, otherwise $Z_m^{-1} Z D$ would have a pole of $O(g_R^2)$. Later we will do better by use of the renormalization group: In fact the same information implies that $D = O((n-4)^3)$.

To continue the proof, we define column vectors of the dimension-4 operators

$$\Phi_0 = (Z^2 \phi^4 / 4!, \tfrac{1}{2} Z (\partial\phi)^2, \tfrac{1}{2} Z \phi^2, \tfrac{1}{2} Z \Box \phi^2)^T , \tag{5.13}$$

$$\Phi_R = (\mu^{4-n} N[\phi^4]/4!, \tfrac{1}{2} N[(\partial\phi)^2], \tfrac{1}{2} N[\phi^2], \tfrac{1}{2} \Box N[\phi^2])^T . \tag{5.14}$$

So Eqs. (5.1) to (5.4) are equivalent to

$$\Phi_R = M \Phi_0 , \tag{5.15}$$

where M is the matrix

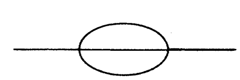

FIG. 1. Two-loop diagrams for A and Z have this topology. No one-loop diagrams contribute.

$$\begin{bmatrix} \frac{\partial g_B}{\partial g_R} Z^{-2} & -\frac{\partial Z}{\partial g_R} Z^{-1} & \frac{\partial m_B{}^2}{\partial g_R} Z^{-1} & AZ^{-1}/g_R \\ 2Z^{-2}\left(g_R\frac{\partial g_B}{\partial g_R} - g_B\right) & 1 - 2g_R Z^{-1}\frac{\partial Z}{\partial g_R} & 2g_R Z^{-1}\frac{\partial m_B{}^2}{\partial g_R} & BZ^{-1} \\ 0 & 0 & Z_m Z^{-1} & 0 \\ 0 & 0 & 0 & Z_m Z^{-1} \end{bmatrix} . \quad (5.16)$$

Consider Green's functions G_{N,X_R} with an insertion of the operator $X_R\Phi_R$ where X_R is a row vector of numbers. Define

$$X_0 = X_R M \quad (5.17)$$

so that $X_0\Phi_0 = X_R\Phi_R$. Then the RG equation for G_{N,X_R} is

$$0 = \left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g_R} - \gamma_m m_R{}^2\frac{\partial}{\partial m_R{}^2} + \frac{N}{2}\gamma + \gamma_X\frac{\partial}{\partial X_R}\right)G_{N,X_R}, \quad (5.18)$$

where

$$\begin{aligned} \gamma_X &= \mu\frac{\partial}{\partial\mu}X_R\Big|_{\text{fixed } g_0, m_0{}^2, X_0} \\ &= -X_R\left(\mu\frac{\partial M}{\partial\mu}\right)_{g_0, m_0{}^2} M^{-1} \\ &= -X_R\left(\mu\frac{\partial M}{\partial\mu} + \beta\frac{\partial M}{\partial g_R} - \gamma_m m_R{}^2\frac{\partial M}{\partial m_R{}^2}\right)M^{-1} . \end{aligned} \quad (5.19)$$

A tedious calculation with the aid of Eqs. (2.5) to (2.7) gives

$$\gamma_X = X_R \begin{bmatrix} \beta' - 2g_R\gamma' & \gamma' & -m_R{}^2(\gamma' + \gamma'_m) & -\alpha \\ 2g_R\beta' - 2\beta - 4g_R{}^2\gamma' & 2g_R\gamma' & -2g_R m_R{}^2(\gamma' + \gamma'_m) & -2g_R\alpha - \gamma_m/2 \\ 0 & 0 & -\gamma_m & 0 \\ 0 & 0 & 0 & -\gamma_m \end{bmatrix} , \quad (5.20)$$

where the finite RG coefficient α is defined by

$$\alpha = Z_m{}^{-1}(A\beta/g_R)' - \gamma A Z_m{}^{-1}/g_R + \tfrac{1}{2}\gamma' - \tfrac{1}{2}\gamma' Z Z_m{}^{-1} . \quad (5.21)$$

Thus by Eq. (5.12)

$$\beta E' + \gamma_m E = \beta(\alpha - \tfrac{1}{2}\gamma') . \quad (5.22)$$

Suppose the first nonzero term in $\alpha - \frac{1}{2}\gamma'$ is of order $g_R{}^m$. Note that by their definitions α and γ depend only on g_R. Then Eq. (5.22) shows that the lowest-order divergent term in E is $O(g_R{}^{m+2})$, and it is a single pole. We now use the explicit form (4.15) for the leading terms in $Z_m{}^{-1}Z$. In order for θ^μ_μ and hence $\theta_{\mu\nu}$ to be finite, it follows from Eq. (5.10) that $D = O((n-4)^{m+1})$ with nonzero coefficient of $(n-4)^{m+1}$. Thus to take $H_0 = \frac{1}{4}(n-2)/(n-1)$ would give correctly only $\eta_0, \eta_1, \ldots, \eta_m$ (in the notation of Sec. IV). The corresponding $\theta_{\mu\nu}$ is finite at the $(m+1)$-loop level and correct at m loops.[27] [Note that η_{m+1} contributes to the finite part but not to the divergence at the $(m+1)$-loop level.]

So far, we know that $\alpha = \frac{1}{2}\gamma'$ to order g_R. So m is at least 2. Next we calculate $\alpha - \frac{1}{2}\gamma'$ to $O(g_R{}^2)$, and find it to be nonzero, so that $m = 2$.

To lowest order we have[21]

$$\begin{aligned} g_B &= \mu^{4-n}\{g_R - 3g_R{}^2/[16\pi^2(n-4)] + O(g_R{}^3)\} , \\ Z &= 1 + \tfrac{1}{12}g_R^2/[(16\pi^2)^2(n-4)] + O(g_R{}^3) . \end{aligned} \quad (5.23)$$

From earlier remarks it follows that

$$A = \tfrac{1}{12}g_R{}^2/[(16\pi^2)^2(n-4)] + O(g_R{}^3) . \quad (5.24)$$

Since Z and A are independent[19] of m_R, we set $m_R = 0$ throughout. The three-loop diagrams contributing to Z are given in Fig. 2. They give

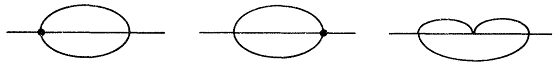

FIG. 2. Diagrams for Z at order $g_R{}^3$. A dot denotes a counterterm.

$$Z = 1 + \frac{1}{n-4}\left[\frac{1}{12}\left(\frac{g_R}{16\pi^2}\right)^2 - \frac{1}{24}\left(\frac{g_R}{16\pi^2}\right)^3\right] - \frac{1}{6}\frac{1}{(n-4)^2}\left(\frac{g_R}{16\pi^2}\right)^3 + O(g_R{}^4)\,, \qquad (5.25)$$

and hence

$$\gamma = \tfrac{1}{6} g_R{}^2 (16\pi^2)^{-2} - \tfrac{1}{8} g_R{}^3 (16\pi^2)^{-3} + O(g_R{}^4)\,. \qquad (5.26)$$

The one-particle irreducible (1PI) diagrams with divergent q dependence for $\langle T\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{N}[\phi^4](q)\rangle$ at order $g_R{}^3$ are given in Fig. 3. Counterterms for the inserted operator are given by Eq. (5.2). The diagrams give[26]

$$A = [\tfrac{1}{12}(g_R/16\pi^2)^2 - \tfrac{17}{144}(g_R/16\pi^2)^3]/(n-4) - \tfrac{1}{4}(g_R/16\pi^2)^3/(n-4)^2 + O(g_R{}^4)\,. \qquad (5.27)$$

Hence

$$\alpha = \tfrac{1}{6} g_R/(16\pi^2)^2 - \tfrac{17}{48} g_R{}^2/(16\pi^2)^3 + O(g_R{}^3)\,. \qquad (5.28)$$

So

$$\alpha - \tfrac{1}{2}\gamma' = -\tfrac{1}{6} g_R{}^2/(16\pi^2)^3 + O(g_R{}^3)\,, \qquad (5.29)$$

and the results claimed earlier about H_0 are proved.

VI. SOFTNESS OF θ^μ_μ AT A FIXED POINT

Callan, Coleman, and Jackiw[1] tried to determine the improvement coefficient by requiring the trace of $\theta_{\mu\nu}$ to be a soft operator. Unfortunately there are[2] anomalies in the broken conformal symmetry they used. As Lowenstein[5] proved, no improvement of $\theta_{\mu\nu}$ can make its trace soft, in perturbation theory. However, as Schroer[6] explained, θ^μ_μ can be soft if one sums the perturbation series: For one particular value of the improvement coefficient, θ^μ_μ is soft at a fixed point of the renormalization group (i.e., when $\beta = 0$).

In Secs. II–V we have produced a unique "natural" $\theta_{\mu\nu}$. We now show that it satisfies Schroer's condition that θ^μ_μ is soft at a fixed point.

First, we use the relation $\Box N[\phi^2] = 2N[(\partial\phi)^2] + 2N[\phi\Box\phi]$ and the equation of motion (5.5) to rewrite the right-hand side of Eq. (5.10) in terms of $N[\phi^4]$, $N[(\partial\phi)^2]$, and $N[\phi^2]$. This gives

$$\theta^\mu_\mu = \frac{\mu^{4-n} g_R}{4!} N[\phi^4]\left(-\frac{\beta}{g_R} - 4Y\right) + Y N[(\partial\phi)^2] + m_R{}^2(\tfrac{1}{2}\gamma_m + 1 - Y) N[\phi^2]\,, \qquad (6.1)$$

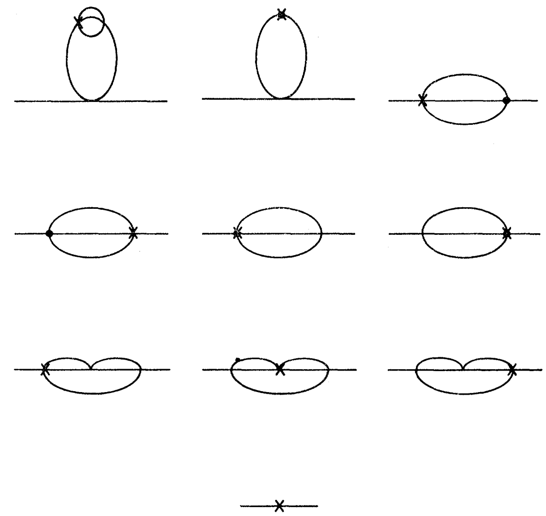

FIG. 3. Diagrams to calculate A. A cross denotes an insertion of ϕ^4 at momentum q, and a cross plus dot denotes one of the counterterms indicated in Eq. (5.2).

where

$$Y = E + 2(n-1)H_0 Z_m{}^{-1} Z + (1 - \tfrac{1}{2}n) Z_m{}^{-1} Z\,, \qquad (6.2)$$

and E is defined by Eq. (5.12). Define H_0 by the finite improvement program. Then Y is certainly finite, since θ^μ_μ is. It is irrelevant whether the trace of $\theta_{\mu\nu}$ is taken before or after the limit $n \to 4$, since it is finite. So it is correct[28] to derive Eq. (6.1) in n dimensions and then set $n = 4$.

Softness at $n = 4$ of θ^μ_μ means that the coefficients of $N[\phi^4]$ and $N[(\partial\phi)^2]$ vanish, i.e., that $\bar{\beta} = 0$ and $Y = 0$. So to show that our $\theta_{\mu\nu}$ agrees with Schroer's we have to prove that $Y = 0$ when $\bar{\beta} = 0$ and $n = 4$. Note that then

$$\theta^\mu_\mu = m_R{}^2(1 + \tfrac{1}{2}\gamma_m) N[\phi^2]\,. \qquad (6.3)$$

Applying $\beta\partial/\partial g_R + \gamma_m$ to Eq. (6.2) and using Eqs. (4.3), (4.4), and (5.22) to substitute for $\beta E' + \gamma_m E$, etc. gives

$$\bar{\beta} Y' + \gamma_m Y = \bar{\beta}(\alpha - \tfrac{1}{2}\gamma') + O(n-4)\,. \qquad (6.4)$$

So if $\gamma_m \neq 0$ at the fixed point then $Y = 0$ there, and so θ^μ_μ is soft, as claimed.

VII. CONCLUSIONS

We have seen that there exists a unique, minimal, finite energy-momentum tensor in ϕ^4 theory. This is perhaps not surprising, given (a) that the homogeneous parts of the RGE's we use are the same,[29] and (b) CCJ's attempted proof[1] of finiteness of their $\theta_{\mu\nu}$. The CCJ proof works for the one-loop diagrams; hence divergences of ${}^{\rm CCJ}\theta_{\mu\nu}$ are nonleading. Thus point (a) plus 't Hooft's methods[14] imply finiteness of ${}^{CCJ}\theta_{\mu\nu}$. Since the CCJ definition is RG-invariant, it must coincide with the definition by RG covariance (unless there is a bad enough singularity at $g_R=0$ or $m_R=0$). Further, point (a) is needed to ensure success of the finite improvement program. It makes consistent the equations for H_0. Uniqueness of the solution of this program implies RG invariance and hence coincidence with all that has gone before.

The argument just presented is of course not a substitute for the proofs given earlier. But it does indicate that the original CCJ proof[1] of finiteness could possibly be considered correct. Add to this the results of Sec. VI and one could say that CCJ's results are as true as they are allowed to be.

Our renormalization of $\theta_{\mu\nu}$ means[10] that the problem of matter interacting with an external gravitational field is dealt with. Only the "vacuum" part[8] of the problem is ignored; this must be investigated.

Another, but much easier, question is to check the way in which our methods carry over to a general renormalizable theory with scalar fields. It would also be useful to know to what extent our results can be duplicated by conventional methods, such as that of Bogoliubov, Parasiuk, Hepp, and Zimmermann. (One can always use the RG covariance idea, but the situation for the other ideas is not clear.)

Since the finite improvement coefficient is independent of the coupling, one might, at first sight, expect the same value to work for an arbitrary theory. (At least one expects this assuming the obvious conjecture that the methods of this paper generalize to any theory.) However, one must recall that the *raison d'être* for H_0 is the existence of a nonzero interaction, though its precise value is irrelevant. So the most reasonable conjecture on the value of H_0 is that it depends on the symmetry class of the theory. Symmetry considerations are the standard way of fixing relations between otherwise arbitrary coupling parameters. For example, to set some particular coupling to zero is, in general, not a condition invariant under change of renormalization prescription. But it is invariant if the coupling is a symmetry-breaking parameter. The idea of some object depending only on the symmetry class is of course reminiscent of the idea of universality in critical phenomena.[30]

The results here can be applied[31] to the renormalization of field theories in curved space. (A curved space is no more than an external gravitational field.) Drummond[32] has explicitly calculated the renormalization constants for massless (Euclidean) ϕ^4 theory on a hypersphere. He used the conformal-invariant form of the free Lagrangian. Up to three-loop order he found that the renormalizations are the same as in flat space. It is a corollary of the results in Sec. V that, in perturbation theory, adjustment of the coefficient of $R\phi^2$ from the conformal value is needed to deal with the four-loop divergences.

In using the present results to renormalize interactions in a curved space, one is treating the curvature as a perturbation around an initial flat space. Such an approach is naturally abhorrent to many general relativists. However, this method permits one to use the standard techniques of renormalization theory in flat space. Even so, it would be useful to reformulate our results directly in curved space. Apart from considerations of elegance, there is liable to be a breakdown of perturbation theory if the curvature is comparable to particle masses. Physically, this occurs even when the radius of curvature is much larger than the Planck length. In such a case our approximation of ignoring quantum corrections to gravity remains valid; this is a situation of at least potential cosmological and astrophysical relevance.

Ultimately any analysis such as ours rests on the assumption that it will not be upset by a proper quantization of gravity. That is, it assumes that to ignore quantum gravity effects (specifically the ultraviolet renormalization) is a sensible approximation. This is a reasonable hypothesis, but its truth or falseness is not known at present.

A preliminary account of the work described in this paper has been given in Ref. 33.

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[26]The calculation is straightforward if one notes that (a) there is no divergence if $q=0$; (b) hence most of the relevant diagrams do not contribute since they have no q-dependent divergences; and (c) by Ref. 19 the divergence in question is independent of m_R, so one can set $m_R=0$.
[27]Since there is a choice of D as a function of n that makes $\theta_{\mu\nu}$ finite, we must have $\beta E' + \gamma_m E = \text{finite}$. However, without the explicit form of the right-hand side of Eq. (5.22) our m-loop calculations would not exclude a pole of order $g_R{}^{m+1}/(n-4)$ in E. Also, the $(m+1)$-loop calculations would not prove nonzero the coefficient of $g_R{}^{m+2}/(n-4)$.
[28]But remember that $g^{\mu\nu} N[\partial_\mu\phi\partial_\nu\phi] \neq N[(\partial\phi)^2]$—see Ref. 24. This is no problem for Eq. (6.1) since $N[\partial_\mu\phi\partial_\nu\phi]$ is not used in deriving it.
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